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A new generalization of Halanay-type inequality and its applications

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Abstract

In this paper, in order to study the dissipativity of nonlinear neutral functional differential equations, a generalization of the Halanay inequality is given. We apply this generalized Halanay inequality to an analysis of the dissipativity of two classes of nonlinear neutral delay integro-differential equations and the sufficient conditions are presented to ensure these systems are dissipative.

Keywords: Neutral delay integro-differential equations; Dynamical systems; Halanay inequality; Functional differential equations

1 Introduction

In 1966, in order to discuss the stability of the zero solution of the delay differential equation

$$u'(t) = Au(t) + Bu(t-\tau), \quad \tau > 0,$$

Halanay introduced the following lemma (see [1] p. 378).

Lemma 1.1 (Basic Halanay inequality) *Assume that* $\tau \geq 0$ *and* $v(t)$ *is a positive function defined on* $[t_0 - \tau, +\infty)$, *with derivative* $v'(t)$ *on* $[t_0, +\infty)$. *If*

$$v'(t) \leq -\alpha v(t) + \beta \sup_{t-\tau \leq s \leq t} v(s), \quad t \geq t_0,$$

where $\alpha > \beta > 0$, *then there exist* $\gamma > 0$ *and* $k > 0$ *such that*

$$v(t) \leq k \exp\bigl(-\gamma (t - t_0)\bigr), \quad t \geq t_0.$$

The Halanay inequality became a powerful tool in the stability theory of delay differential equations, therefore many authors improved or generalized it to more general type and used it for investigating the stability and dissipativity of various functional differential equations. We refer the reader to the papers, for instance, of Baker and Tang [2], Agarwal, Kim and Sen [3, 4], Baker [5], Liz and Trofimchuk [6], Tian [7], Wen, Yu and Wang [8, 9], Liu et al. [10], Wang [11], Hien et al. [12], and Gan [13].

On the other hand, many interesting problems in physics and engineering are modeled by dissipative dynamical systems. These systems are characterized by the property of possessing a bounded absorbing set, which all trajectories enter in a finite time and thereafter remain inside of. In the study of dissipative systems, this asymptotic behavior of the system is of interest and important (see [14]). In 1994, Humphries and Stuart [15] first studied the analytical and numerical dissipativity of initial value problems (IVPs) in ODEs. Hereafter, a number of results on the analytical and numerical dissipativity with respect to various types of differential equations are presented (such as found in [16–21]).

In this paper, we first present a more general Halanay-type inequality in Sect. 2. Then, in Sect. 3, we use this inequality to discuss the analytical dissipativity of two classes of nonlinear neutral delay integro-differential equations (NDIDEs) and some sufficient conditions which ensure the systems to be dissipative are given. Finally, the paper ends with a conclusion.

2 The generalized Halanay inequality

For simplicity of presentation, we denote $f^{[t_1,t_2]} := \sup_{t_1\le\xi\le t_2} f(\xi)$ and $f^{[t_1,+\infty)} := \sup_{\xi\ge t_1} f(\xi)$ for a bounded function f.

Theorem 2.1 *Assume that $\tau \ge 0$ and $u(t), w(t)$ are non-negative functions defined on $[t_0 - \tau, +\infty)$, with derivative $u'(t)$ on $[t_0, +\infty)$. If*

$$\begin{cases} u'(t) \le R_1(t) + A(t)u(t) + B(t)u^{[t-\tau,t]} + C(t)w(t) + D(t)w^{[t-\tau,t]}, \\ w(t) \le R_2(t) + F(t)u(t) + G(t)u^{[t-\tau,t]} + H(t)w^{[t-\tau,t]}, \end{cases} \tag{2.1}$$

for $t \ge t_0$ and

$$w^{[t_0-\tau,t_0]} \le \frac{r_2}{1-H_0} + \frac{F_0+G_0}{1-H_0}u^{[t_0-\tau,t_0]}, \tag{2.2}$$

and there exists a constant $\sigma > 0$ such that

$$A(t) + B(t) + \big(C(t) + D(t)\big)\frac{F_0+G_0}{1-H_0} \le -\sigma, \quad \forall t \ge t_0. \tag{2.3}$$

Then, for $t \ge t_0$, we have

$$\begin{cases} u(t) \le \frac{\gamma^*}{\sigma} + \phi e^{-\mu^*(t-t_0)}, \\ w(t) \le \frac{\gamma_2}{1-H_0} + \frac{F_0+G_0}{1-H_0}\frac{\gamma^*}{\sigma} + \frac{F_0+G_0e^{\mu^*\tau}}{1-H_0e^{\mu^*\tau}}\phi e^{-\mu^*(t-t_0)}, \end{cases} \tag{2.4}$$

where $R_1(t), R_2(t), -A(t), B(t), C(t), D(t), F(t), G(t), H(t)$ are non-negative, continuous and bounded functions defined on $[t_0, +\infty)$;

