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Generalized nonlinear weakly singular retarded integral inequalities with maxima and their applications

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Abstract

This paper deals with a generalized nonlinear weakly singular retarded Wendroff-type integral inequality with maxima of an unknown function of two variables. The key is that a technique of monotonization without separability and monotonicity of given functions is used for estimating the boundedness of unknown functions. Then our outcomes can be helpful to weaken conditions for some known results. By applying our results, the uniqueness of solutions for some singular integral equation with maxima may be proven.

MSC: 26D15; 34A08; 34A34

Keywords: Integral inequalities; Maxima; Monotonicity; Uniqueness

1 Introduction

The Gronwall inequality [1] holds a vital place in studying qualitative properties of the solutions of integral equations and differential equations. Some linear and nonlinear generalizations (e.g. [2–11]) of the Gronwall inequality have been extensively discussed. With further study of fractional differential equations, integral inequalities with weakly singular kernels have attracted more and more attention (see [12–20]). In [14], a new method was presented to analyze the nonlinear singular integral inequalities of Henry type:

$$u(t) \le a(t) + b(t) \int_{t_0}^{t} (t-s)^{\beta-1} s^{\gamma-1} F(s) u(s)\, ds, \quad t \ge 0. \tag{1.1}$$

In 2008, Cheung *et al.* [20] solved the nonlinear weakly singular inequality

$$u^p(x,y) \le a(x,y) + b(x,y) \int_0^x \int_0^y \left(x^\alpha - s^\alpha\right)^{\beta-1} s^{\gamma-1} \left(y^\alpha - t^\alpha\right)^{\beta-1} t^{\gamma-1} \cdot f(s,t) u^q(s,t)\, dt\, ds. \tag{1.2}$$

On the other hand, since differential equations with maxima of the unknown function [21–26] can be applied in control theory, some significant results for integral inequalities containing the maxima of the unknown function [22, 27–30] have been obtained. The integral inequality with maxima

$$\begin{aligned} u(x,y) &\le a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)u^p(s,t)\,dt\,ds \\ &\quad + \int_{\alpha(x_0)}^{\alpha(x)}\int_{y_0}^{y} g(s,t)\Big(\max_{\tilde{\eta}\in[s-h,s]} u^p(\tilde{\eta},t)\Big)\,dt\,ds, \quad x\ge x_0, y\ge y_0, \\ u(x,y) &\le \psi(x,y), \quad x\in\big[\alpha(x_0)-h,x_0\big], y\ge y_0, \end{aligned} \tag{1.3}$$

where f, g, and ψ are nonnegative continuous functions and $a(x,y)>0$ is a nondecreasing continuous function, was discussed in [22].

Combining (1.2) with (1.3), we will consider the integral inequality with maxima

$$\begin{aligned} \varphi\big(u(x,y)\big) &\le a(x,y) + \sum_{j=1}^{m}\int_{b_j(x_0)}^{b_j(x)}\int_{c_j(y_0)}^{c_j(y)} \big(x^{\alpha_j}-s^{\alpha_j}\big)^{\beta_j-1}s^{\gamma_j-1}\big(y^{\bar{\alpha}_j}-t^{\bar{\alpha}_j}\big)^{\bar{\beta}_j-1}t^{\bar{\gamma}_j-1} \\ &\quad \cdot f_j(x,y,s,t)\omega_j\big(u(s,t)\big)\mu_j\Big(\max_{\tilde{\eta}\in[s-h,s]} g\big(u(\tilde{\eta},t)\big)\Big)\,dt\,ds, \\ &\quad (x,y)\in[x_0,x_1)\times[y_0,y_1), \\ u(x,y) &\le \psi(x,y), \quad (x,y)\in\big[b_*(x_0)-h,x_0\big]\times[y_0,y_1), \end{aligned} \tag{1.4}$$

where a, g, ω_j, f_j, b_j, and c_j are nonnegative continuous functions, b_j and c_j are increasing functions and belong to C^1, $b_*(x_0) := \min_{1\le j\le m} b_j(x_0)$, $h>0$ is a constant. Specially, the monotonicity of a, ω_j, μ_j, f_j, and g is not required. Further, ω_j's are used to construct a sequence of stronger monotonized functions. Then the obtained result is applied for considering the uniqueness of solutions to a boundary value problem of an integral equation with maxima.

2 Main result

Let $\mathbb{R} := (-\infty,+\infty)$, $\mathbb{R}_+ := [0,\infty)$, $\Delta := [x_0,x_1)\times[y_0,y_1)$ and $\Xi := [b_*(x_0)-h,x_0]\times[y_0,y_1)$. Define $\Phi_1,\Phi_2 : B\subset\mathbb{R}\to\mathbb{R}\setminus\{0\}$. As in [4], if Φ_1/Φ_2 is nondecreasing on B, then $\Phi_1\propto\Phi_2$. Considering inequality (1.4), we make the following assumptions for all $j=1,\dots,m$:

- (A_1) $b_j\in C^1([x_0,x_1),\mathbb{R}_+)$ and $c_j\in C^1([y_0,y_1),[y_0,y_1))$ are nondecreasing such that $b_j(x)\le x$ and $c_j(y)\le y$, and $c_j(y_0)=y_0$;
- (A_2) $a\in C(\Delta,\mathbb{R}_+), f_j\in C(\Delta\times[b_*(x_0),x_1)\times[y_0,y_1),\mathbb{R}_+)$, $\omega_j,\mu_j\in C(\mathbb{R}_+,\mathbb{R}_+)$ with $\omega_j(t)>0$, $\mu_j(t)>0$ for $t>0$;
- (A_3) $g,\varphi\in C(\mathbb{R}_+,\mathbb{R}_+)$ and $\psi\in C(\Xi,\mathbb{R}_+)$, and φ is strictly increasing such that $\lim_{t\to\infty}\varphi(t)=\infty$;
- (A_4) $\alpha_j,\bar{\alpha}_j\in(0,1]$, $\beta_j,\bar{\beta}_j\in(0,1)$, $\gamma_j>1-\frac{1}{p}$, $\bar{\gamma}_j>1-\frac{1}{p}$ such that $\frac{1}{p}+\alpha_j(\beta_j-1)+\gamma_j-1\ge 0$, $\frac{1}{p}+\bar{\alpha}_j(\bar{\beta}_j-1)+\bar{\gamma}_j-1\ge 0$, $p(\beta_j-1)+1>0$, $p(\bar{\beta}_j-1)+1>0$, $p>1$.

For those ω_j's, μ_j's given in (A_4), define $\tilde{\omega}_j(t)$ inductively by

$$\tilde{\omega}_j(t) := \begin{cases} \hat{\omega}_1(t)\max_{\tau\in[0,t]}\{\hat{\mu}_1(\tilde{g}(\tau))\}, & t\ge 0, j=1, \\ \max_{\tau\in[0,t]}\{\frac{\hat{\omega}_j(\tau)\hat{\mu}_{j+1}(\tilde{g}((\tau))}{\tilde{\omega}_{i-1}(\tau)}\}\tilde{\omega}_{i-1}(t), & t\ge 0, j=2,\dots,m, \end{cases} \tag{2.1}$$

where $\hat{\omega}_j(t) := \max_{\tau\in[0,t]}\{\bar{\omega}_j(\tau)\}$, $\hat{\mu}_j(t) := \max_{\tau\in[0,t]}\{\bar{\mu}_j(\tau)\}$, $\tilde{g}(t) := \max_{\tau\in[0,t]}\{g(\tau)\}$, $\bar{\omega}_j(t) := \omega_j(t)+\varepsilon_j$, $\bar{\mu}_j(t) := \mu_j(t)+\varepsilon_j$ for $t\geq 0$, $\epsilon_j := \varepsilon$ if $\omega_j(0)=0$ or $:= 0$ if $\omega_j(0)\neq 0$ for all $j=1,2,\dots,m$, where $\varepsilon > 0$ is an arbitrarily given constant.

