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Applications of a q-Salagean type operator on multivalent functions

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Abstract

In this paper, we introduce a new class $k\text{-}\mathcal{US}(q,\gamma,m,p)$, $\gamma \in \mathbb{C}\backslash\{0\}$, of multivalent functions using a newly defined q-analogue of a Salagean type differential operator. We investigate the coefficient problem, Fekete–Szego inequality, and some other properties related to subordination. Relevant connections of the results presented here with those obtained in the earlier work are also pointed out.

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1 Introduction

For a positive integer p, let $\mathcal{A}_p$ denote the set of all functions $f(z)$ which are analytic and p-valent in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$ and have series expansion of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n. \tag{1.1}$$

Also, let $f * g$ denote the convolution (or Hadamard product) of $f, g \in \mathcal{A}_p$ defined as follows:

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n,$$

where $f(z)$ is given by (1.1) and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$.

Quite recently, q-analysis has influenced the researchers a lot due to rapid applications in mathematics and related fields. In the last century many well-known researchers (for details, see [1, 4, 6–10, 13, 14, 21, 22, 32]) did great work on q-calculus and found numerous applications. It is worth mentioning that convolution theory helps many researchers to investigate a number of properties of analytic univalent and multivalent functions. Several differential and integral operators were defined using ordinary derivative; for details, see [29].

Due to growing applications of q-calculus, investigators are interested in studying properties of functions using q-operators instead of ordinary differential operators; for comprehensive study, we refer to Kanas and Reducanu [15], Mahmood and Darus [19], and

Mahmood and Sokol [20]. In this paper we define a q-analogue of a Salagean type operator and study its effect on multivalent functions in conic domains.

For any non-negative integer n, the q-integer number n denoted by $[n]_q$ is defined by

$$[n]_q = \frac{1-q^n}{1-q}, \qquad [0]_q = 0.$$

For a non-negative integer n, the q-number shift factorial is defined as

$$[n]_q! = [1]_q[2]_q[3]_q \dots [n]_q \quad \bigl([0]_q! = 1\bigr).$$

We note that when $q \to 1$, $[n]_q!$ reduces to the classical definition of factorial. In general, $[t]_q$ is defined as follows:

$$[t]_q = \frac{1-q^t}{1-q}, \qquad [0]_q = 0, \quad q \in (0,1).$$

For $f \in A$, in [5], the q-derivative operator or q-difference operator is defined as follows:

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}.$$

It can easily be seen that

$$\partial_q z^n = [n]_q z^{n-1}, \qquad \partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}.$$

Taking motivation from the above mentioned work, we define new convolution operators as follows.

Let

$$\Phi(p,q,m,z) = z^p + \sum_{n=p+1}^{\infty} \bigl[n + (p-1)\bigr]_q^m z^n. \tag{1.2}$$

Using the functions $\Phi(p,q,m,z)$ and the definition of q-derivative along with the idea of convolution, we now define the following differential operator $\mathcal{S}_{q,p}^m f(z) : \mathcal{A}_p \to \mathcal{A}_p$ for multivalent functions

$$\begin{aligned} \mathcal{S}_{q,p}^m f(z) &= \Phi(p,q,m,z) * f(z), \quad m \in N \cup \{0\}, \\ &= z^p + \sum_{n=p+1}^{\infty} \bigl[n + (p-1)\bigr]_q^m a_n z^n, \\ &= z^p + \sum_{n=p+1}^{\infty} \psi_n a_n z^n, \end{aligned} \tag{1.3}$$

where

$$\psi_n = \bigl[n + (p-1)\bigr]_q^m. \tag{1.4}$$

For $p = 1$, the operator $\mathcal{S}_{q,p}^{m}f(z)$ reduces to the Salagean q-differential operator defined by Govindaraj and Sivasubramanian [11], and for $p = 1$, $q \to 1$, the operator $\mathcal{S}_{q,p}^{m}f(z)$ reduces to the Salagean differential operator defined by Salagean [26].

Taking motivation from [12] and using (1.3), we define a new class k-$\mathcal{US}(q,\gamma,m,p)$ of multivalent functions as follows.

Throughout paper we shall assume $k \geq 0$, $m \in N \cup \{0\}$, $q \in (0,1)$, $\gamma \in \mathbb{C}\backslash\{0\}$, and $p \in N$.