$$\begin{cases} F_0 = F^{[t_0,+\infty)}, \quad G_0 = G^{[t_0,+\infty)}, \quad C_0 = C^{[t_0,+\infty)}, \quad D_0 = D^{[t_0,+\infty)}, \\ H_0 = H^{[t_0,+\infty)}, \quad \gamma_1 = R_1^{[t_0,+\infty)}, \quad \gamma_2 = R_2^{[t_0,+\infty)}, \quad \phi = u^{[t_0-\tau,t_0]}, \end{cases}$$

and $0 < H(t) \le H_0 < 1$, $\gamma^* = \gamma_1 + \frac{C_0+D_0}{1-H_0}\gamma_2$. The constant $\mu^* > 0$ is defined as

$$\mu^* = \inf_{t\ge t_0}\left\{\mu(t) : \mu(t) + A(t) + B(t)e^{\mu(t)\tau} + \big(C(t) + D(t)e^{\mu(t)\tau}\big)\frac{F_0+G_0e^{\mu(t)\tau}}{1-H_0e^{\mu(t)\tau}} = 0\right\}.$$

Specially, if $R_1(t) = R_2(t) \equiv 0$, (2.4) degenerates into following form:

$$\begin{cases} u(t) \le \phi e^{-\mu^*(t-t_0)}, \\ w(t) \le \frac{F_0+G_0e^{\mu^*\tau}}{1-H_0e^{\mu^*\tau}}\phi e^{-\mu^*(t-t_0)}, \end{cases} \quad t \ge t_0. \tag{2.5}$$

Proof First, if $\tau = 0$, from the second formula of (2.1), we have

$$w(t) \le \frac{R_2(t)}{1-H(t)} + \frac{F(t)+G(t)}{1-H(t)}u(t). \tag{2.6}$$

Substituting (2.6) into the first formula of (2.1) shows that (2.1) degenerates into a differential inequality

$$u'(t) \le R(t) + \widetilde{A}(t)u(t), \quad t \ge t_0,$$

where

$$\begin{cases} R(t) = R_1(t) + (C(t)+D(t))\frac{R_2(t)}{1-H(t)}, \\ \widetilde{A}(t) = A(t)+B(t)+(C(t)+D(t))\frac{F(t)+G(t)}{1-H(t)}. \end{cases}$$

Noting the condition (2.3), it is can be proved that

$$u(t) \le \frac{\gamma^*}{\sigma}\left(1-\exp\left(\int_{t_0}^t \widetilde{A}(s)\,ds\right)\right) + u(t_0)\exp\left(\int_{t_0}^t \widetilde{A}(s)\,ds\right).$$

The combination of this formula and (2.6) shows that (2.4) holds with

$$\mu^* = \inf_{t\ge t_0}\left\{-A(t)-B(t)-\bigl(C(t)+D(t)\bigr)\frac{F(t)+G(t)}{1-H_0}\right\}.$$

It is obvious that $\mu^* > 0$ under the assumption (2.3).

In the following we assume that $\tau > 0$. For any given $t \in [t_0, +\infty)$, we define function $E(\mu)$ on $[0, \frac{1}{\tau}\ln\frac{1}{H_0})$ by

$$E(\mu) := \mu + A(t) + B(t)e^{\mu\tau} + \bigl(C(t)+D(t)e^{\mu\tau}\bigr)\frac{F_0+G_0e^{\mu\tau}}{1-H_0e^{\mu\tau}}. \tag{2.7}$$

From (2.7) we can see that

$$E(0) < 0, \qquad \lim_{\mu\to\frac{1}{\tau}\ln\frac{1}{H_0}-0} E(\mu) = +\infty, \qquad E'(\mu) > 0.$$

Therefore, there exists a unique $\mu \in (0, \frac{1}{\tau}\ln\frac{1}{H_0})$ such that

$$\mu + A(t) + B(t)e^{\mu\tau} + \bigl(C(t)+D(t)e^{\mu\tau}\bigr)\frac{F_0+G_0e^{\mu\tau}}{1-H_0e^{\mu\tau}} = 0, \tag{2.8}$$

which defines an implicit function $\mu(t)$ for $t \ge t_0$. It is obvious that $\mu^* \ge 0$. Now we prove that $\mu^* > 0$.

In fact, if this is not true. Let $\widetilde{H}_0$ satisfying $0 < H_0 < \widetilde{H}_0 < 1$ and let $0 < \varepsilon_1 < \min\{\frac{\sigma}{2}, -\frac{1}{\tau}\ln\widetilde{H}_0, \frac{1}{\tau}\ln(\frac{\sigma}{2Q}+1)\}$, where

$$Q = B_0 + \frac{C_0\widetilde{H}_0(G_0 + F_0H_0) + D_0\widetilde{H}_0(F_0 + G_0) + D_0G_0(1 - H_0)}{(\widetilde{H}_0 - H_0)(1 - H_0)},$$

and $B_0 = B^{[t_0,+\infty)}$.

