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Norm inequalities related to the Heinz means

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Abstract

Let $(I, |||\cdot|||)$ be a two-sided ideal of operators equipped with a unitarily invariant norm $|||\cdot|||$. We generalize the results of Kapil's, using a new contractive map in I to obtain a norm inequality. And we give a new inequality, which is a comparison between the Heinz means and other related inequalities; moreover, we will obtain some correlative conclusions.

Keywords: Norm inequalities; Contractive maps; Unitarily invariant norm

1 Introduction and preliminaries

In this paper, let $B(H)$ denote the algebra of all bounded linear operators on a complex separable Hilbert space, $B(H)_+$ denote the cone of positive operators and $K(H)$ denote the ideal of compact operators in $B(H)$. And $(I, |||\cdot|||)$ is a two-sided ideal of $B(H)$ equipped with a unitarily invariant norm $|||\cdot|||$. We shall denote this by I instead of $(I, |||\cdot|||)$ for convenience.

For any compact operator $A \in K(H)$, let $S_1(A), S_2(A), \ldots$ be the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$ arranged in decreasing order. If $A \in M_n$, which is the algebra of all $n \times n$ matrices over $\mathbf{C}$, we take $S_k(A) = 0$ for $k > n$. A unitarily invariant norm in $K(H)$ is a map $|||\cdot||| : K(H) \longrightarrow [0, \infty]$ given by $|||A||| = g(S(A))$, $A \in K(H)$, where g is a symmetric gauge function, cf. [6, 9]. The Schatten p-norms $\|A\|_p = (\sum_j S_j^p(A))^{\frac{1}{p}}$ for $p \geq 1$ are significant examples of the unitarily invariant norm, and $\|\cdot\|_2$ is a special unitarily invariant norm which is the Hilbert–Schmidt norm defined, for $A \in M_n$, as follows:

$$\|A\|_2^2 = \sum_{i,j} |a_{ij}|^2 = \operatorname{tr}(A^*A).$$

It is well known that the arithmetic–geometric mean inequality

$$\sqrt{ab} \leq \frac{a+b}{2}$$

for positive numbers a and b has been generalized in various directions.

Bhatia et al. [2] have obtained the result that if A, B, and X are $n \times n$ matrices with A and B are positive definite, then

$$\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| \leq \frac{1}{2}|||AX + XB|||.$$

The Heinz means

$$H_\nu(a,b)=\frac{a^{1-\nu}b^\nu+a^\nu b^{1-\nu}}{2}$$

is an interpolation between the arithmetic and geometric means for $a,b\geq 0$ and $\nu\in[0,1]$. It is easy to see that H_ν is symmetric and convex function for $\nu\in[0,1]$ and attains its minimum at $\nu=\frac{1}{2}$; thus we have

$$\sqrt{ab}\leq H_\nu(a,b)\leq\frac{a+b}{2}.$$

The matrix version has been proved in [2]: if A, B and X are positive definite, then for every unitarily invariant norm the function

$$g(\nu)=\left|\left|\left|A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu\right|\right|\right|$$

is convex on $[0,1]$, and attains its minimum at $\nu=\frac{1}{2}$. Thus we have

$$2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\leq\left|\left|\left|A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu\right|\right|\right|\leq|||AX+XB|||.$$

The Heron means is defined by

$$F_\alpha(a,b)=(1-\alpha)\sqrt{ab}+\alpha\frac{a+b}{2}$$

for $0\leq\alpha\leq 1$, see [1]. This family is the linear interpolation between the geometric and the arithmetic mean. Clearly, $F_\alpha\leq F_\beta$ whenever $\alpha\leq\beta$.

Bhatia and Davis have proved that the inequality

$$\left|\left|\left|(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}}+\alpha\left(\frac{AX+XB}{2}\right)\right|\right|\right|\leq\left|\left|\left|(1-\beta)A^{\frac{1}{2}}XB^{\frac{1}{2}}+\beta\left(\frac{AX+XB}{2}\right)\right|\right|\right|$$

is always true for $0\leq\alpha\leq\beta\leq 1$, $\beta\geq\frac{1}{2}$, and this restriction on β is necessary in [2].