Lemma 1 ([16]) *Let α, β, γ, and p be positive constants. Then*

$$\int_0^t \left(t^\alpha - s^\alpha\right)^{p(\beta-1)} s^{p(\gamma-1)}\,ds = \frac{t^\theta}{\alpha}B\left(\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right), \quad t\in\mathbb{R}_+,$$

where $\theta := p[\alpha(\beta-1)+\gamma-1]+1$, $B(\xi,\eta)=\int_0^1 s^{\xi-1}(1-s)^{\eta-1}\,ds$ ($\mathrm{Re}\,\xi>0$, $\mathrm{Re}\,\eta>0$) is the beta function.

Lemma 2 *Suppose that*

(C1) *$b_j\in C^1([x_0,x_1),\mathbb{R}_+)$ and $c_j\in C^1([y_0,y_1),[y_0,y_1))$ are nondecreasing with $b_j(x)\leq x$ on $[x_0,x_1)$, $c_j(y)\leq y$ on $[y_0,y_1)$ and $c_j(y_0)=y_0$ for all $j=1,\dots,m$;*

(C2) *$\psi\in C(\Xi,\mathbb{R}_+)$, $g_j\in C(\Delta\times\mathbb{R}_+^2,\mathbb{R}_+)$ are nondecreasing functions in x and y for all $j=1,\dots,m$;*

(C3) *$h_j,\bar{h}_j\in C(\mathbb{R}_+,\mathbb{R}_+)$ ($j=1,\dots,m$) are all nondecreasing with $h_j(t)>0$, $\bar{h}_j(t)>0$ for $t>0$, and $h_j\bar{h}_j \propto h_{j+1}\bar{h}_{j+1}$ ($j=1,\dots,m-1$);*

(C4) *$b\in C(\Delta,\mathbb{R}_+)$, $b_x,b_y\in(\Delta,\mathbb{R})$, and $\max_{s\in[b_*(x_0)-h,x_0]}\psi(s,t)\leq b(x_0,t)$ for all $t\in[y_0,y_1)$.*

If $u\in C([b_*(x_0)-h,x_1)\times[y_0,y_1),\mathbb{R}_+)$ satisfies the integral inequality

$$\begin{aligned} u(x,y) &\leq b(x,y)+\sum_{j=1}^m\int_{b_j(x_0)}^{b_j(x)}\int_{c_j(y_0)}^{c_j(y)} g_j(x,y,s,t) \\ &\quad\times h_j\bigl(u(s,t)\bigr)\tilde{h}_j\Bigl(\max_{\tilde{\eta}\in[s-h,s]}u(\tilde{\eta},t)\Bigr)\,dt\,ds, \quad (x,y)\in\Delta, \\ u(x,y) &\leq \psi(x,y), \quad (x,y)\in\Xi, \end{aligned} \tag{2.2}$$

then

$$u(x,y)\leq H_m^{-1}\left(H_m\bigl(\eta_m(x,y)\bigr)+\int_{b_m(x_0)}^{b_m(x)}\int_{c_m(y_0)}^{c_m(y)} g_m(x,y,s,t)\,dt\,ds\right) \tag{2.3}$$

for all $(x,y)\in[x_0,X_1^*]\times[y_0,Y_1^*]$, where H_j^{-1} is the inverse of the function

$$H_j(t) := \int_{t_j}^t \frac{ds}{h_j(s)\bar{h}_j(s)}, \quad t\geq t_j>0, j=1,\dots,m, \tag{2.4}$$

t_j is a given constant, and η_j is defined by

$$\begin{aligned} \eta_1(x,y) &:= b(x_0,y_0)+\int_{x_0}^x \bigl|b_x(s,y_0)\bigr|\,ds+\int_{y_0}^y\bigl|b_x(x,t)\bigr|\,dt, \\ \eta_{j+1}(x,y) &:= H_j^{-1}\left(H_j\bigl(\eta_j(x,y)\bigr)+\int_{b_j(x_0)}^{b_j(x)}\int_{c_j(y_0)}^{c_j(y)} g_j(x,y,s,t)dt\,ds\right) \end{aligned} \tag{2.5}$$

for $j = 1, \ldots, m-1$, *and* $x_0 \le X_1^* < x_1$, $y_0 \le Y_1^* < y_1$ *are chosen such that*

$$H_j\bigl(\eta_j\bigl(X_1^*, Y_1^*\bigr)\bigr) + \int_{a_j(x_0)}^{a_j(X_1^*)} \int_{b_j(y_0)}^{b_j(Y_1^*)} g_j\bigl(X_1^*, Y_1^*, s, t\bigr)\,dt\,ds \le \int_{u_j}^{\infty} \frac{ds}{h_j(s)\tilde{h}(s)} \tag{2.6}$$

for $j = 1, \ldots, m$.

Proof Let b be positive on Δ. It means that $\eta_1(x,y)$ is positive on Δ. Under such a circumstance, η_1 is nondecreasing on Δ and $\eta_1(x,y) > 0$,

$$\eta_1(x,y) \ge b(x_0,y_0) + \int_{x_0}^{x} b_x(s,y_0)\,ds + \int_{y_0}^{y} b_y(x,t)\,dt = b(x,y). \tag{2.7}$$

From (2.2) and (2.7), we have

$$\begin{aligned} u(x,y) &\le \eta_1(x,y) + \sum_{j=1}^{m} \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} g_j(x,y,s,t) \\ &\quad \cdot h_j\bigl(u(s,t)\bigr)\bar{h}_j\Bigl(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Bigr)\,dt\,ds, \quad (x,y) \in \Delta, \\ u(x,y) &\le \psi(x,y), \quad (x,y) \in \Xi. \end{aligned} \tag{2.8}$$

Concerning (2.8), we consider the auxiliary inequality

$$\begin{aligned} u(x,y) &\le \eta_1(x,y) + \sum_{j=1}^{m} \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} g_j(\xi,\eta,s,t) \\ &\quad \times h_j\bigl(u(s,t)\bigr)\bar{h}_j\Bigl(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Bigr)\,dt\,ds, \quad (x,y) \in [x_0,\xi] \times [y_0,\eta], \\ u(x,y) &\le \psi(x,y), \quad (x,y) \in \bigl[b_*(x_0)-h, x_0\bigr] \times [y_0,\eta], \end{aligned} \tag{2.9}$$

where $x_0 \le \xi \le X_1^*$ and $y_0 \le \eta \le Y_1^*$ are chosen arbitrarily. Having (2.9) we claim

$$u(x,y) \le H_m^{-1}\biggl(H_m\bigl(\eta_m(\xi,\eta,x,y)\bigr) + \int_{b_m(x_0)}^{b_m(x)} \int_{c_m(y_0)}^{c_m(y)} g_m(\xi,\eta,s,t)\,dt\,ds\biggr) \tag{2.10}$$

for $x_0 \le x \le \min\{\xi, X_2^*\}$, $y_0 \le y \le \min\{\eta, Y_2^*\}$, where $\tilde{\eta}_j(\xi,\eta,x,y)$ is defined inductively by $\tilde{\eta}_1(\xi,\eta,x,y) := \eta_1(x,y)$ and

$$\tilde{\eta}_j(\xi,\eta,x,y) := H_{j-1}^{-1}\biggl(H_{j-1}\bigl(\tilde{\eta}_{j-1}(\xi,\eta,x,y)\bigr) + \int_{b_{j-1}(x_0)}^{b_{j-1}(x)} \int_{c_{j-1}(y_0)}^{c_{j-1}(y)} g_{j-1}(\xi,\eta,s,t)\,dt\,ds\biggr)$$

for $j = 2, \ldots, m$, and $X_2^* \in [x_0, x_1)$, $Y_2^* \in [y_0, y_1)$ are chosen such that

$$\begin{aligned} &H_j\bigl(\tilde{\eta}_j\bigl(\xi,\eta,X_2^*,Y_2^*\bigr)\bigr) + \int_{b_j(x_0)}^{b_j(X_2^*)} \int_{c_j(y_0)}^{c_j(Y_2^*)} g_j(\xi,\eta,s,t) \\ &\quad \le \int_{t_j}^{\infty} \frac{ds}{h_j(s)\bar{h}_j(s)} \end{aligned} \tag{2.11}$$

for $j = 1, 2, \ldots, m$. Note that $X_2^* \geq X_1^*$ and $Y_2^* \geq Y_1^*$. In fact, both $\tilde{\eta}_j(\xi, \eta, x, y)$ and $g_j(\xi, \eta, x, y)$ are nondecreasing in ξ and η. Thus X_2^*, Y_2^* satisfying (2.11) will get smaller as ξ, η are chosen larger.