Then there would exist $t^* \geq t_0$ such that $\hat{\mu} := \mu(t^*) < \varepsilon_1$ and

$$\hat{\mu} + A(t^*) + B(t^*)e^{\hat{\mu}\tau} + \bigl(C(t^*) + D(t^*)e^{\hat{\mu}\tau}\bigr)\frac{F_0 + G_0e^{\hat{\mu}\tau}}{1 - H_0e^{\hat{\mu}\tau}} = 0. \tag{2.9}$$

Substituting (2.3) into (2.9) gives

$$\begin{aligned}
0 &= \hat{\mu} + A(t^*) + B(t^*)e^{\hat{\mu}\tau} + \bigl(C(t^*) + D(t^*)e^{\hat{\mu}\tau}\bigr)\frac{F_0 + G_0e^{\hat{\mu}\tau}}{1 - H_0e^{\hat{\mu}\tau}} \\
&\leq \varepsilon_1 - \sigma + \left(B_0 + \frac{C_0(F_0H_0 + G_0) + D_0(F_0 + G_0) + D_0G_0e^{\varepsilon_1\tau}(1 - H_0))}{(1 - H_0e^{\varepsilon_1\tau})(1 - H_0)}\right)\bigl(e^{\varepsilon_1\tau} - 1\bigr) \\
&\leq \varepsilon_1 - \sigma + Q\bigl(e^{\varepsilon_1\tau} - 1\bigr) < \varepsilon_1 - \sigma + \frac{\sigma}{2} < 0,
\end{aligned}$$

which is a contradiction.

In order to verify (2.4), we first show that, for any $\varepsilon > 0$,

$$\begin{cases} u(t) < \frac{\gamma^*}{\sigma} + \varepsilon + \phi e^{-\mu^*(t-t_0)}, \\ w(t) < \frac{\gamma_2}{1-H_0} + \frac{F_0+G_0}{1-H_0}(\frac{\gamma^*}{\sigma} + \varepsilon) + \frac{F_0+G_0e^{\mu\tau}}{1-H_0e^{\mu\tau}}\phi e^{-\mu^*(t-t_0)}, \end{cases} \quad t \geq t_0. \tag{2.10}$$

In fact, when $t = t_0$, (2.10) is evident by using (2.2).

If we suppose (2.10) is not true for $t > t_0$, then there would exist some $\varepsilon_0 > 0$ and $\varsigma > t_0$ such that when $t < \varsigma$

$$\begin{cases} u(t) < \frac{\gamma^*}{\sigma} + \varepsilon_0 + \phi e^{-\mu^*(t-t_0)}, \\ w(t) < \frac{\gamma_2}{1-H_0} + \frac{G_0+F_0}{1-H_0}(\frac{\gamma^*}{\sigma} + \varepsilon_0) + \frac{F_0+G_0e^{\mu^*\tau}}{1-H_0e^{\mu^*\tau}}\phi e^{-\mu^*(t-t_0)}, \end{cases} \tag{2.11}$$

while when $t = \varsigma$, at least one of the following two equalities is true:

$$u(\varsigma) = \frac{\gamma^*}{\sigma} + \varepsilon_0 + \phi e^{-\mu^*(\varsigma-t_0)} \tag{2.12}$$

and

$$w(\varsigma) = \frac{\gamma_2}{1 - H_0} + \frac{G_0 + F_0}{1 - H_0}\left(\frac{\gamma^*}{\sigma} + \varepsilon_0\right) + \frac{F_0 + G_0e^{\mu^*\tau}}{1 - H_0e^{\mu^*\tau}}\phi e^{-\mu^*(\varsigma-t_0)}. \tag{2.13}$$

However, from the second formula of (2.1), when $\varsigma - \tau \geq t_0$, we have

$$w(\varsigma) \leq R_2(\varsigma) + F(\varsigma)u(\varsigma) + G(\varsigma)\sup_{\varsigma-\tau\leq\xi\leq\varsigma} u(\xi) + H(\varsigma)\sup_{\varsigma-\tau\leq\xi\leq\varsigma} w(\xi)$$