The Heinz means and the Heron means satisfy the inequality

$$H_\nu(a,b)\leq F_{\alpha(\nu)}(a,b)$$

for $0\leq\nu\leq 1$. And $\alpha(\nu)=1-4(\nu-\nu^2)$, this is a convex function, its minimum value is $\alpha(\frac{1}{2})=0$, and its maximum value is $\alpha(0)=\alpha(1)=1$.

In [10] the authors have presented the result that

$$\frac{1}{2}\left|\left|\left|A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu\right|\right|\right|\leq\left|\left|\left|(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}}+\alpha\left(\frac{AX+XB}{2}\right)\right|\right|\right|$$

for $\frac{1}{4}\leq\nu\leq\frac{3}{4}$ and $\alpha\in[\frac{1}{2},\infty]$ and they used the properties of contractive map on I to prove it.

A different version of the Heinz inequality,

$$\left|\left|\left|A^\alpha XB^{1-\alpha}-A^{1-\alpha}XB^\alpha\right|\right|\right|\leq|2\alpha-1|\,|||AX-XB|||,$$

for $\alpha\in[0,1]$ was proved by Bhatia and Davis [3] in 1995.

A further generalization, namely

$$\frac{2+t}{2}\left|\left|\left|A^{\alpha}XB^{1-\alpha}+A^{1-\alpha}XB^{\alpha}\right|\right|\right| \leq \left|\left|\left|AX+XB+tA^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|,$$

was proved for $t \in [-2,2]$, and $\alpha \in [\frac{1}{4},\frac{3}{4}]$. For more details refer to [13] and [14].

Contractive maps on I play a key role to prove the inequalities, more details refer to [4, 5, 7, 8, 11]. This in turn proves the corresponding Schur multiplier maps to be contractive. We use the ideals to establish the fact that some special maps on I are contractive.

Our paper consists of three parts. In the second part, we will give a new norm inequality which is proved by the properties of contractive maps on I. In the third part, we will present a inequality related to the Heinz means and we notice that this inequality is a comparison between the Heinz means and the geometric mean. Our results are extensions of some previous conclusions about norm inequalities.

2 Norm inequalities with contractive maps

Let L_X, R_Y denote the left and right multiplication maps on $B(H)$, respectively, that is, $L_X(T) = XT$ and $R_Y(T) = TY$ and we have

$$e^{L_X+R_Y}(T) = e^X Te^Y. \tag{1}$$

Neeb [12] has proved that $(L_X \pm R_Y)^{-1}\sin(L_X \pm R_Y)$ is an expansive map on I by using the Weierstrass factorization theorem. We use X_1 and Y_1 to denote two selfadjoint operators in $B(H)$ and $D = L_{X_1} - R_{Y_1}$. The following proposition is a result for a contractive map in I; for more details refer to [10].

Proposition 2.1 ([10, Proposition 2.4]) *Let $0 \leq r \leq s$ or $s \leq r \leq 0$, then each of the following operator maps is a contraction on I.*

(1) $\frac{(1+t)\cosh(\frac{r}{2}D)}{t+\cosh(sD)}$ *for* $|t| \leq 1$.
(2) $\frac{t+\cosh(rD)}{t+\cosh(sD)}$ *for* $|t| \leq 1$.
(3) $\frac{\cosh(rD)}{\cosh(sD)}$. *In particular* $\frac{1}{r}\int \frac{\cosh(rD)}{\cosh(sD)}\,dr = \frac{\sinh(rD)}{rD\cosh(sD)}$ *is a contraction.*
(4) $\frac{s\sinh(rD)}{r\sinh(sD)}$. *The case* $r = 0$ *would become* $\frac{sD}{\sinh(sD)}$.