Since $\max_{s\in[b^*(x_0)-h,x_0]} \psi(s,t) \leq b(x_0,t)$ and $b(x_0,t) \leq \eta_1(x_0,t) \leq \eta_1(x,t)$, we obtain

$$\max_{s\in[b_*(x_0)-h,x_0]} \psi(s,t) \leq \eta_1(x,t), \quad (x,t) \in [x_0, x_1) \times [y_0, y_1). \tag{2.12}$$

First, (2.10) holds for $m = 1$. In fact,(2.9) for $m = 1$ is written as

$$u(x,y) \leq z_1(x,y), \quad (x,y) \in \bigl[b_*(x_0) - h, \xi\bigr] \times [y_0, \eta], \tag{2.13}$$

where

$$z_1(x,y) = \begin{cases} \eta_1(x,y) + \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} g_1(\xi, \eta, s, t) h_1(u(s,t)) \\ \quad \times \bar{h}_1(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta}, t))\, dt\, ds, \quad (x,y) \in [x_0, \xi] \times [y_0, \eta] \\ \eta_1(x_0, y), \quad (x,y) \in [b_*(x_0) - h, x_0] \times [y_0, \eta], \end{cases} \tag{2.14}$$

$z_1(x,y)$ is a nondecreasing function on $[x_0, \xi] \times [y_0, \eta]$. Then

$$\begin{aligned} \frac{\partial}{\partial x} z_1(x,y) = \frac{\partial}{\partial x} \eta_1(x,y) &+ \int_{c_1(y_0)}^{c_1(y)} g_1\bigl(\xi, \eta, b_1(x), t\bigr) h_1\bigl(u\bigl(b_1(x), t\bigr)\bigr) \\ &\times \bar{h}_1\Bigl(\max_{\tilde{\eta}\in[b_1(x)-h, b_1(x)]} u(\tilde{\eta}, t)\Bigr)\, dt b'(x) \end{aligned}$$

for all $(x,y) \in [x_0, \xi] \times [y_0, \eta]$. We have $0 < h_1(u(s,t))\bar{h}_1(u(s,t)) \leq h_1(z_1(s,t))\bar{h}_1(z_1(s,t)) \leq h_1(z_1(x,y))\bar{h}_1(z_1(x,y))$ by (C3) and (2.13) $s \leq b_1(x) \leq x$, $t \leq c_1(y) \leq y$ and both z_1 and $h_1\tilde{h}_1$ are nondecreasing. Thus

$$\begin{aligned} &\frac{\frac{\partial}{\partial x} z_1(x,y)}{h_1(z_1(x,y))\bar{h}_1(z_1(x,y))} \\ &\quad \leq \frac{\frac{\partial}{\partial x} \eta_1(x,y)}{h_1(\eta_1(x,y))\bar{h}_1(\eta_1(x,y))} + \frac{b'(x)}{h_1(z_1(x,y))\bar{h}_1(z_1(x,y))} \\ &\qquad \times \int_{c_1(y_0)}^{c_1(y)} g_1\bigl(\xi, \eta, b_1(x), t\bigr) h_1\bigl(u\bigl(b_1(x), t\bigr)\bigr) \bar{h}_1\Bigl(\max_{\tilde{\eta}\in[b_1(x)-h, b_1(x)]} u(\tilde{\eta}, t)\Bigr)\, dt \\ &\quad \leq \frac{\frac{\partial}{\partial x} \eta_1(x,y)}{h_1(\eta_1(x,y))\bar{h}_1(\eta_1(x,y))} + b'(x) \int_{c_1(y_0)}^{c_1(y)} g_1\bigl(\xi, \eta, b_1(x), t\bigr)\, dt. \end{aligned} \tag{2.15}$$

Integrating inequality (2.15) from x_0 to x, from (2.4) we get

$$\begin{aligned} H_1\bigl(Z_1(x,y)\bigr) &\leq H_1\bigl(\eta_1(x,y)\bigr) + \int_{x_0}^{x} b'(s) \int_{c_1(y_0)}^{c_1(y)} g_1\bigl(\xi, \eta, b_1(s), t\bigr)\, dt\, ds \\ &= H_1\bigl(\eta_1(x,y)\bigr) + \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} g_1(\xi, \eta, s, t)\, dt\, ds \end{aligned} \tag{2.16}$$

for all $(x,y) \in [x_0,\xi] \times [y_0,\eta]$. From (2.14), (2.16), and the monotonicity of H_1^{-1}, we have

$$u(x,y)) \le H_1^{-1}\left(H_1\big(\eta_1(x,y)\big) + \int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} g_1(\xi,\eta,s,t)\,dt\,ds\right) \tag{2.17}$$

for $x_0 \le x \le \xi < X_2^*$, $Y_0 \le y \le \eta < Y_2^*$, implying that (2.7) is true for $m = 1$.

Assume that (2.10) holds for $m = k$. Consider

$$\begin{aligned} u(x,y) &\le \eta_1(x,y) + \sum_{j=1}^{k+1}\int_{b_j(x_0)}^{b_j(x)}\int_{c_j(y_0)}^{c_j(y)} g_j(\xi,\eta,s,t) \\ &\quad \times h_j\big(u(s,t)\big)\bar{h}_j\Big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Big)\,dt\,ds, \quad (x,y) \in [x_0,\xi] \times [y_0,\eta] \\ u(x,y) &\le \psi(x,y), \quad (x,y) \in \big[b_*(x_0) - h, x_0\big] \times [y_0,\eta]. \end{aligned} \tag{2.18}$$

Let

$$z_2(x,y) = \begin{cases} \eta_1(x,y) + \sum_{j=1}^{k+1}\int_{b_j(x_0)}^{b_j(x)}\int_{c_j(y_0)}^{c_j(y)} g_j(\xi,\eta,s,t)h_j(u(s,t)) \\ \quad \cdot \bar{h}_j(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t))\,dt\,ds, \quad (x,y) \in [x_0,\xi] \times [y_0,\eta], \\ \eta_1(x_0,y), \quad (x,y) \in [b_*(x_0) - h, x_0] \times [y_0,\eta]. \end{cases} \tag{2.19}$$

Then z_2 is a nondecreasing function on $[x_0,x] \times [y_0,\eta]$. By (2.19) and the definition of z_2, it follows that

$$u(x,y) \le z_2(x,y), \quad (x,y) \in \big[b_*(x_0) - h, \xi\big] \times [y_0,\eta]. \tag{2.20}$$