Definition 1.1 A function $f(z) \in \mathcal{A}_p$ is in the class k-$\mathcal{US}(q,\gamma,m,p)$ if it satisfies the condition

$$\operatorname{Re}\left\{1 + \frac{1}{\gamma}\left\{\frac{1}{[p]_q}\left(\frac{z\partial_q S_{q,p}^{m}f(z)}{S_{q,p}^{m}f(z)}\right) - 1\right\}\right\} > k\left|\frac{1}{\gamma}\left\{\frac{1}{[p]_q}\left(\frac{z\partial_q S_{q,p}^{m}f(z)}{S_{q,p}^{m}f(z)}\right) - 1\right\}\right|, \quad z \in E.$$

By taking specific values of parameters, we obtain many important subclasses studied by various authors in earlier papers. Here we enlist some of them.

(i) For $p = 1$, the class k-$\mathcal{US}(q,\gamma,m,p)$ reduces to the class k-$\mathcal{US}(q,\gamma,m)$ studied by Saqib et al. [12].

(ii) For $p = 1$, $m = 0$, $k = 0$, and $\gamma \in \mathbb{C}\backslash\{0\}$, the class k-$\mathcal{US}(q,\gamma,m,p)$ reduces to the class $\mathcal{S}_q^*(\gamma)$ studied by Seoudy and Aouf [27].

(iii) For $p = 1$, $m = 0$, $k = 0$, and $\gamma = \frac{1}{1-\alpha}$, with $0 \leq \alpha < 1$, the class k-$\mathcal{US}(q,\gamma,m,p)$ reduces to the class $\mathcal{S}_q^*(\alpha)$ studied by Agrawal and Sahoo [2].

(iv) For $p = 1$, $m = 0$, $q \to 1$, and $\gamma = \frac{1}{1-\alpha}$, with $0 \leq \alpha < 1$, the class k-$\mathcal{US}(q,\gamma,m,p)$ reduces to the class $\mathcal{SD}(k,\alpha)$ studied by Shams et al. [28].

(v) For $p = 1$, $m = 0$, $q \to 1$, and $\gamma = \frac{2}{1-\alpha}$, with $0 \leq \alpha < 1$, the class k-$\mathcal{US}(q,\gamma,m,p)$ reduces to the class $\mathcal{KD}(k,\alpha)$ studied by Owa et al. [24].

(vi) For $p = 1$, $k = 1$, $m = 0$, $q \to 1$, and $\gamma = \frac{1}{1-\alpha}$, with $0 \leq \alpha < 1$, the class k-$\mathcal{US}(q,\gamma,m,p)$ reduces to the class $\mathcal{S}(\alpha)$ studied by Ali et al. [3].

(vii) For $p = 1$, $k = 1$, $m = 0$, $q \to 1$, and $\gamma = \frac{2}{1-\alpha}$, with $0 \leq \alpha < 1$, the class k-$\mathcal{US}(q,\gamma,m,p)$ reduces to the class $\mathcal{K}(\alpha)$ studied by Ali et al. [3].

(viii) For $p = 1$, $m = 0$, $q \to 1$, the class k-$\mathcal{US}(q,\gamma,m,p)$ reduces to the class $\mathcal{K}$-$\mathcal{ST}$ introduced by Kanas and Wisniowska [17].

(ix) For $p = 1$, $k = 0$, $m = 0$, $q \to 1$, and $\gamma = \frac{1}{1-\alpha}$, with $0 \leq \alpha < 1$, the class k-$\mathcal{US}(q,\gamma,m,p)$ reduces to the class $\mathcal{S}^*(\alpha)$, a well-known class of starlike functions of order α, respectively.

Geometric interpretation. A function $f(z) \in \mathcal{A}_p$ is in the class k-$\mathcal{US}(q,\gamma,m,p)$ if and only if $\frac{1}{[p]_q}(\frac{z\partial_q S_{q,p}^{m}f(z)}{S_{q,p}^{m}f(z)})$ takes all the values in the conic domain $\Omega_{k,\gamma} = h_{k,\gamma}(E)$ such that

$$\Omega_{k,\gamma} = \gamma\Omega_k + (1-\alpha),$$

where

$$\Omega_k = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\}.$$

Since $h_{k,\gamma}(z)$ is convex univalent, so the above definition can be written as