$$
\begin{aligned}
&< R_2(\varsigma) + F(\varsigma)\left(\frac{\gamma^*}{\sigma} + \varepsilon_0 + \phi e^{-\mu^*(\varsigma - t_0)}\right) + G(\varsigma)\left(\frac{\gamma^*}{\sigma} + \varepsilon_0 + \phi e^{-\mu^*(\varsigma - \tau - t_0)}\right) \\
&\quad + H(\varsigma)\left(\frac{\gamma_2}{1 - H_0} + \frac{F_0 + G_0}{1 - H_0}\left(\frac{\gamma^*}{\sigma} + \varepsilon_0\right) + \frac{F_0 + G_0 e^{\mu^*\tau}}{1 - H_0 e^{\mu^*\tau}}\phi e^{-\mu^*(\varsigma - \tau - t_0)}\right) \\
&= R_2(\varsigma) + H(\varsigma)\frac{\gamma_2}{1 - H_0} + \left(F(\varsigma) + G(\varsigma) + H(\varsigma)\frac{F_0 + G_0}{1 - H_0}\right)\left(\frac{\gamma}{\sigma} + \varepsilon_0\right) \\
&\quad + \left(F(\varsigma) + G(\varsigma)e^{\mu^*\tau} + \frac{F_0 + G_0 e^{\mu^*\tau}}{1 - H_0 e^{\mu^*\tau}}H_0 e^{\mu^*\tau}\right)\phi e^{-\mu^*(\varsigma - t_0)} \\
&\le \frac{\gamma_2}{1 - H_0} + \frac{F_0 + G_0}{1 - H_0}\left(\frac{\gamma^*}{\sigma} + \varepsilon_0\right) + \frac{F_0 + G_0 e^{\mu^*\tau}}{1 - H_0 e^{\mu^*\tau}}\phi e^{-\mu^*(\varsigma - t_0)},
\end{aligned}
\tag{2.14}
$$

and, when $\varsigma - \tau < t_0$, we have

$$
\begin{aligned}
w(\varsigma) &\le R_2(\varsigma) + F(\varsigma)u(\varsigma) + G(\varsigma)\max\Big\{\sup_{t_0 - \tau \le \xi \le t_0} u(\xi), \sup_{t_0 \le \xi \le \varsigma} u(\xi)\Big\} \\
&\quad + H(\varsigma)\max\Big\{\sup_{t_0 - \tau \le \xi \le t_0} w(\xi), \sup_{t_0 \le \xi \le \varsigma} w(\xi)\Big\} \\
&< R_2(\varsigma) + F(\varsigma)\left(\frac{\gamma^*}{\sigma} + \varepsilon_0 + \phi e^{-\mu^*(\varsigma - t_0)}\right) + G(\varsigma)\left(\frac{\gamma^*}{\sigma} + \varepsilon_0 + \phi\right) \\
&\quad + H(\varsigma)\left(\frac{\gamma_2}{1 - H_0} + \frac{F_0 + G_0}{1 - H_0}\left(\frac{\gamma^*}{\sigma} + \varepsilon_0\right) + \frac{F_0 + G_0 e^{\mu^*\tau}}{1 - H_0 e^{\mu^*\tau}}\phi\right) \\
&\le \frac{\gamma_2}{1 - H_0} + \frac{F_0 + G_0}{1 - H_0}\left(\frac{\gamma^*}{\sigma} + \varepsilon_0\right) + \frac{F_0 + G_0 e^{\mu^*\tau}}{1 - H_0 e^{\mu^*\tau}}\phi e^{-\mu^*(\varsigma - t_0)}.
\end{aligned}
\tag{2.15}
$$

Hence (2.14) and (2.15) show that (2.13) is not true. Therefore we need only consider the case that (2.12) holds and we shall obtain a contradiction. Set

$$
v(t) = \frac{\gamma^*}{\sigma} + \varepsilon_0 + \phi e^{-\mu^*(t - t_0)}, \qquad z(t) = v(t) - u(t).
$$

Then $z(t) > 0$ for $t < \varsigma$ and $z(\varsigma) = 0$ and $z'(\varsigma) \le 0$. Hence from the first formula of (2.1) we have

$$
\begin{aligned}
z'(\varsigma) &= v'(\varsigma) - u'(\varsigma) \\
&\ge -\phi\mu^* e^{-\mu^*(\varsigma - t_0)} \\
&\quad - \left(R_1(\varsigma) + A(\varsigma)u(\varsigma) + B(\varsigma)u^{[\varsigma - \tau, \varsigma]} + C(\varsigma)w(\varsigma) + D(\varsigma)w^{[\varsigma - \tau, \varsigma]}\right).
\end{aligned}
\tag{2.16}
$$

If $\varsigma - \tau \ge t_0$, it follows from (2.11), (2.12), (2.16) and the definition of γ^* that

$$
\begin{aligned}
z'(\varsigma) &\ge -\gamma^* - \left(\frac{\gamma^*}{\sigma} + \varepsilon_0\right)\left(A(\varsigma) + B(\varsigma) + \big(C(\varsigma) + D(\varsigma)\big)\frac{F_0 + G_0}{1 - H_0}\right) \\
&\quad - \phi e^{-\mu^*(\varsigma - t_0)} \\
&\quad \times \left(\mu^* + A(\varsigma) + B(\varsigma)e^{\mu^*\tau} + \big(C(\varsigma) + D(\varsigma)e^{\mu^*\tau}\big)\frac{F_0 + G_0 e^{\mu^*\tau}}{1 - H_0 e^{\mu^*\tau}}\right).
\end{aligned}
\tag{2.17}
$$