Corollary 2.2 ([10, Corollary 2.5]) *The following maps are contractive on I.*

(1) $\frac{(t+1)\cosh((2\nu-1)D)}{t+\cosh D}$ *for* $\frac{1}{4} \leq \nu \leq \frac{3}{4}$, *and* $|t| \leq 1$.
(2) $\frac{2\cosh(rD)}{\cosh(s_1D)+\cosh(s_2D)}$ *for* $0 \leq r \leq \max\{\frac{s_1+s_2}{2},\frac{s_1-s_2}{2}\}$ *or* $\min\{\frac{s_1+s_2}{2},\frac{s_1-s_2}{2}\} \leq r \leq 0$.
(3) $\frac{(s_1+s_2)\sinh(rD)}{r(\sinh(s_1D)+\sinh(s_2D))}$ *for* $0 \leq r \leq \frac{s_1+s_2}{2}$ *or* $\frac{s_1+s_2}{2} \leq r \leq 0$.

Corollary 2.3 *Let $0 \leq \nu \leq 1$ and $|t| \leq 1$. Then the map $\frac{(t+1)\cosh((2\nu-1)D)}{t+\cosh(2D)}$ is contractive on I.*

Proof This map is a special case of (1) of Proposition 2.1 when $|s| = 2$. From the condition $0 \leq \nu \leq 1$, we can obtain $|2\nu - 1| \leq 1$. Letting $\frac{r}{2} = 2\nu - 1$, we get the desired result. □

With these above results about contractive maps in I, we obtain a new norm inequality which derives from the Heinz means and other related inequalities. Let R be an invertible operator in $B(H)_+$, then there exists a selfadjoint operator $S \in B(H)$ such that $R = e^S$. To avoid repetitions, we denote the two invertible operators A and B in $B(H)_+$ by e^{2X_1} and

e^{2Y_1}, respectively, where X_1 and Y_1 in $B(H)$ are selfadjoint. The corresponding operator map $L_{X_1} - R_{Y_1}$ is denoted by D.

Theorem 2.4 *Let $\nu \in [0,1]$ and $\alpha \in [\frac{1}{2}, \infty)$. Supposed that A and B are any two invertible operators in $B(H)_+$, $X \in I$, then*

$$\frac{1}{2}\left|\left|\left|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\right|\right|\right| \leq \left|\left|\left|(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \alpha\frac{A^{\frac{3}{2}}XB^{-\frac{1}{2}} + A^{-\frac{1}{2}}XB^{\frac{3}{2}}}{2}\right|\right|\right|. \tag{2}$$

Proof Put $A^{\frac{1}{2}}XB^{\frac{1}{2}} = T$ and use (1) we can notice that

$$\begin{aligned}
&\frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2} \\
&\quad = \frac{A^{\frac{2\nu-1}{2}}A^{\frac{1}{2}}XB^{\frac{1}{2}}B^{\frac{1-2\nu}{2}} + A^{\frac{1-2\nu}{2}}A^{\frac{1}{2}}XB^{\frac{1}{2}}B^{\frac{2\nu-1}{2}}}{2} \\
&\quad = \frac{A^{\frac{2\nu-1}{2}}TB^{\frac{1-2\nu}{2}} + A^{\frac{1-2\nu}{2}}TB^{\frac{2\nu-1}{2}}}{2} \\
&\quad = \frac{e^{(2\nu-1)(L_{X_1}-R_{Y_1})}T + e^{-(2\nu-1)(L_{X_1}-R_{Y_1})}T}{2} \\
&\quad = \cosh\bigl((2\nu-1)D\bigr)T.
\end{aligned} \tag{3}$$

And we also have

$$\begin{aligned}
&(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \alpha\frac{A^{\frac{3}{2}}XB^{-\frac{1}{2}} + A^{-\frac{1}{2}}XB^{\frac{3}{2}}}{2} \\
&\quad = (1-\alpha)T + \alpha\frac{AA^{\frac{1}{2}}XB^{\frac{1}{2}}B^{-1} + A^{-1}A^{\frac{1}{2}}XB^{\frac{1}{2}}B}{2} \\
&\quad = (1-\alpha)T + \alpha\frac{ATB^{-1} + A^{-1}TB}{2} \\
&\quad = (1-\alpha)T + \alpha\frac{e^{2(L_{X_1}-R_{Y_1})}T + e^{-2(L_{X_1}-R_{Y_1})}T}{2} \\
&\quad = \bigl[(1-\alpha) + \alpha\cosh(2D)\bigr]T.
\end{aligned} \tag{4}$$