$$\frac{1}{[p]_q}\left(\frac{z\partial_q S_{q,p}^{m}f(z)}{S_{q,p}^{m}f(z)}\right) \prec h_{k,\gamma}(z), \tag{1.5}$$

where

$$h_{k,\gamma}(z)=\begin{cases}\frac{1+z}{1-z}, & \text{for } k=0,\\ 1+\frac{2\gamma}{\pi^2}(\log\frac{1+\sqrt{z}}{1-\sqrt{z}})^2, & \text{for } k=1,\\ 1+\frac{2\gamma}{1-k^2}\sinh^2\{(\frac{2}{\pi}\arccos k)\arctan h\sqrt{z}\}, & \text{for } 0<k<1,\\ 1+\frac{\gamma}{k^2-1}\sin(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}\,dx)+\frac{\gamma}{1-k^2}, & \text{for } k>1.\end{cases} \tag{1.6}$$

The boundary $\partial\Omega_{k,\gamma}$ of the above set becomes an imaginary axis when $k=0$, and a hyperbola when $0<k<1$. For $k=1$, the boundary $\partial\Omega_{k,\gamma}$ becomes a parabola and it is an ellipse when $k>1$ and in this case where

$$u(z)=\frac{z-\sqrt{t}}{1-\sqrt{t}z},\quad z\in E,$$

and $t\in(0,1)$ is chosen such that $k=\cosh(\pi K'(t)/(4K(t)))$. Here $K(t)$ is Legendre's complete elliptic integral of the first kind and $K'(t)=K(\sqrt{1-t^2})$, and $K'(t)$ is the complementary integral of $K(t)$ (for details, see [16, 17, 23]). Moreover, $h_{k,\gamma}(E)$ is convex univalent in E, see [16, 17]. All of these curves have the vertex at the point $\frac{k+\gamma}{k+1}$.

2 A set of lemmas

Each of the following lemmas will be needed in our present investigation.

Lemma 2.1 ([25]) *Let $h(z)=\sum_{n=1}^{\infty}h_nz^n\prec F(z)=\sum_{n=1}^{\infty}d_nz^n$ in E. If $F(z)$ is convex univalent in E, then*

$$|h_n|\le|d_1|,\quad n\ge1.$$

Lemma 2.2 ([31]) *Let $k\in[0,\infty)$ and let $h_{k,\gamma}$ be defined (1.6). If*

$$h_{k,\gamma}(z)=1+Q_1z+Q_2z^2+\cdots, \tag{2.1}$$

$$Q_1=\begin{cases}\frac{2\gamma A^2}{1-k^2}, & 0\le k<1,\\ \frac{8\gamma}{\pi^2}, & k=1,\\ \frac{\pi^2\gamma}{4(1+t)\sqrt{t}K^2(t)(k^2-1)}, & k>1,\end{cases} \tag{2.2}$$

$$Q_2=\begin{cases}\frac{A^2+2}{3}Q_1, & 0\le k<1,\\ \frac{2}{3}Q_1, & k=1,\\ \frac{4K^2(t)(t^2+6t+1)-\pi^2}{24K^2(t)(1+t)\sqrt{t}}Q_1, & k>1,\end{cases} \tag{2.3}$$

where $A=\frac{2\cos^{-1}k}{\pi}$, and $t\in(0,1)$ is chosen such that $k=\cosh(\frac{\pi K'(t)}{K(t)})$, $K(t)$ is Legendre's complete elliptic integral of the first kind.

Lemma 2.3 ([18]) *Let $h(z)=1+\sum_{n=1}^{\infty}c_nz^n$ be analytic in E and satisfy* $\operatorname{Re}\{h(z)\}>0$ *for z in E. Then the following sharp estimate holds*:

$$\left|c_2-\mu c_1^2\right|\le2\max\left\{1,|2\mu-1|\right\},\quad\forall\mu\in\mathbb{C}.$$

3 Main results

In this section, we will prove our main results.

Theorem 3.1 *Let* $f(z) \in k - \mathcal{US}(q,\gamma,m,p)$. *Then*

$$S_{q,p}^m f(z) \prec z \exp \int_0^z \frac{p\{h_{k,\gamma}(w(z))\} - 1}{\zeta}\, d\xi, \tag{3.1}$$

where $w(z)$ *is analytic in* E *with* $w(0) = 0$ *and* $|w(z)| < 1$. *Moreover, for* $|z| = \rho$, *we have*

$$\exp\left(\int_0^1 \frac{p\{h_{k,\gamma}(-\rho)\} - 1}{\rho}\, d\rho\right) \le \left|\frac{S_{q,p}^m f(z)}{z}\right| \le \exp\left(\int_0^1 \frac{p\{h_{k,\gamma}(\rho)\} - 1}{\rho}\, d\rho\right), \tag{3.2}$$

where $h_{k,\gamma}(z)$ *is defined by* (1.6).