From the definition of the function $\mu(t)$, we have

$$\mu(\varsigma) + A(\varsigma) + B(\varsigma)e^{\mu(\varsigma)\tau} + \bigl(C(\varsigma) + D(\varsigma)e^{\mu(\varsigma)\tau}\bigr)\frac{F_0 + G_0e^{\mu(\varsigma)\tau}}{1 - H_0e^{\mu(\varsigma)\tau}} = 0.$$

Therefore, it is easy to see that

$$\begin{aligned}
&\mu^* + A(\varsigma) + B(\varsigma)e^{\mu^*\tau} + \bigl(C(\varsigma) + D(\varsigma)e^{\mu^*\tau}\bigr)\frac{F_0 + G_0e^{\mu^*\tau}}{1 - H_0e^{\mu^*\tau}} \\
&\quad = \mu^* - \mu(\varsigma) + B(\varsigma)\bigl(e^{\mu^*\tau} - e^{\mu(\varsigma)\tau}\bigr) + C(\varsigma)\biggl(\frac{F_0 + G_0e^{\mu^*\tau}}{1 - H_0e^{\mu^*\tau}} - \frac{F_0 + G_0e^{\mu(\varsigma)\tau}}{1 - H_0e^{\mu(\varsigma)\tau}}\biggr) \\
&\qquad + D(\varsigma)\biggl(\frac{F_0 + G_0e^{\mu^*\tau}}{1 - H_0e^{\mu^*\tau}}e^{\mu^*\tau} - \frac{F_0 + G_0e^{\mu(\varsigma)\tau}}{1 - H_0e^{\mu(\varsigma)\tau}}e^{\mu(\varsigma)\tau}\biggr) \\
&\quad \le 0,
\end{aligned}$$

which substituting into (2.17) and noting the condition (2.3), gives

$$w'(\varsigma) = v'(\varsigma) - u'(\varsigma) \ge \sigma\varepsilon_0 > 0. \tag{2.18}$$

If $\varsigma - \tau < t_0$, it follows from (2.16) that

$$\begin{aligned}
z'(\varsigma) \ge{}& -\phi\mu^*e^{-\mu^*(\varsigma - t_0)} - R_1(\varsigma) - A(\varsigma)u(\varsigma) - B(\varsigma)\max\bigl\{\phi, u^{[t_0,\varsigma]}\bigr\} \\
&- C(\varsigma)w(\varsigma) - D(\varsigma)\max\bigl\{w^{[t_0-\tau,t_0]}, w^{[t_0,\varsigma]}\bigr\}.
\end{aligned}$$

Thus we also can get (2.18) by simple derivation. This is in contradiction with our result $w'(\varsigma) \le 0$. Therefore the inequality (2.10) must hold for any given $\varepsilon > 0$. Since $\varepsilon > 0$ is arbitrary, we let $\varepsilon \to 0$ and obtain (2.4), which completes the proof of Theorem 2.1. □

Remark 2.2 If $R_1(t) = R_2(t) = C(t) = F(t) \equiv 0$, we can obtain expression (2.5). Particularly, if we further assume that $C(t) = F(t) \equiv 0$, then (2.5) degenerates into a conclusion which is present in [11].

3 Dissipativity of two classes of nonlinear neutral functional differential equations

In this section, we consider several simple applications of Theorem 2.1 to the study of dissipativity for two classes of nonlinear neutral functional differential equations.

Let X be a real or complex, finite-dimensional or infinite-dimensional Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$.

3.1 Dissipativity of nonlinear neutral delay integro-differential equations (NNDIDEs)

Consider the IVPs in NNDIDEs as follows:

$$\begin{cases} y'(t) = f(t, y(t), y(t-\tau), y'(t-\tau), \int_{t-\tau}^{t} g(t,\xi,y(\xi))\,d\xi), & t \ge t_0, \\ y(t) = \phi(t), \qquad y'(t) = \phi'(t), & t_0 - \tau \le t \le t_0, \end{cases} \tag{3.1}$$

where τ are positive constant, the functions $f : [t_0, +\infty) \times X \times X \times X \times X \to X$, $g : [t_0, +\infty) \times [t_0 - \tau, +\infty) \times X \to X$, $\phi : [t_0 - \tau, t_0] \to X$ are assumed to be continuous functions and for any $t \geq t_0, y, u, v, w \in X$, f and g satisfy the conditions:

$$\begin{cases} 2\operatorname{Re}\langle f(t,y,u,v,w), y\rangle \leq \alpha\|y\|^2 + \beta\|f(t,0,u,v,w)\|^2, \\ \|f(t,y,u,v,w)\|^2 \leq \gamma_1 + L_y\|y\|^2 + \omega\|f(t,0,u,v,w)\|^2, \\ \|f(t,0,u,v,w)\|^2 \leq \gamma_2 + L_u\|u\|^2 + L_v\|v\|^2 + L_w\|w\|^2, \end{cases} \tag{3.2}$$

and

$$\|g(t,\xi,u)\| \leq \lambda\|u\|, \quad t-\tau \leq \xi \leq t, t \geq t_0, \tag{3.3}$$

where $-\alpha, \beta, \gamma_1, \gamma_2, \omega, \lambda, L_y, L_u, L_v, L_w$ are all non-negative real constants.