Since $h_j\bar{h}_j$ is nondecreasing and $z_2(x,y) > 0$, $b_j'(x) \ge 0$, and $b_j(x) \le x$, we have

$$\begin{aligned} &\frac{\frac{\partial}{\partial x} z_2(x,y)}{h_1(z_2(x,y))\bar{h}_1(z_2(x,y))} \\ &\quad \le \frac{\frac{\partial}{\partial x}\eta_1(x,y)}{h_1(z_2(x,y))\bar{h}_1(z_2(x,y))} + \sum_{j=1}^{k+1}\frac{b_j'(x)}{h_1(z_2(x,y))\bar{h}_1(z_2(x,y))} \\ &\qquad \cdot \int_{c_j(y_0)}^{c_j(y)} g_j\big(X,Y,b_j(x),t\big)h_j\big(u\big(b_j(x),t\big)\big)h_j\Big(\max_{\xi\in[b_j(x)-h,b_j(x)]} u(\tilde{\eta},t)\Big)\,dt \\ &\quad \le \frac{\frac{\partial}{\partial x}\eta_1(x,y)}{h_1(\eta_1(x,y))\bar{h}_1(\eta_1(x,y))} + \sum_{j=1}^{k+1}\frac{b_j'(x)}{h_j(z_2(x,y))\bar{h}_j(z_2(x,y))} \\ &\qquad \cdot \int_{c_j(y_0)}^{c_j(y)} g_j\big(\xi,\eta,b_j(x),t\big)h_j\big(z_2\big(b_j(x),t\big)\big)\bar{h}_j\Big(\max_{\tilde{\eta}\in[b_j(x)-h,b_j(x)]} z_2(\tilde{\eta},t)\Big)\,dt \\ &\quad \le \frac{\frac{\partial}{\partial x}\eta_1(x,y)}{h_1(\eta_1(x,y))\bar{h}_1(\eta_1(x,y))} + b_1'(x)\int_{c_1(y_0)}^{c_1(y)} g_1\big(\xi,\eta,b_1(x),t\big)\,dt + \sum_{j=1}^{k} b_{j+1}'(x) \\ &\qquad \cdot \int_{c_j(y_0)}^{c_j(y)} g_{j+1}\big(\xi,\eta,b_{j+1}(x),t\big)\tilde{h}_{j+1}\big(z_2\big(b_{j+1}(x),t\big)\big)\hat{h}_{j+1}\Big(\max_{\tilde{\eta}\in[b_j(x)-h,b_j(x)]} z_2(\tilde{\eta},t)\Big)\,dt \end{aligned}$$

for all $(x, y) \in [x_0, X_1^*] \times [y_0, Y_1^*]$, where $\tilde{h}_{j+1}(u) := h_{j+1}(u)/h_1(u)$, $\hat{h}_{j+1}(u) := \bar{h}_{j+1}(u)/\bar{h}_1(u)$, $j = 1, \ldots, k$. Integrating the above inequality from x_0 to x, we can obtain

$$\begin{aligned} H_1\big(z_2(x,y)\big) \le{} & H_1\big(\eta_1(x,y)\big) + \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} g_1(\xi,\eta,s,t)\,dt\,ds \\ & + \sum_{j=1}^{k} \int_{b_{j+1}(x_0)}^{b_{j+1}(x)} \int_{c_{j+1}(y_0)}^{c_{j+1}(y)} g_{j+1}(\xi,\eta,s,t)\tilde{h}_{j+1}\big(z_2(s,t)\big) \\ & \cdot \hat{h}_{j+1}\Big(\max_{\tilde{\eta}\in[s-h,s]} z_2(\tilde{\eta},t)\Big)\,dt\,ds \end{aligned} \tag{2.21}$$

for all $(x, y) \in [x_0, X] \times [y_0, Y]$. Let

$$\begin{aligned} &\eta(x,y) := H_1\big(z_2(x,y)\big), \\ &\varrho_1(x,y) := H_1\big(\eta_1(x,y)\big) + \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} g_1(\xi,\eta,s,t)\,dt\,ds. \end{aligned} \tag{2.22}$$

Then inequality (2.21) can be rewritten as

$$\begin{aligned} \eta(x,y) \le{} & \varrho_1(x,y) + \sum_{j=1}^{k} \int_{b_{j+1}(x_0)}^{b_{j+1}(x)} \int_{c_{j+1}(y_0)}^{c_{j+1}(y)} g_{j+1}(\xi,\eta,s,t)\tilde{h}_{j+1}\big(H_1^{-1}\big(z_2(s,t)\big)\big) \\ & \cdot \hat{h}_{j+1}\Big(\max_{\tilde{\eta}\in[s-h,s]} H_1^{-1}\big(z_2(\tilde{\eta},t)\big)\Big)\,dt\,ds, \quad (x,y) \in [x_0,X]\times[y_0,Y], \end{aligned} \tag{2.23}$$

$$\eta(x,y) = H_1\big(\eta(x_0,y)\big) \le \varrho_1(x_0,y), \quad (x,y) \in \big[b_*(x_0)-h, x_0\big]\times[y_0,Y],$$

the same form as (2.9) for $m = k$. By (C3), each $(\bar{h}_{j+1} \circ H_1^{-1})(\tilde{h}_{j+1} \circ H_1^{-1})$ $(j = 1, \ldots, k)$ is a nonnegative continuous and increasing function on $\mathbb{R}_+$ and positive on $(0, +\infty)$. Moreover, $(\tilde{h}_j \circ H_1^{-1}) \propto (\hat{h}_{j+1} \circ H_1^{-1})$ for all $j = 2, \ldots, k$. By the inductive assumption, we have

$$\eta(x,y) \le \bar{H}_{k+1}^{-1}\left(\bar{H}_{k+1}\big(\varrho_k(x,y)\big) + \int_{b_{k+1}(x_0)}^{b_{k+1}(x)} \int_{c_{k+1}(y_0)}^{c_{k+1}(y)} g_{k+1}(\xi,\eta,s,t)\,dt\,ds\right) \tag{2.24}$$

for $x_0 \le x \le \min\{\xi, X_3^*\}$, $y_0 \le y \le \min\{\eta, Y_3^*\}$, where

$$\bar{H}_{j+1}(t) := \int_{\tilde{t}_{j+1}}^{t} \frac{ds}{\tilde{h}_{j+1}(H_1^{-1}(s))\hat{h}_{j+1}(H_1^{-1}(s))}, \quad t > 0, \tag{2.25}$$

$\tilde{t}_{j+1} = H_1(t_{j+1})$, $\bar{H}_{j+1}^{-1}$ is the inverse of $\bar{H}_{j+1}$, $j = 1, \ldots, k$,

$$\varrho_{j+1}(x,y) := \bar{H}_{j+1}^{-1}\left(\bar{H}_{j+1}\big(\varrho_j(x,y)\big) + \int_{b_{j+1}(x_0)}^{b_{j+1}(x)} \int_{c_{j+1}(y_0)}^{c_{j+1}(y)} g_{j+1}(\xi,\eta,s,t)\,dt\,ds\right), \tag{2.26}$$

$j = 1, \ldots, k-1$, and X_3^*, Y_3^* are chosen such that

$$\begin{aligned} & \bar{H}_{j+1}\big(\varrho_j\big(X_3^*, Y_3^*\big)\big) + \int_{b_{j+1}(x_0)}^{b_{j+1}(X_3^*)} \int_{c_{j+1}(y_0)}^{c_{j+1}(Y_3^*)} g_{j+1}(\xi,\eta,t,s)\,dt\,ds \\ & \quad \le \int_{\tilde{t}_{j+1}}^{H_1(\infty)} \frac{ds}{\tilde{h}_{j+1}(H_1^{-1}(s))\hat{h}_{j+1}(H_1^{-1}(s))}, \quad j = 1, \ldots, k. \end{aligned} \tag{2.27}$$

Note that

$$\begin{aligned}
\bar{H}_j(t) &= \int_{\tilde{t}_j}^{t} \frac{ds}{\tilde{h}_j(H_1^{-1}(s))\hat{h}_j(H_1^{-1}(s))} \\
&= \int_{H_1(t_j)}^{t} \frac{h_1(H_1^{-1}(s))\bar{h}_1(H_1^{-1}(s))\,ds}{h_j(H_1^{-1}(s))\bar{h}_j(H_1^{-1}(s))} \\
&= \int_{H_1(t_j)}^{t} \frac{h_1(H_1^{-1}(s))\bar{h}_1(H_1^{-1}(s))\,ds}{h_j(H_1^{-1}(s))\bar{h}_j(H_1^{-1}(s))} \\
&= \int_{t_j}^{H_1^{-1}(t)} \frac{ds}{h_j(s)\bar{h}_j(s)} = H_j\bigl(H_1^{-1}(t)\bigr), \quad j = 2,\dots,k+1. \qquad (2.28)
\end{aligned}$$