Proof If $f(z) \in k - \mathcal{US}(q,\gamma,m,p)$, then using identity (1.5), we obtain

$$\begin{aligned}
&\frac{1}{p}\left(\frac{z\partial_q S_{q,p}^m f(z)}{S_{q,p}^m f(z)}\right) \prec h_{k,\gamma}(z), \\
&\frac{1}{p}\left(\frac{z\partial_q S_{q,p}^m f(z)}{S_{q,p}^m f(z)}\right) = h_{k,\gamma}\bigl(w(z)\bigr), \\
&\frac{\partial_q S_{q,p}^m f(z)}{S_{q,p}^m f(z)} - \frac{1}{z} = \frac{p\{h_{k,\gamma}(w(z))\} - 1}{z}.
\end{aligned} \tag{3.3}$$

For some function $w(z)$ is analytic in E with $w(0) = 0$ and $|w(z)| < 1$. Integrating (3.3) and after some simplification, we have

$$S_{q,p}^m f(z) \prec z \exp \int_0^z \frac{p\{h_{k,\gamma}(w(z))\} - 1}{\zeta}\, d\xi. \tag{3.4}$$

This proves (3.1). Noting that the univalent function $h_{k,\gamma}(z)$ maps the disk $|z| < \rho$ $(0 < \rho \le 1)$ onto a region which is convex and symmetric with respect to the real axis, we see

$$h_{k,\gamma}\bigl(-\rho|z|\bigr) \le \operatorname{Re}\bigl\{h_{k,\gamma}(w(\rho z)\bigr\} \le h_{k,\gamma}\bigl(\rho|z|\bigr) \quad (0 < \rho \le 1, z \in E). \tag{3.5}$$

Using (3.4) and (3.5) gives

$$\begin{aligned}
\int_0^1 \frac{p\{h_{k,\gamma}(-\rho|z|)\} - 1}{\rho}\, d\rho &\le \operatorname{Re} \int_0^1 \frac{p\{h_{k,\gamma}(w(\rho(z))\} - 1}{\rho}\, d\rho \\
&\le \int_0^1 \frac{p\{h_{k,\gamma}(\rho|z|)\} - 1}{\rho}\, d\rho
\end{aligned}$$

for $z \in E$. Consequently, subordination (3.4) leads to

$$\begin{aligned}
&\int_0^1 \frac{p\{h_{k,\gamma}(-\rho|z|)\} - 1}{\rho}\, d\rho \le \log\left|\frac{S_{q,p}^m f(z)}{z}\right| \le \int_0^1 \frac{p\{h_{k,\gamma}(\rho|z|)\} - 1}{\rho}\, d\rho, \\
&h_{k,\gamma}(-\rho) \le h_{k,\gamma}\bigl(-\rho|z|\bigr), \qquad h_{k,\gamma}\bigl(\rho|z|\bigr) \le h_{k,\gamma}(\rho)
\end{aligned}$$

implies that

$$\exp \int_0^1 \frac{p\{h_{k,\gamma}(-\rho)\}-1}{\rho}\, d\rho \leq \left|\frac{S_{q,p}^m f(z)}{z}\right| \\ \leq \exp \int_0^1 \frac{p\{h_{k,\gamma}(\rho)\}-1}{\rho}\, d\rho.$$

This completes the proof. □

When $p = 1$, we have the following known result proved by Saqib et al. in [12].

Corollary 3.2 *Let $f(z) \in k-\mathcal{US}(q,\gamma,m)$. Then*

$$S_q^m f(z) \prec z \exp \int_0^z \frac{h_{k,\gamma}(w(\xi))-1}{\zeta}\, d\xi,$$

where $w(z)$ is analytic in E with $w(0) = 0$ and $|w(z)| < 1$. Moreover, for $|z| = \rho$, we have

$$\exp\left(\int_0^1 \frac{h_{k,\gamma}(-\rho)-1}{\rho}\, d\rho\right) \leq \left|\frac{S_q^m f(z)}{z}\right| \\ \leq \exp\left(\int_0^1 \frac{h_{k,\gamma}(\rho)-1}{\rho}\, d\rho\right),$$

where $h_{k,\gamma}(z)$ is defined by (1.6).