Setting $\frac{1}{1+t} = \alpha$ in (4) and applying Corollary 2.3, we can obtain (2) from (3) and (4). □

3 Norm inequality related to the Heinz means

According to the above results, we set

$$\begin{aligned}
H_{\nu}(a,b) &= \frac{a^{\nu}b^{1-\nu} + a^{1-\nu}b^{\nu}}{2}, \\
G_{\nu}(a,b) &= (1-\nu)a^{\frac{1}{2}}b^{\frac{1}{2}} + \nu\frac{a^{\frac{3}{2}}b^{-\frac{1}{2}} + a^{-\frac{1}{2}}b^{\frac{3}{2}}}{2}.
\end{aligned}$$

Then we have the following result, which is a interpolation of H_{ν} and G_{ν}.

Theorem 3.1 *Let $a, b > 0$ and $\nu \in [\frac{1}{2}, 1]$. Then*

$$H_\nu(a,b) \le \left(\frac{H_{\frac{1}{2}}(a,b)}{H_1(a,b)}\right)^{1-\nu} G_\nu(a,b). \tag{5}$$

Proof Without loss of generality, we can suppose that $a = 1$. Then the inequality (5) can be simplified to

$$\frac{b^\nu + b^{1-\nu}}{2} \le \left(\frac{2\sqrt{b}}{1+b}\right)^{1-\nu} \left[(1-\nu)\sqrt{b} + \nu \frac{b^{-\frac{1}{2}} + b^{\frac{3}{2}}}{2}\right].$$

To prove this inequality, let

$$f(\nu) = \log\bigl(b^\nu + b^{1-\nu}\bigr) - (1-\nu)\log\frac{2\sqrt{b}}{1+b} - \log\bigl[2(1-\nu)\sqrt{b} + \nu\bigl(b^{-\frac{1}{2}} + b^{\frac{3}{2}}\bigr)\bigr].$$

Calculations show that

$$f''(\nu) = \frac{4b^\nu b^{1-\nu}\log^2 b}{(b^\nu + b^{1-\nu})^2} + \frac{(b^{-\frac{1}{2}} + b^{\frac{3}{2}} - 2b^{\frac{1}{2}})^2}{[(1-\nu)b^{\frac{1}{2}} + \nu(b^{-\frac{1}{2}} + b^{\frac{3}{2}})]^2}.$$

We can notice that $f''(\nu) \ge 0$ for $\frac{1}{2} \le \nu \le 1$, thus f is convex on $[\frac{1}{2}, 1]$. Then we have $f(\nu) \le \max\{f(\frac{1}{2}), f(1)\}$. By calculation, we have

$$f\left(\frac{1}{2}\right) = \log\left(\frac{\sqrt{2}b^{\frac{1}{4}}(1+b)^{\frac{1}{2}}}{b^{\frac{1}{2}} + \frac{1}{2}b^{-\frac{1}{2}} + \frac{1}{2}b^{\frac{3}{2}}}\right) \le 0$$

because of $\frac{\sqrt{2}b^{\frac{1}{4}}(1+b)^{\frac{1}{2}}}{b^{\frac{1}{2}} + \frac{1}{2}b^{-\frac{1}{2}} + \frac{1}{2}b^{\frac{3}{2}}} \le 1$ $(b > 0)$, which equals $\sqrt{2}b^{\frac{1}{4}}(1+b)^{\frac{1}{2}} \le b^{\frac{1}{2}} + \frac{1}{2}b^{-\frac{1}{2}} + \frac{1}{2}b^{\frac{3}{2}}$, that is, $8(1+b)b^{\frac{3}{2}} \le 6b^2 + b^4 + 4b^3 + 4b + 1$.