Theorem 3.1 *Let problem* (3.1) *satisfy* (3.2) *and* (3.3) *with* $L_v\omega < 1$, *and initial value function* $\phi(t)$ *satisfy*

$$\max_{t_0-\tau\leq t\leq t_0} \|\phi'(t)\|^2 \leq \frac{\gamma_1 + \omega\gamma_2}{1-\omega L_v} + \frac{L_y + \omega(L_u + \lambda^2\tau^2 L_w)}{1-\omega L_v} \max_{t_0-\tau\leq t\leq t_0} \|\phi(t)\|^2.$$

Let $y(t)$ *be the solution of* (3.1). *Assume that there exists a constant* $\sigma > 0$ *such that*

$$\alpha + \frac{\beta(L_u + L_v L_y + \lambda^2\tau^2 L_w)}{1 - L_v\omega} \leq -\sigma. \tag{3.4}$$

Then

(1) *for any* $t \geq t_0$ *we have*

$$\begin{cases} \|y(t)\|^2 \leq \frac{\beta(\gamma_2 + L_v\gamma_1)}{(1-L_v\omega)\sigma} + \phi_0 e^{-\mu^*(t-t_0)}, \\ \|y'(t)\|^2 \leq \frac{\gamma_1+\omega\gamma_2}{1-\omega L_v} + \frac{L_y+\omega(L_u+\lambda^2\tau^2 L_w)}{1-\omega L_v}\frac{\beta(\gamma_2+L_v\gamma_1)}{(1-L_v\omega)\sigma} \\ \qquad + \frac{L_y+\omega(L_u+\lambda^2\tau^2L_w)e^{\mu^*\tau}}{1-\omega L_v e^{\mu^*\tau}}\phi_0 e^{-\mu^*(t-t_0)}, \end{cases} \tag{3.5}$$

where $\phi_0 = \max_{t_0-\tau\leq t\leq t_0} \|\phi(t)\|^2$, $\mu^* > 0$ *is given as follows*:

$$\mu^* = \inf_{t\geq t_0}\Big\{\mu(t) : \mu(t) + \alpha + \beta\big(L_u + \lambda^2\tau^2 L_w\big)e^{\mu(t)\tau} + \frac{L_y + \omega(L_u + L_w\lambda^2\tau^2)e^{\mu(t)\tau}}{1-\omega L_v e^{\mu(t)\tau}}\beta L_v e^{\mu(t)\tau} = 0\Big\}. \tag{3.6}$$

(2) *the system is dissipative, for any* $\varepsilon > 0$ *the open ball*

$$B = B\left(0, \sqrt{\frac{\beta(\gamma_2 + L_v\gamma_1)}{(1-L_v\omega)\sigma} + \varepsilon}\right)$$

is an absorbing set.

Proof Let

$$\begin{cases} u(t) = \|y(t)\|^2, \\ w(t) = \|y'(t)\|^2, \end{cases} \quad t \geq t_0 - \tau \tag{3.7}$$

and

$$z(t) = \int_{t-\tau}^{t} g\bigl(t, \xi, y(\xi)\bigr)\, d\xi, \quad t \geq t_0.$$

Then when $t \geq t_0$, from (3.2) we have

$$\begin{aligned} u'(t) &= \frac{d}{dt}\bigl\langle y(t), y(t)\bigr\rangle \\ &= 2\,\mathrm{Re}\bigl\langle y(t), f\bigl(t, y(t), y(t-\tau), y'(t-\tau), z(t)\bigr)\bigr\rangle \\ &\leq \alpha u(t) + \beta \bigl\| f\bigl(t, 0, y(t-\tau), y'(t-\tau), z(t)\bigr)\bigr\|^2 \\ &\leq \alpha u(t) + \beta\bigl(\gamma_2 + L_u u(t-\tau) + L_v w(t-\tau) + L_w \bigl\|z(t)\bigr\|^2\bigr). \end{aligned} \tag{3.8}$$

Noting (3.3) one obtains

$$\begin{aligned} \bigl\|z(t)\bigr\| &\leq \lambda \int_{t-\tau}^{t} \bigl\|y(\xi)\bigr\|\, d\xi \\ &\leq \lambda\tau \max_{t-\tau \leq \xi \leq t} \bigl\|y(\xi)\bigr\|, \end{aligned}$$

which gives

$$\bigl\|z(t)\bigr\|^2 \leq \lambda^2\tau^2 \max_{t-\tau \leq \xi \leq t} u(\xi). \tag{3.9}$$

Substituting (3.9) into (3.8), we have

$$u'(t) \leq \beta\gamma_2 + \alpha u(t) + \beta\bigl(L_u + L_w\lambda^2\tau^2\bigr)u^{[t-\tau,t]} + \beta L_v w(t-\tau). \tag{3.10}$$