Theorem 3.3 *If $f(z) \in k-\mathcal{US}(q,\gamma,m,p)$, then*

$$|a_{p+1}| \leq \frac{\delta}{\{[p+1]_q - p\}\psi_{p+1}} \tag{3.6}$$

and

$$|a_{n+p-1}| \leq \frac{\delta}{\{[n+p-1]_q - p\}\psi_{n+p-1}} \prod_{j=1}^{n-2}\left(1 + \frac{\delta}{\{[j+p]_q - p\}}\right) \quad \textit{for } n = 3,4,\ldots, \tag{3.7}$$

where $\delta = p|Q_1|$ with Q_1 given by (2.2).

Proof Let

$$\begin{aligned} \frac{1}{[p]_q}\left(\frac{z\partial_q S_{q,p}^m f(z)}{S_{q,p}^m f(z)}\right) &= h(z), \\ z\partial_q S_{q,p}^m f(z) &= [p]_q S_{q,p}^m f(z) h(z), \end{aligned} \tag{3.8}$$

where $h(z)$ is analytic in E and $h(0) = 1$. Let $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $S_{q,p}^m f(z)$ be given by (1.3). Then (3.8) becomes

$$z^p + \sum_{n=p+1}^{\infty} [n]_q \psi_n a_n z^n = p\left(\sum_{n=0}^{\infty} c_n z^n\right)\left(z^p + \sum_{n=p+1}^{\infty} \psi_n a_n z^n\right).$$

Now comparing the coefficients of z^{n+p-1}, we obtain

$$[n+p-1]_q\psi_{n+p-1}a_{n+p-1} = p\psi_{n+p-1}a_{n+p-1} + p\{c_1\psi_{n+p-2}a_{n+p-2} + \cdots + c_{n-1}\},$$
$$\bigl\{[n+p-1]_q - p\bigr\}\psi_{n+p-1}a_{n+p-1} = p\{c_1\psi_{n+p-2}a_{n+p-2} + \cdots + c_{n-1}\}.$$

Taking the absolute on both sides and then applying the coefficient estimates $|c_n| \leq |Q_1|$, see in [23], we have

$$|a_{n+p-1}| \leq \frac{p|Q_1|}{\{[n+p-1]_q - p\}\psi_{n+p-1}}\bigl\{1 + \psi_{p+1}|a_{p+1}| + \cdots + \psi_{n+p-2}|a_{n+p-2}|\bigr\}.$$

Let us take $\delta = p|Q_1|$, then we have

$$|a_{n+p-1}| \leq \frac{\delta}{\{[n+p-1]_q - p\}\psi_{n+p-1}}\bigl\{1 + \psi_{p+1}|a_{p+1}| + \cdots + \psi_{n+p-2}|a_{n+p-2}|\bigr\}. \tag{3.9}$$

We apply mathematical induction on (3.9), so for $n = 2$ in (3.9), we have

$$|a_{p+1}| \leq \frac{\delta}{\{[p+1]_q - p\}\psi_{p+1}}, \tag{3.10}$$

which shows that (3.7) holds for $n = 2$. Now consider the case $n = 3$ in (3.9), we have

$$|a_{p+2}| \leq \frac{\delta}{\{[p+2]_q - p\}\psi_{p+2}}\bigl\{1 + \psi_{p+1}|a_{p+1}|\bigr\}.$$

Using (3.10), we have

$$|a_{p+2}| \leq \frac{\delta}{\{[p+2]_q - p\}\psi_{p+2}}\left\{1 + \frac{\delta}{[p+1]_q - p}\right\},$$

which shows that (3.7) holds for $n = 3$. Let us assume that (3.7) is true for $n \leq t$, that is,

$$|a_{t+p-1}| \leq \frac{\delta}{\{[t+p-1]_q - p\}\psi_{t+p-1}}\prod_{j=1}^{t-2}\left(1 + \frac{\delta}{[j+p]_q - p}\right) \quad \text{for } n = 3, 4, \ldots.$$