Then, from (2.20), (2.24), and (2.28), we get

$$\begin{aligned}
u(x,y) &\le H_1^{-1}\bigl(\eta(x,y)\bigr) \\
&\le H_{k+1}^{-1}\left(H_{k+1}\bigl(H_1^{-1}\bigl(\varrho_k(x,y)\bigr)\bigr) + \int_{b_{k+1}(x_0)}^{b_{k+1}(x)}\int_{c_{k+1}(y_0)}^{c_{k+1}(y)} g_{k+1}(\xi,\eta,s,t)\,dt\,ds\right) \qquad (2.29)
\end{aligned}$$

for $x_0 \le x \le \min\{X, X_3^*\}$, $y_0 \le y \le \min\{Y, Y_3^*\}$. Let $\tilde{\varrho}_j(x,y) = H_1^{-1}(\varrho_j(x,y))$. Then

$$\begin{aligned}
\tilde{\varrho}_1(x,y) &= H_1\bigl(\varrho_1(x,y)\bigr) \\
&= H_1^{-1}\left(H_1\bigl(\eta_1(x,y)\bigr) + \int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} g_1(\xi,\eta,s,t)\,dt\,ds\right) \\
&= H_1^{-1}\left(H_1\bigl(\tilde{\eta}_1(\xi,\eta,x,y)\bigr) + \int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} g_1(\xi,\eta,s,t)\,dt\,ds\right) \\
&= \tilde{\eta}_2(X,Y,x,y). \qquad (2.30)
\end{aligned}$$

Moreover, with the assumption that $\tilde{\varrho}_k(x,y) = \tilde{\eta}_{k+1}(\xi,\eta,x,y)$, we get

$$\begin{aligned}
\tilde{\varrho}_{k+1}(x,y) &= H_1^{-1}\left(\bar{H}_{k+1}^{-1}\left(\bar{H}_{k+1}\bigl(\varrho_k(x,y)\bigr) + \int_{b_{k+1}(x_0)}^{b_{k+1}(x)}\int_{c_{k+1}(y_0)}^{c_{k+1}(y)} g_{k+1}(\xi,\eta,t,s)\,dt\,ds\right)\right) \\
&= H_{k+1}^{-1}\left(H_{k+1}\bigl(H_1^{-1}\bigl(\varrho_k(x,y)\bigr)\bigr) + \int_{b_{k+1}(x_0)}^{b_{k+1}(x)}\int_{c_{k+1}(y_0)}^{c_{k+1}(y)} g_{k+1}(\xi,\eta,t,s)\,dt\,ds\right) \\
&= H_{k+1}^{-1}\left(H_{k+1}\bigl(\tilde{\varrho}_k(x,y)\bigr) + \int_{b_{k+1}(x_0)}^{b_{k+1}(x)}\int_{c_{k+1}(y_0)}^{c_{k+1}(y)} g_{k+1}(\xi,\eta,t,s)\,dt\,ds\right) \\
&= H_{k+1}^{-1}\left(H_{k+1}\bigl(\tilde{\eta}_{k+1}(\xi,\eta,x,y)\bigr) + \int_{b_{k+1}(x_0)}^{b_{k+1}(x)}\int_{c_{k+1}(y_0)}^{c_{k+1}(y)} g_{k+1}(\xi,\eta,t,s)\,dt\,ds\right) \\
&= \tilde{\eta}_{k+2}(\xi,\eta,x,y). \qquad (2.31)
\end{aligned}$$

This proves that

$$\tilde{\varrho}_j(x,y) = \tilde{\eta}_{j+1}(\xi,\eta,x,y), \quad j = 1,\dots,k. \qquad (2.32)$$

Therefore, (2.27) becomes

$$\begin{aligned}
&H_{j+1}\big(\tilde{\eta}_{j+1}\big(\xi,\eta,X_3^*,Y_3^*\big)\big)+\int_{b_{j+1}(x_0)}^{b_{j+1}(X_3^*)}\int_{c_{j+1}(y_0)}^{c_{j+1}(Y_3^*)} g_{j+1}(\xi,\eta,t,s)\,dt\,ds \\
&\quad\le \int_{\tilde{t}_{j+1}}^{H_1(\infty)} \frac{ds}{\tilde{h}_{j+1}(H_1^{-1}(s))\hat{h}_{j+1}(H_1^{-1}(s))} \\
&\quad= \int_{t_{j+1}}^{\infty} \frac{ds}{h_{j+1}(s)\bar{h}_{j+1}(s)}, \quad j=1,\ldots,k,
\end{aligned} \tag{2.33}$$

which implies that $X_2^* = X_3^*$, $\xi \le X_3^*$, $Y_2^* = Y_3^*$, $\eta \le Y_3^*$. From (2.29) we obtain

$$u(x,y) \le H_{k+1}^{-1}\left(H_{k+1}\big(\tilde{\eta}_{k+1}(\xi,\eta,x,y)\big)+\int_{b_{k+1}(x_0)}^{b_{k+1}(x)}\int_{c_{k+1}(y_0)}^{c_{k+1}(y)} g_{k+1}(\xi,\eta,s,t)\,dt\,ds\right)$$

for $x_0 \le x \le \min\{X, X_2^*\}$, $y_0 \le y \le \min\{Y, Y_2^*\}$. This proves (2.10) by induction.

Taking $x = \xi,\eta$, $y = \xi,\eta$ in (2.10), we have

$$\begin{aligned}
u(\xi,\eta) &\le H_m^{-1}\left(H_m\big(\tilde{\eta}_m(\xi,\eta,\xi,\eta)\big)+\int_{b_m(x_0)}^{b_m(X)}\int_{c_m(y_0)}^{c_m(\eta)} g_m(\xi,\eta,s,t)\,dt\,ds\right) \\
&= H_m^{-1}\left(H_m\big(\eta_m(\xi,\eta)\big)+\int_{b_m(x_0)}^{b_m(\xi)}\int_{c_m(y_0)}^{c_m(\eta)} g_m(\xi,\eta,s,t)\,dt\,ds\right)
\end{aligned} \tag{2.34}$$

for $x_0 \le \xi \le X_1^*$, $y_0 \le \eta \le Y_1^*$, since $x_2^* \ge X_1^*$, $Y_2^* \ge Y_1^*$ and $\tilde{\eta}_m(\xi,\eta,\xi,\eta) = \eta_m(\xi,\eta)$. Since ξ, η are arbitrary, replacing ξ and η with x and y, respectively, we have

$$u(x,y) \le H_m^{-1}\left(H_m\big(\eta_m(x,y)\big)+\int_{b_m(x_0)}^{b_m(x)}\int_{c_m(y_0)}^{c_m(y)} g_m(x,y,s,t)\,dt\,ds\right) \tag{2.35}$$

for all $(x,y) \in [x_0, X_1^*] \times [y_0, Y_1^*]$.

Let $b(x,y) = 0$ for some $(x,y) \in \Delta$. Let $\eta_{1,\epsilon}(x,y) := r_1(x,y) + \epsilon$ for any $\epsilon > 0$. Then $\eta_{1,\epsilon}(x,y) > 0$. Using the same arguments as above, where $\eta_1(x,y)$ is replaced with $\eta_{1,\epsilon}(x,y)$, we get

$$u(x,y) \le H_m^{-1}\left(H_m\big(\eta_{n,\epsilon}(x,y)\big)+\int_{b_m(x_0)}^{b_m(x)}\int_{c_m(y_0)}^{c_m(y)} g_m(x,y,s,t)\,dt\,ds\right)$$

for $x_0 \le x \le X_1^*$, $y_0 \le Y_1^*$. Then consider the continuity of $\eta_{i,\epsilon}$ in ϵ and the continuity of H_j and H_j^{-1} for $j = 1,\ldots,m$, and let $\epsilon \to 0^+$. Then we obtain (2.7). This completes the proof. □