Setting $b = x^2$ $(x > 0)$, thus we have $x^8 + 4x^6 - 8x^5 + 6x^4 - 8x^3 + 4x^2 + 1 \ge 0$, that is, $(x-1)^2(x^6 + 2x^5 + 7x^4 + 4x^3 + 7x^2 + 2x + 1) \ge 0$, which is always true when $x > 0$, then we get the desired results.

In the same way,

$$f(1) = \log\left(\frac{1+b}{b^{-\frac{1}{2}} + b^{\frac{3}{2}}}\right) \le 0$$

because of $\frac{1+b}{b^{-\frac{1}{2}} + b^{\frac{3}{2}}} \le 1$ $(b > 0)$, which equals $1 + b \le b^{-\frac{1}{2}} + b^{\frac{3}{2}}$, that is $(b^{\frac{1}{2}} - 1)(b^{\frac{3}{2}} - 1) \ge 0$, which is always true when $b > 0$. Hence the values of $f(\frac{1}{2})$ and $f(1)$ are both less than 0, that is, $f(\nu) \le 0$. Then we complete the proof. □

Remark 3.2 The inequality which we obtained from Theorem 3.1 can also be written as

$$H_\nu(a,b) \le \left(\frac{2\sqrt{ab}}{a+b}\right)^{1-\nu} G_\nu,$$

and we notice that it is a new comparison between the Heinz means and the geometric means.

Corollary 3.3 *Let $A, B \in M_n^+$, $X \in M_n$ and $\frac{1}{2} \le \nu \le 1$. If there are two positive numbers m, M such that $m \le A, B \le M$, then*

$$\left\| \frac{A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu}{2} \right\|_2 \le \left(\frac{m+M}{2\sqrt{mM}} \right)^{\nu-1} \left\| (1-\nu)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \nu \frac{A^{\frac{3}{2}}XB^{-\frac{1}{2}} + A^{-\frac{1}{2}}XB^{\frac{3}{2}}}{2} \right\|_2 .$$

Proof Let $A = U \operatorname{diag}(\lambda_i)U^*$ and $B = V \operatorname{diag}(\mu_j)V^*$ be the spectral decompositions of A and B, respectively. Letting $U^*XV = Y$, we have

$$\begin{aligned} &\frac{A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu}{2} \\ &\quad = U \frac{\operatorname{diag}(\lambda_i^\nu) Y \operatorname{diag}(\mu_j^{1-\nu}) + \operatorname{diag}(\lambda_i^{1-\nu}) Y \operatorname{diag}(\mu_j^\nu)}{2} V^* \\ &\quad = U \frac{[\lambda_i^\nu \mu_j^{1-\nu} + \lambda_i^{1-\nu}\mu_j^\nu] \circ [y_{ij}]}{2} V^*, \end{aligned} \tag{6}$$

where $\circ$ denotes the Schur product. Applying (5) we have

$$\begin{aligned} &\left\| \frac{A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu}{2} \right\|_2^2 \\ &\quad = \sum_{i,j} \left(\frac{\lambda_i^\nu \mu_j^{1-\nu} + \lambda_i^{1-\nu}\mu_j^\nu}{2} \right)^2 |y_{ij}|^2 \\ &\quad \le \sum_{i,j} \left(\frac{\lambda_i + \mu_j}{2\sqrt{\lambda_i \lambda_j}} \right)^{2(\nu-1)} \left((1-\nu)\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}} + \nu \frac{\lambda_i^{\frac{3}{2}}\mu_j^{-\frac{1}{2}} + \lambda_i^{-\frac{1}{2}}\mu_j^{\frac{3}{2}}}{2} \right) |y_{ij}|^2 \\ &\quad \le \left(\frac{m+M}{2\sqrt{mM}} \right)^{2(\nu-1)} \left\| (1-\nu)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \nu \frac{A^{\frac{3}{2}}XB^{-\frac{1}{2}} + A^{-\frac{1}{2}}XB^{\frac{3}{2}}}{2} \right\|_2^2, \end{aligned} \tag{7}$$

where we have used the fact that $m \le \lambda_i, \mu_j \le M$ to obtain the last inequality. □

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Authors' contributions
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