Consider

$$\begin{aligned} |a_{t+p}| &\leq \frac{\delta}{\{[t+p]_q - p\}\psi_{t+p}}\bigl\{1 + \psi_{p+1}|a_{p+1}| + \cdots \psi_{t+p-1}|a_{t+p-1}|\bigr\} \\ &\leq \frac{\delta}{\{[t+p]_q - p\}\psi_{t+p}}\left\{\begin{array}{c} 1 + \frac{\delta}{[p+1]_q-p} + \frac{\delta}{[p+2]_q-p}(1 + \frac{\delta}{[p+1]_q-p}) + \cdots \\ + \frac{\delta}{\{[t+p-1]_q-p\}}\prod_{j=1}^{t-2}(1 + \frac{\delta}{[j+p]_q-p}) \end{array}\right\} \\ &= \frac{\delta}{\{[t+p]_q - p\}\psi_{t+p}}\prod_{j=1}^{t-1}\left(1 + \frac{\delta}{[j+p]_q - p}\right), \end{aligned}$$

which proves the assertion of theorem $n = t + 1$. Hence (3.7) holds for all n, $n \geq 3$.

This completes the proof. □

When $p = 1$, we have the following known result proved by Saqib et al. in [12].

Corollary 3.4 ([12]) *If $f(z) \in k-\mathcal{US}(q,\gamma,m)$, then*

$$|a_2| \le \frac{\delta}{\{[2]_q - 1\}[2]_q^m}$$

and

$$|a_n| \le \frac{\delta}{\{[n]_q - 1\}[n]_q^m} \prod_{j=1}^{n-2}\left(1 + \frac{\delta}{[j+1]_q - 1}\right) \quad \text{for } n = 3,4,\ldots,$$

where $\delta = |Q_1|$ with Q_1 given by (2.2).

Theorem 3.5 *Let $0 \le k < \infty$ be fixed and let $f(z) \in k-\mathcal{US}(q,\gamma,m,p)$ with the form (1.1). Then, for a complex number μ,*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{pQ_1}{2[2p+1]_q^m\{[p+2]_q - p\}} \max\left[1, |2\nu - 1|\right], \tag{3.11}$$

where

$$\nu = \frac{1}{2}\left\{1 - \frac{Q_2}{Q_1} - Q_1\left(\frac{4p}{\{[p+1]_q - p\}} - \mu\frac{4p[2p+1]_q^m\{[p+2]_q - p\}}{([2p]_q^m)^2\{[p+1]_q - p\}}\right)\right\}, \tag{3.12}$$

and $\delta = p|Q_1|$, with Q_1 and Q_2 given by (2.2) and (2.3).

Proof Let $f(z) \in k-\mathcal{US}(q,\gamma,m,p)$, then there exists a Schwarz function $w(z)$, with $w(0) = 0$ and $|w(z)| < 1$, such that

$$\begin{aligned} &\frac{1}{[p]_q}\left(\frac{z\partial_q S_{q,p}^m f(z)}{S_{q,p}^m f(z)}\right) \prec h_{k,\gamma}(z), \quad z \in E, \\ &\frac{1}{[p]_q}\left(\frac{z\partial_q S_{q,p}^m f(z)}{S_{q,p}^m f(z)}\right) = h_{k,\gamma}\big(w(z)\big). \end{aligned} \tag{3.13}$$

Let $h(z) \in \mathcal{P}$ be a function defined as

$$h(z) = \frac{1 + w(z)}{1 - w(z)},$$

which gives

$$w(z) = \frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots$$

and

$$h_{k,\gamma}\big(w(z)\big) = 1 + \frac{Q_1 c_1}{2}z + \left\{\frac{Q_2 c_1^2}{4} + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)Q_1\right\}z^2 + \cdots. \tag{3.14}$$

Using (3.14) in (3.13) and along with (1.3), we obtain

$$a_{p+1}=\frac{pQ_1c_1}{[2p]_q^m\{[p+1]_q-p\}}$$

and

$$a_{p+2}=\frac{p}{[2p+1]_q^m\{[p+2]_q-p\}}\left\{\frac{Q_1c_2}{2}+\frac{c_1^2}{4}\left(Q_2-Q_1+\frac{4pQ_1^2}{\{[p+1]_q-p\}}\right)\right\}.$$