Theorem 2.1 *Suppose that* (A_1)–(A_4) *hold.* $\max_{s\in[b_*(x_0)-h,x_0]} \psi(s,y) \le \varphi^{-1}((1+m)^{1-1/q}a(x_0,y))$ *for* $y \in [y_0,y_1)$ *and* $u \in C([b_*(x_0)-h,x_1) \times [y_0,y_1),\mathbb{R}_+)$ *are satisfied* (1.4). *Then, for all* $(x,y) \in [x_0,X_1) \times [y_0,Y_1)$, *we have*

$$u(x,y) \le \varphi^{-1}\left(\left(W_m^{-1}\big(W_m\big(r_m(x,y)\big)\big)+\int_{\alpha_m(x_0)}^{\alpha_m(x)}\int_{\beta_m(y_0)}^{\beta_m(y)} \tilde{f}_m(x,y,s,t)\,dt\,ds\right)^{1/q}\right), \tag{2.36}$$

where W_j^{-1} is the inverse of the function

$$W_j(t) := \int_{t_j}^{t} \frac{ds}{\tilde{\omega}_j^q(\varphi^{-1}(s^{1/q}))}, \quad t \geq t_j > 0, j = 1, \ldots, m. \tag{2.37}$$

In (2.36) *and* (2.37), *t_j is a given constant,* $\frac{1}{p} + \frac{1}{q} = 1$, *$\tilde{\omega}_j$ ($j = 1, 2, \ldots, m$) are defined by* (2.1),

$$\begin{aligned} & r_1(x,y) := (1+m)^{q-1}\Big(\max_{(\tau,\xi)\in[x_0,x]\times[y_0,y]}\{a(\tau,\xi)\}\Big)^q, \\ & r_j(x,y) := W_{j-1}^{-1}\bigg[W_{j-1}\big(r_{j-1}(x,y)\big) + \int_{b_{i-1}(x_0)}^{b_{i-1}(x)} \int_{c_{i-1}(y_0)}^{c_{i-1}(y)} \tilde{f}_{i-1}(x,y,s,t)\,dt\,ds\bigg], \\ & \quad j = 2, \ldots, m, \end{aligned} \tag{2.38}$$

$$\begin{aligned} & \tilde{f}_j(x,y,s,t) := (1+m)^{q-1}\big(M_j x^{\theta_j} \bar{M}_j y^{\bar{\theta}_j}\big)^{q/p}\Big(\max_{(\iota,\xi)\in[x_0,x]\times[y_0,y]} f_j(\iota,\xi,s,t)\Big)^q, \\ & \quad (x,y) \in [x_0,x_1) \times [y_0,y_1), \end{aligned} \tag{2.39}$$

$$\begin{aligned} & M_j = \alpha_j^{-1} B\bigg(\frac{p(\gamma_j - 1) + 1}{\alpha_j}, p(\beta_j - 1) + 1\bigg), \\ & \bar{M}_j = \bar{\alpha}_j^{-1} B\bigg(\frac{p(\bar{\gamma}_j - 1) + 1}{\bar{\alpha}_j}, p(\beta_j - 1) + 1\bigg), \\ & \theta_j = p\big(\alpha_j(\beta_j - 1) + \gamma_j - 1\big) + 1, \\ & \bar{\theta}_j = p\big(\bar{\alpha}_j(\bar{\beta}_j - 1) + \bar{\gamma}_j - 1\big) + 1, \quad j = 1, \ldots, m, \end{aligned} \tag{2.40}$$

$X_1 \in [x_0, x_1)$, $Y_1 \in [y_0, y_1)$ are chosen such that

$$W_j\big(r_j(X_1,Y_1)\big) + \int_{b_j(x_0)}^{b_j(X_1)} \int_{c_j(y_0)}^{c_j(Y_1)} \tilde{f}_j(x,y,s,t)\,dt\,ds \leq \int_{t_j}^{\infty} \frac{ds}{\tilde{\omega}_j^q(\varphi^{-1}(s^{1/q}))} \tag{2.41}$$

for $j = 1, \ldots, m$.

Proof Above all, we monotonize functions f_j, ω_j, μ_j, g, and a in (1.4). Let

$$\hat{a}(x,y) := \max_{(\tau,\xi)\in[x_0,x]\times[y_0,y]}\big\{a(\tau,\xi)\big\}, \quad (x,y) \in [x_0,x_1) \times [y_0,y_1),$$

which is increasing in x and y. The sequence $\{\tilde{\omega}_j\}$, defined by $\omega_j(s)$ and $\mu_j(s)$ in (2.1), consists of nonnegative and nondecreasing functions on $\mathbb{R}_+$ and satisfies

$$\omega_j(t) \leq \hat{\omega}_j(t), \qquad \mu_j(t) \leq \hat{\mu}_j(t), \qquad \hat{\omega}_j(t)\hat{\mu}_j\big(\tilde{g}(t)\big) \leq \tilde{\omega}_j(t), \quad j = 1, \ldots, m. \tag{2.42}$$

Moreover, because the ratios $\tilde{\omega}_{j+1}/\tilde{\omega}_j$ ($j = 1, \ldots, m-1$) are all nondecreasing, we have

$$\tilde{\omega}_j \propto \tilde{\omega}_{j+1}, \quad j = 1, 2, \ldots, m-1. \tag{2.43}$$

Let

$$\hat{f}_j(x,y,s,t) := \max_{(\iota,\xi)\in[x_0,x]\times[y_0,y]} f_j(\iota,\xi,s,t), \tag{2.44}$$

which are increasing in x and y and satisfy $\tilde{f}_j(x,y,s,t) \geq f_j(x,y,s,t) \geq 0$ for $j = 1,2,\ldots,m$. Since $\tilde{g}$ is nondecreasing, we obtain

$$\max_{\tilde{\eta}\in[s-h,s]} g\big(u(\xi,y)\big) \leq \max_{\tilde{\eta}\in[s-h,s]} \tilde{g}\big(u(\xi,y)\big) \leq \tilde{g}\Big(\max_{\tilde{\eta}\in[s-h,s]} u(\xi,y)\Big) \tag{2.45}$$

for all $(s,y) \in [b_*(x_0),x_1) \times [y_0,y_1)$. From (1.4), (2.42), (2.45), and the definition of $\hat{f}_j$, we can obtain

$$\begin{aligned} \varphi\big(u(x,y)\big) &\leq \hat{a}(x,y) + \sum_{j=1}^{m} \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \big(x^{\alpha_j} - s^{\alpha_j}\big)^{\beta_j-1} s^{\gamma_j-1} \big(y^{\bar{\alpha}_j} - t^{\bar{\alpha}_j}\big)^{\bar{\beta}_j-1} t^{\bar{\gamma}_j-1} \\ &\quad \times \hat{f}_j(x,y,s,t)\hat{\omega}_j\big(u(s,t)\big)\hat{\mu}_j\Big(\tilde{g}\Big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Big)\Big)\,dt\,ds, \\ &\quad (x,y) \in [x_0,x_1) \times [y_0,y_1), \\ u(x,y) &\leq \psi(x,y), \quad (x,y) \in \big[b_*(x_0)-h,x_0\big] \times [y_0,y_1). \end{aligned} \tag{2.46}$$

Let $\frac{1}{p} + \frac{1}{q} = 1, p > 1$, then $q > 0$. By Lemma 1, Hölder's inequality, (A_4) and (2.46), we obtain, for all $(x,y) \in [x_0,x_1) \times [y_0,y_1)$,

$$\begin{aligned} &\varphi\big(u(x,y)\big) \\ &\quad\leq \hat{a}(x,y) + \sum_{j=1}^{m} \bigg(\int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \big(x^{\alpha_j} - s^{\alpha_j}\big)^{p(\beta_j-1)} s^{p(\gamma_j-1)} \big(y^{\bar{\alpha}_j} - t^{\bar{\alpha}_j}\big)^{p(\bar{\beta}_j-1)} t^{(\bar{\gamma}_j-1)}\,dt\,ds\bigg)^{1/p} \\ &\qquad \cdot \bigg(\int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \hat{f}_j^q(x,y,s,t)\hat{\omega}_j^q\big(u(s,t)\big)\Big(\hat{\mu}_j\Big(\tilde{g}\Big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Big)\Big)\Big)^q\,dt\,ds\bigg)^{1/q} \\ &\quad\leq \hat{a}(x,y) + \sum_{j=1}^{m} \bigg(\int_0^x \int_0^y \big(x^{\alpha_j} - s^{\alpha_j}\big)^{p(\beta_j-1)} s^{p(\gamma_j-1)} \big(y^{\bar{\alpha}_j} - t^{\bar{\beta}_j}\big)^{p(\bar{\gamma}_j-1)} t^{p(\bar{\gamma}_j-1)}\,dt\,ds\bigg)^{1/p} \\ &\qquad \cdot \bigg(\int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \hat{f}_j^q(x,y,s,t)\hat{\omega}_j^q\big(u(s,t)\big)\Big(\hat{\mu}_j\Big(\tilde{g}\Big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Big)\Big)\Big)^q\,dtds\bigg)^{1/q} \\ &\quad\leq \hat{a}(x,y) + \sum_{j=1}^{m} \big(M_j x^{\theta_j} \bar{M}_j y^{\bar{\theta}_j}\big)^{1/p} \bigg(\int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \hat{f}_j^q(x,y,s,t) \\ &\qquad \cdot \hat{\omega}_j^q\big(u(s,t)\big)\Big(\hat{\mu}_j\Big(\tilde{g}\Big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Big)\Big)\Big)^q\,dt\,ds\bigg)^{1/q}, \end{aligned} \tag{2.47}$$