On the other hand, from the second formula of (3.2) and (3.9) we have

$$\begin{aligned} w(t) &= \bigl\| f\bigl(t, y(t), y(t-\tau), y'(t-\tau), z(t)\bigr)\bigr\|^2 \\ &\leq \gamma_1 + L_y u(t) + \omega\bigl(\gamma_2 + L_u u(t-\tau) + L_v w(t-\tau) + L_w \bigl\|z(t)\bigr\|^2\bigr) \\ &\leq \gamma_1 + \omega\gamma_2 + L_y u(t) + \omega L_v w(t-\tau) + \omega\bigl(L_u + L_w\lambda^2\tau^2\bigr)u^{[t-\tau,t]}. \end{aligned} \tag{3.11}$$

Therefore, combining of (3.10) and (3.11), for $t \geq t_0$ we have

$$\begin{cases} u'(t) \leq \beta\gamma_2 + \alpha u(t) + \beta(L_u + L_w\lambda^2\tau^2)u^{[t-\tau,t]} + \beta L_v w^{[t-\tau,t]}, \\ w(t) \leq \gamma_1 + \omega\gamma_2 + L_y u(t) + \omega(L_u + L_w\lambda^2\tau^2)u^{[t-\tau,t]} + \omega L_v w^{[t-\tau,t]}. \end{cases} \tag{3.12}$$

Let

$$\begin{cases} R_1 = \beta\gamma_2, \quad A = \alpha, \quad B = \beta(L_u + L_w\lambda^2\tau^2), \quad C = 0, \quad D = \beta L_v, \\ R_2 = \gamma_1 + \omega\gamma_2, \quad F = L_y, \quad G = \omega(L_u + L_w\lambda^2\tau^2), \quad H = \omega L_v. \end{cases}$$

From Theorem 2.1 we can obtain (3.5) immediately. This completes the proof of Theorem 3.1. □

3.2 Dissipativity of nonlinear neutral Volterra integro-differential equations (NNVIDEs)

Consider the IVPs in NNVIDEs as follows:

$$\begin{cases} y'(t) = f(t, y(t), y(t-\tau), \int_{t-\tau}^{t} K(t,s,y(s),y'(s))\,ds), \quad t \ge t_0, \\ y(t) = \phi(t), \quad y'(t) = \phi'(t), \quad t \in [t_0 - \tau, t_0], \end{cases} \tag{3.13}$$

where $\tau > 0$ is constant, ϕ is a continuous function, and the functions $f : [t_0, +\infty) \times X \times X \times X \to X$ and $K : [t_0, +\infty) \times [t_0 - \tau, +\infty) \times X \times X \to X$ satisfy the conditions for any $t \ge t_0, y, u, v \in X$:

$$\begin{cases} 2\,\mathrm{Re}\langle f(t,y,u,v), y\rangle \le \gamma + \alpha\|y\|^2 + \beta_1\|f(t,0,u,v)\|^2, \\ \|f(t,y,u,v)\|^2 \le L_y\|y\|^2 + \beta_2\|f(t,0,u,v)\|^2, \\ \|f(t,0,u,v)\|^2 \le L_u\|u\|^2 + L_v\|v\|^2, \\ \|K(t,s,u,v)\| \le \mu\|u\| + L_k\|v\|, \quad (t,s) \in D, \end{cases} \tag{3.14}$$

where $D = \{(t,s) : t \in [t_0, +\infty), s \in [t-\tau, t]\}$, $\gamma, \beta_1, \beta_2, \mu, L_y, L_u, L_v$ are non-negative real constants and $\alpha \le 0$.

Theorem 3.2 *Assume that* (3.13) *satisfies* (3.14) *with* $2\beta_2\tau^2 L_v L_k^2 < 1$, *and initial value function* $\phi(t)$ *satisfies*

$$\max_{t_0-\tau\le t\le t_0} \|\phi'(t)\|^2 \le \frac{L_y + \beta_2(L_u + 2\tau^2\mu^2 L_v)}{1 - 2\beta_2\tau^2 L_v L_k^2} \max_{t_0-\tau\le t\le t_0} \|\phi(t)\|^2.$$

Assume there exists a constant $\sigma > 0$ *such that*

$$\alpha + \beta_1 \frac{L_u + 2\tau^2 L_v(L_k^2 L_y + \mu^2)}{1 - 2\beta_2\tau^2 L_v L_k^2} \le -\sigma. \tag{3.15}$$