Using any complex number μ and the above coefficients, we have

$$a_{p+2}-\mu a_{p+1}^2=\frac{pQ_1}{2[2p+1]_q^m\{[p+2]_q-p\}}\{c_2-\nu c_1^2\}. \tag{3.15}$$

Using Lemma 2.3 on (3.15), we have

$$\left|a_{p+2}-\mu a_{p+1}^2\right|\le\frac{pQ_1}{2[2p+1]_q^m\{[p+2]_q-p\}}\max\left[1,|2\nu-1|\right],$$

where

$$\nu=\frac{1}{2}\left\{1-\frac{Q_2}{Q_1}-Q_1\left(\frac{4p}{\{[p+1]_q-p\}}-\mu\frac{4p[2p+1]_q^m\{[p+2]_q-p\}}{([2p]_q^m)^2\{[p+1]_q-p\}}\right)\right\}.$$

This is our required result (3.11). □

When $p=1$, we have the following known result proved by Saqib et al. in [12].

Corollary 3.6 ([12]) *Let $0\le k<\infty$ be fixed and let $f(z)\in k-\mathcal{US}(q,\gamma,m)$ with the form* (1.1). *Then, for a complex number μ,*

$$\left|a_3-\mu a_2^2\right|\le\frac{Q_1}{2[3]_q^m\{[3]_q-1\}}\max\left[1,|2\nu-1|\right],$$

where

$$\nu=\frac{1}{2}\left\{1-\frac{Q_2}{Q_1}-Q_1\left(\frac{4}{[2]_q-1}-\mu\frac{4[3]_q^m\{[3]_q-1\}}{([2]_q^m)^2\{[2]_q-1\}}\right)\right\},$$

and Q_1 and Q_2 are given by (2.2) *and* (2.3).

Theorem 3.7 *If a function $f(z)\in\mathcal{A}_p$ has the form* (1.1) *and satisfies the condition*

$$\sum_{n=p+1}^{\infty}\left\{\left\{\left|[n]_q-p\right|\right\}(k+1)+p|\gamma|\right\}|\psi_n||a_n|\le|\gamma||p|, \tag{3.16}$$

then $f(z)\in k-\mathcal{US}(q,\gamma,m,p)$.

Proof Let

$$\begin{aligned}\left|\frac{1}{p}\left(\frac{z\partial_q S_{q,p}^m f(z)}{S_{q,p}^m f(z)}\right)-1\right| &= \left|\frac{z\partial_q S_{q,p}^m f(z)-pS_{q,p}^m f(z)}{pS_{q,p}^m f(z)}\right| \\ &= \left|\frac{\sum_{n=p+1}^{\infty}\psi_n\{[n]_q-p\}a_n z^n}{pz^p+p\sum_{n=p+1}^{\infty}\psi_n a_n z^n}\right| \\ &\leq \frac{\sum_{n=p+1}^{\infty}|\psi_n\{[n]_q-p\}||a_n|}{|p|-\sum_{n=p+1}^{\infty}p|\psi_n||a_n|}.\end{aligned}\tag{3.17}$$

From (3.16), it follows that

$$p-\sum_{n=p+1}^{\infty}p|\psi_n||a_n|>0.$$

To show that $f(z)\in k-\mathcal{US}(q,\gamma,m,p)$, it is enough to prove that

$$\left|\frac{k}{\gamma}\left\{\frac{1}{p}\left(\frac{z\partial_q S_{q,p}^m f(z)}{S_{q,p}^m f(z)}\right)-1\right\}\right|-\operatorname{Re}\left\{\frac{1}{\gamma}\left\{\frac{1}{p}\left(\frac{z\partial_q S_{q,p}^m f(z)}{S_{q,p}^m f(z)}\right)-1\right\}\right\}\leq 1.$$