where $0 \leq b_j(t) \leq t$, $0 \leq c_j(t) \leq t$, M_j, $\bar{M}_j$, θ_j, and $\bar{\theta}_j$ are given by (2.40) for $j = 1,\ldots,m$.

By Jensen's inequality and (2.47), we get, for all $(x,y) \in \Delta$,

$$\begin{aligned} \varphi^q\big(u(x,y)\big) &\leq (1+m)^{q-1}\bigg(\hat{a}^q(x,y) + \sum_{j=1}^{m} \big(M_j x^{\theta_j} \bar{M}_j y^{\bar{\theta}_j}\big)^{q/p} \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \hat{f}_j^q(x,y,s,t) \\ &\quad \times \hat{\omega}_j^q\big(u(s,t)\big)\Big(\hat{\mu}_j\Big(\tilde{g}\Big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Big)\Big)\Big)\bigg)^q\,dt\,ds. \end{aligned} \tag{2.48}$$

Then, from (2.38), r_1 is increasing on Δ. Then, by the definition of r_1 and $\tilde{f}_j$, from (2.48) we have

$$\varphi^q\bigl(u(x,y)\bigr) \le r_1(x,y) + \sum_{j=1}^{m} \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \tilde{f}_j(x,y,s,t)\hat{\omega}_j^q\bigl(u(s,t)\bigr) \cdot \Bigl(\hat{\mu}_j\Bigl(\tilde{g}\Bigl(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Bigr)\Bigr)\Bigr)^q \,dt\,ds, \quad (x,y)\in\Delta. \tag{2.49}$$

According to (2.49), we consider the inequalities

$$\begin{aligned} &\varphi^q\bigl(u(x,y)\bigr) \le r_1(X,Y) + \sum_{j=1}^{m} \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \tilde{f}_j(X,Y,s,t) \\ &\qquad \cdot \hat{\omega}_j^q\bigl(u(s,t)\bigr)\Bigl(\hat{\mu}_j\Bigl(\hat{g}\Bigl(\max_{\tilde{\eta}\in[\tilde{\eta}-h,s]} u(\tilde{\eta},t)\Bigr)\Bigr)\Bigr)^q \,dt\,ds, \\ &\qquad (x,y)\in[x_0,X]\times[y_0,Y], \\ &u(x,y)\le\psi(x,y), \quad (x,y)\in\bigl[b_*(x_0)-h,x_0\bigr]\times[y_0,Y], \end{aligned} \tag{2.50}$$

where $x_0 \le X \le X_1$ and $y_0 \le Y \le Y_1$ are chosen arbitrarily.

Since $\max_{s\in[b_*(x_0)-h,x_0]}\psi(s,y) \le \varphi^{-1}((1+m)^{1-1/q}a(x_0,y))$ for $y\in[y_0,y_1)$, $a(x_0,y)\le\hat{a}(x_0,y)$, we have $\max_{s\in[b_*(x_0)-h,x_0]}\psi(s,y) \le \varphi^{-1}(r_1^{1/q}(X,Y))$, $y\in[y_0,Y]$. Define a function $z(x,y):[b_*(x_0)-h,X)\times[y_0,Y)\to\mathbb{R}_+$ by

$$z(x,y) = \begin{cases} r_1(X,Y) + \sum_{j=1}^{m} \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \tilde{f}_j(X,Y,s,t)\hat{\omega}_j^q(u(s,t)) \\ \quad \times (\hat{\mu}_j(\hat{g}(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t))))^q \,dt\,ds, \quad (x,y)\in[x_0,X]\times[y_0,Y], \\ r_1(X,Y), \quad (x,y)\in[b_*(x_0)-h,x_0]\times[y_0,Y]. \end{cases}$$

Clearly, $z(x,y)$ is increasing in x. By the definition of $z(x,y)$ and (2.50), we have

$$u(x,y)\le\varphi^{-1}\bigl(z^{1/q}(x,y)\bigr), \quad (x,y)\in\bigl[b_*(x_0)-h,X\bigr]\times[y_0,Y]. \tag{2.51}$$

Since $\varphi(t)$ is strictly increasing and $z(x,y)$ is nondecreasing, from (2.51) we get, for $(s,y)\in[b_*(x_0),X]\times[y_0,Y]$,

$$\max_{\xi\in[s-h,s]} u(\xi,y) \le \max_{\xi\in[s-h,s]} \varphi^{-1}\bigl(z^{1/q}(\xi,y)\bigr) \le \varphi^{-1}\bigl(z^{1/q}(s,y)\bigr). \tag{2.52}$$

From the definition of $z(x,y)$, (2.42), (2.51), and (2.52), it follows that

$$\begin{aligned} z(x,y) &\le r_1(X,Y) + \sum_{j=1}^{m} \int_{b_j(x_0)}^{b_j(x)} \int_{b_j(y_0)}^{b_j(y)} \tilde{f}_j(X,Y,s,t)\hat{\omega}_j^q\bigl(\varphi^{-1}\bigl(z^{1/q}(s,t)\bigr)\bigr) \\ &\quad \cdot \Bigl(\hat{\mu}_j\Bigl(\hat{g}\Bigl(\max_{\tilde{\eta}\in[s-h,s]} \varphi^{-1}\bigl(z^{1/q}(\tilde{\eta},t)\bigr)\Bigr)\Bigr)\Bigr)^q \,dt\,ds \\ &\le r_1(X,Y) + \sum_{j=1}^{m} \int_{b_j(x_0)}^{b_j(x)} \int_{b_j(y_0)}^{b_j(y)} \tilde{f}_j(X,Y,s,t)\hat{\omega}_j^q\bigl(\varphi^{-1}\bigl(z^{1/q}(s,t)\bigr)\bigr) \\ &\quad \cdot \bigl(\hat{\mu}_j\bigl(\tilde{g}\bigl(\varphi^{-1}\bigl(z^{1/q}(s,t)\bigr)\bigr)\bigr)\bigr)^q \,dt\,ds \end{aligned} \tag{2.53}$$

$$\begin{aligned}&\le r_1(X,Y)+\sum_{j=1}^{m}\int_{b_j(x_0)}^{b_j(x)}\int_{b_j(y_0)}^{b_j(y)}\tilde f_j(X,Y,s,t)\tilde\omega_j^q\big(\varphi^{-1}\big(z^{1/q}(s,t)\big)\big)\\&\quad\cdot\vartheta_j\Big(\max_{\tilde\eta\in[s-h,s]}u(\tilde\eta,t)\Big)\,dt\,ds,\quad (x,y)\in[x_0,X]\times[y_0,Y],\\ z(x,y)&\le r_1(X,Y),\quad (x,y)\in\big[b_*(x_0)-h,x_0\big]\times[y_0,Y],\end{aligned}$$

where $\vartheta_j(t)\equiv 1$, $t\ge 0$.