Let $y(t)$ *be the solution of* (3.13). *Then*

(1) *for any* $t \ge t_0$ *we have*

$$\begin{cases} \|y(t)\|^2 \le \frac{\gamma}{\sigma} + \phi_0 e^{-\mu^*(t-t_0)}, \\ \|y'(t)\|^2 \le \frac{L_y+\beta_2(L_u+2\tau^2\mu^2L_v)}{1-2\beta_2\tau^2L_vL_k^2}\frac{\gamma}{\sigma} + \frac{L_y+\beta_2(L_u+2\tau^2\mu^2L_v)e^{\mu^*\tau}}{1-2\beta_2\tau^2L_vL_k^2e^{\mu^*\tau}}\phi_0 e^{-\mu^*(t-t_0)}, \end{cases}$$

where $\phi_0 = \sup_{t_0-\tau\le\xi\le t_0}\|\phi(\xi)\|^2$, $\mu^* > 0$ *is defined as*

$$\mu^* = \inf_{t\ge t_0}\left\{\mu(t) : \mu(t) + \alpha + \frac{L_u + 2\tau^2 L_v(L_k^2 L_y + \mu^2)}{1 - 2\beta_2\tau^2 L_v L_k^2 e^{\mu(t)\tau}}\beta_1 e^{\mu(t)\tau} = 0\right\}.$$

(2) *the system is dissipative, for any* $\varepsilon > 0$ *the open ball*

$$\mathbf{B} = \mathbf{B}\left(0, \sqrt{\frac{\gamma}{\sigma} + \varepsilon}\right)$$

is an absorbing set.

Proof Let

$$\begin{cases} u(t) = \|y(t)\|^2, \\ w(t) = \|y'(t)\|^2, \end{cases} \quad t \geq t_0 - \tau$$

and

$$z(t) = \int_{t-\tau}^{t} K\big(t, s, y(s), y'(s)\big)\, ds), \quad t \geq t_0.$$

From (3.14) we can obtain

$$\begin{aligned} \|z(t)\|^2 &\leq \tau^2 \Big(\mu \max_{t-\tau \leq \xi \leq t} \|y(\xi)\| + L_k \max_{t-\tau \leq \xi \leq t} \|y'(\xi)\|\Big)^2 \\ &\leq 2\tau^2 \big(\mu^2 u^{[t-\tau,t]} + L_k^2 w^{[t-\tau,t]}\big) \end{aligned}$$

and

$$\begin{aligned} u'(t) &= 2\,\mathrm{Re}\big\langle f\big(t, y(t), y(t-\tau), z(t)\big), y(t)\big\rangle \\ &\leq \gamma + \alpha u(t) + \beta_1 \big\|f\big(t, 0, y(t-\tau), z(t)\big)\big\|^2 \\ &\leq \gamma + \alpha u(t) + \beta_1 \big(L_u u(t-\tau) + L_v \|z(t)\|^2\big) \\ &\leq \gamma + \alpha u(t) + \beta_1 \big(L_u + 2\mu^2\tau^2 L_v\big) u^{[t-\tau,t]} + 2\beta_1 \tau^2 L_v L_k^2 w^{[t-\tau,t]} \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} w(t) &= \big\|f\big(t, y(t), y(t-\tau), z(t)\big)\big\|^2 \\ &\leq L_y \|y(t)\|^2 + \beta_2 \big\|f\big(t, 0, y(t-\tau), z(t)\big)\big\|^2 \\ &\leq L_y \|y(t)\|^2 + \beta_2 \big[L_u u(t-\tau) + L_v \|z(t)\|^2\big] \\ &\leq L_y u(t) + \beta_2 \big(L_u + 2L_v \tau^2 \mu^2\big) u^{[t-\tau,t]} + 2\beta_2 \tau^2 L_v L_k^2 w^{[t-\tau,t]}. \end{aligned} \tag{3.17}$$

It can be summarized from (3.16) and (3.17) that

$$\begin{cases} u'(t) \leq \gamma + \alpha u(t) + \beta_1 (L_u + 2\tau^2\mu^2 L_v) u^{[t-\tau,t]} + 2\beta_1 \tau^2 L_v L_k^2 w^{[t-\tau,t]}, \\ w(t) \leq L_y u(t) + \beta_2 (L_u + 2\tau^2\mu^2 L_v) u^{[t-\tau,t]} + 2\beta_2 \tau^2 L_v L_k^2 w^{[t-\tau,t]}. \end{cases} \tag{3.18}$$

We denote

$$\gamma_1 = \gamma, \qquad A = \alpha, \qquad B = \beta_1 \big(L_u + 2\tau^2\mu^2 L_v\big), \qquad C = 0, \qquad D = 2\beta_1 \tau^2 L_v L_k^2,$$

$$\gamma_2 = 0, \qquad F = L_y, \qquad G = \beta_2\big(L_u + 2\tau^2\mu^2 L_v\big), \qquad H = 2\beta_2\tau^2 L_v L_k^2.$$

Then from Theorem 2.1 we can complete the proof of Theorem 3.2. □

Remark 3.3 From a numerical point of view, it is important to study the potential of numerical methods in preserving the qualitative behavior of the analytical solutions. Therefore, the results of Theorem 3.1 and Theorem 3.2 presented in this paper, provide the theoretical foundation for analyzing the dissipativity of the numerical methods when they are applied to the underlying systems.

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