From (3.17), we have

$$\begin{aligned}&\left|\frac{k}{\gamma}\left\{\frac{1}{p}\left(\frac{z\partial_q S_{q,p}^m f(z)}{S_{q,p}^m f(z)}\right)-1\right\}\right|-\operatorname{Re}\left\{\frac{1}{\gamma}\left\{\frac{1}{p}\left(\frac{z\partial_q S_{q,p}^m f(z)}{S_{q,p}^m f(z)}\right)-1\right\}\right\} \\ &\quad\leq \frac{k}{|\gamma|}\left|\frac{1}{p}\left(\frac{z\partial_q S_{q,p}^m f(z)}{S_{q,p}^m f(z)}\right)-1\right|+\frac{1}{|\gamma|}\left|\frac{1}{p}\left(\frac{z\partial_q S_{q,p}^m f(z)}{S_{q,p}^m f(z)}\right)-1\right| \\ &\quad\leq \frac{(k+1)}{|\gamma|}\left|\frac{1}{p}\left(\frac{z\partial_q S_{q,p}^m f(z)}{S_{q,p}^m f(z)}\right)-1\right| \\ &\quad= \frac{(k+1)}{|\gamma|}\left|\frac{z\partial_q S_{q,p}^m f(z)-pS_{q,p}^m f(z)}{pS_{q,p}^m f(z)}\right| \\ &\quad\leq \frac{(k+1)}{|\gamma|}\left\{\frac{\sum_{n=p+1}^{\infty}|\psi_n\{[n]_q-p\}||a_n|}{|p|-\sum_{n=p+1}^{\infty}p|\psi_n||a_n|}\right\} \\ &\quad\leq 1.\end{aligned}$$

□

When $p=1$, we have the following known result proved by Hussain et al. in [12].

Corollary 3.8 ([12]) *If a function $f(z)\in\mathcal{A}$ has the form* (1.1) *and satisfies the condition*

$$\sum_{n=2}^{\infty}\left\{\left|[n]_q-1\right|(k+1)+|\gamma|\right\}[n]_q^m|a_n|\leq|\gamma|,$$

then $f(z)\in k-\mathcal{US}(q,\gamma,m)$.

When $q\to 1$, $p=1$, $m=0$, $\gamma=1-\alpha$, with $0\leq\alpha<1$, we have the following known result, proved by Shams et al. in [28].

Corollary 3.9 *A function $f \in A$ of the form* (1.1) *is in the class $\mathcal{SD}(k,\alpha)$ if it satisfies the condition*

$$\sum_{n=2}^{\infty}\bigl\{n(k+1)-(k+\alpha)\bigr\}|a_n| \leq 1-\alpha,$$

where $0 \leq \alpha < 1$ and $k \geq 0$.

When $q \to 1$, $p = 1$, $m = 0$, $\gamma = 1-\alpha$, with $0 \leq \alpha < 1$ and $k = 0$, we have the following known result proved by Silverman in [30].

Corollary 3.10 *A function $f \in A$ of the form* (1.1) *is in the class $\mathcal{SD}(\alpha)$ if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{n-\alpha\}|a_n| \leq 1-\alpha.$$

Theorem 3.11 *Let $f(z) \in k-\mathcal{US}(q,\gamma,m,p)$. Then $f(E)$ contains an open disk of radius*

$$r = \frac{\{[p+1]_q - p\}\psi_{p+1}}{(p+1)\{[p+1]_q - p\}\psi_{p+1} + \delta},$$

where $\delta = p|Q_1|$ with Q_1 given by (2.2).

Proof Let $w_0 \neq 0$ be a complex number such that $f(z) \neq w_0$ for $z \in E$. Then

$$f_1(z) = \frac{w_0 f(z)}{w_0 - f(z)} = z + \left(a_{p+1} + \frac{1}{w_0}\right)z^{p+1} + \cdots.$$

Since $f_1(z)$ is univalent, so

$$\left|a_{p+1} + \frac{1}{w_0}\right| \leq p+1.$$

Now, by using (3.6), we have

$$\left|\frac{1}{w_0}\right| \leq \frac{(p+1)\{[p+1]_q - p\}\psi_{p+1} + \delta}{\{[p+1]_q - p\}\psi_{p+1}}.$$

Hence we have

$$|w_0| \geq \frac{\{[p+1]_q - p\}\psi_{p+1}}{(p+1)\{[p+1]_q - p\}\psi_{p+1} + \delta}.$$ □

When $p = 1$, we have the following known result proved by Saqib et al. in [12].

Corollary 3.12 ([12]) *Let $f(z) \in k-\mathcal{US}(q,\gamma,m)$. Then $f(E)$ contains an open disk of radius*

$$r = \frac{\{[2]_q - 1\}[2]_q^m}{2[2]_q^m\{[2]_q - 1\} + Q_1},$$

where Q_1 is given by (2.2).

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Authors' contributions
SH came with the main thoughts and helped to draft the manuscript, SK and MAZ proved the main theorems, MD revised the paper. All authors read and approved the final manuscript.

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