Let $\nu(t):=\varphi^{-1}(t^{1/q})$, which is a continuous and increasing function on $\mathbb{R}_+$. Thus $\tilde\omega_j^q(h(t))$ ($j=1,\dots,m$) are continuous and increasing on $\mathbb{R}_+$ and satisfy $\tilde\omega_j(\nu(t))>0$ for $t>0$. Moreover, since $\tilde\omega_j(t)\propto\tilde\omega_{j+1}(t)$, $\tilde\omega_{j+1}^q(\nu(t))/\tilde\omega_j^q(\nu(t))$ are continuous and increasing on $\mathbb{R}_+$ and positive on $(0,\infty)$, then $(\tilde\omega_j\circ\nu)\vartheta_j\propto(\tilde\omega_{j+1}\circ\nu)\vartheta_{j+1}$ for $j=1,2,\dots,m-1$.

Applying Lemma 2 to specified $g_j(x,y,s,t)=\tilde f_j(X,Y,s,t)$, $h_j(t)=\tilde\omega_j^q(\varphi^{-1}(t^{1/q}))$, $\bar h_j(t)=\vartheta_j(t)\equiv 1$ ($j=1,2,\dots,m$), and (2.53), we obtain

$$\begin{aligned}z(x,y)&\le W_m^{-1}\bigg[W_n\big(\tilde r_m(X,Y,x,y)\big)\\&\quad+\int_{b_m(x_0)}^{b_m(x)}\int_{c_m(y_0)}^{c_m(y)}\tilde f_m(X,Y,s,t)\,dt\,ds\bigg]\end{aligned}\tag{2.54}$$

for $x_0\le x\le\min\{X,X_2\}$, $y_0\le y\le\min\{Y,Y_2\}$, where $\tilde r_j$ is defined inductively by $\tilde r_1(X,Y,x,y):=\gamma_1(X,Y)$ and

$$\tilde r_j(X,Y,x,y):=W_{i-1}^{-1}\bigg(W_{i-1}\big(\tilde r_{i-1}(X,Y,x,y)\big)+\int_{b_{i-1}(x_0)}^{b_{i-1}(x)}\int_{c_{i-1}(y_0)}^{c_{i-1}(y)}\tilde f_{i-1}(X,Y,s,t)\,dt\,ds\bigg)$$

for $j=2,\dots,m$, and $\bar X_1$, $\bar Y_1$ are chosen such that

$$\begin{aligned}&W_j\big(\tilde r_j(X,Y,\bar X_1,\bar Y_1)\big)+\int_{b_j(x_0)}^{b_j(X_2)}\int_{c_j(y_0)}^{c_j(\bar Y_1)}\tilde f_j(X,Y,s,t)\\&\quad\le\int_{t_j}^{\infty}\frac{ds}{\tilde\omega_j^q(\varphi^{-1}(s^{1/q}))}\end{aligned}\tag{2.55}$$

for $j=1,\dots,m$.

Note that $X_2\ge X_1$ and $Y_2\ge Y_1$. In fact, both $\tilde r_j(X,Y,x,y)$ and $\tilde f_j(X,Y,x,y)$ are increasing in X and Y. Thus X_2, Y_2 satisfying (2.55) get smaller as X, Y are chosen larger.

According to (2.51) and (2.54),

$$\begin{aligned}u(x,y)&\le\varphi^{-1}\bigg(W_m^{-1}\bigg(W_n\big(\tilde r_m(X,Y,x,y)\big)\\&\quad+\int_{\alpha_m(x_0)}^{\alpha_m(x)}\int_{\beta_m(y_0)}^{\beta_m(y)}\tilde f_m(X,Y,s,t)\,dt\,ds\bigg)\bigg)\end{aligned}\tag{2.56}$$

for $x_0\le x\le\min\{X,X_2\}$, $y_0\le y\le\min\{Y,Y_2\}$.

Taking $x = X$, $y = Y$ in (2.56), we have

$$u(X,Y) \le \varphi^{-1}\biggl(W_m^{-1}\biggl(W_n\bigl(\tilde{r}_m(X,Y,X,Y)\bigr) + \int_{b_m(x_0)}^{b_m(X)}\int_{c_m(y_0)}^{c_m(Y)} \tilde{f}_m(X,Y,s,t)\,dt\,ds\biggr)\biggr) \tag{2.57}$$

for $x_0 \le X \le X_1$, $y_0 \le Y \le Y_1$. It is easy to verify $\tilde{r}_m(X,Y,X,Y) = r_m(X,Y)$. Thus, (2.57) can be written as

$$u(X,Y) \le \varphi^{-1}\biggl(W_m^{-1}\biggl(W_n\bigl(r_m(X,Y)\bigr) + \int_{b_m(x_0)}^{b_m(X)}\int_{c_m(y_0)}^{c_m(Y)} \tilde{f}_m(X,Y,s,t)\,dt\,ds\biggr)\biggr). \tag{2.58}$$

Since X, Y are arbitrary, replacing Y and X with y and x, respectively, we have

$$u(x,y) \le \varphi^{-1}\biggl(W_m^{-1}\biggl(W_n\bigl(r_m(x,y)\bigr) + \int_{b_m(x_0)}^{b_m(x)}\int_{c_m(y_0)}^{c_m(y)} \tilde{f}_m(x,y,s,t)\,dt\,ds\biggr)\biggr) \tag{2.59}$$

for all $(x,y) \in [x_0, X_1^*] \times [y_0, Y_1^*]$.

This completes the proof. □

Theorem 2.2 *We make the following assumptions*:

- (S_1) *$c(x,y) \in C(\Delta,\mathbb{R}_+)$ and $b_j \in C^1([x_0,x_1),\mathbb{R}_+)$, and $c_j \in C^1([y_0,y_1),[y_0,y_1))$ are nondecreasing with $b_j(x) \le x$ on $[x_0,x_1)$ and $c_j(y) \le y$ on $[y_0,y_1)$, and $c_j(y_0) = y_0$ for $j = 1,\ldots,m$;*
- (S_2) *$\hat{\psi} \in C(\Xi,\mathbb{R}_+)$, $\hat{g}_j \in C(\Delta \times [b_*(x_0),x_1) \times [y_0,y_1),\mathbb{R}_+)$ ($j = 1,2,\ldots,m$);*
- (S_3) *$\phi_j, \hat{\phi}_j \in C(\mathbb{R}_+,\mathbb{R}_+)$ ($j = 1,\ldots,m$) are all nondecreasing with $\{\phi_j,\hat{\phi}_j\}(t) > 0$ for $t > 0$, and $\phi_j\hat{\phi}_j \propto \phi_{j+1}\hat{\phi}_{j+1}$ ($j = 1,\ldots,m-1$);*
- (S_4) *$k \ge 1$, $\alpha_j, \bar{\alpha}_j \in (0,1]$, $\beta_j, \bar{\beta}_j \in (0,1)$, $\gamma_j > 1 - \frac{1}{p}$, $\bar{\gamma}_j > 1 - \frac{1}{p}$ such that $\frac{1}{p} + \alpha_j(\beta_j - 1) + \gamma_j - 1 \ge 0$, $\frac{1}{p} + \bar{\alpha}_j(\bar{\beta}_j - 1) + \bar{\gamma}_j - 1 \ge 0$, $p(\beta_j - 1) + 1 > 0$, $p(\bar{\beta}_j - 1) + 1 > 0$, $p > 1$ for all $j = 1,\ldots,m$.*

If $u \in C([b_*(x_0) - h, x_1) \times [y_0,y_1),\mathbb{R}_+)$ satisfies the integral inequality

$$\begin{aligned} u^k(x,y) \le{}& c(x,y) + \sum_{j=1}^{M}\int_{b_j(x_0)}^{b_j(x)}\int_{c_j(y_0)}^{c_j(y)} \bigl(x^{\alpha_j} - s^{\alpha_j}\bigr)^{\beta_j - 1} s^{\gamma_j - 1}\bigl(y^{\bar{\alpha}_j} - t^{\bar{\alpha}_j}\bigr)^{\bar{\beta}_j - 1} \\ &\times t^{\bar{\gamma}_j - 1}\hat{g}_j(x,y,s,t)\phi_j\bigl(u(s,t)\bigr)\hat{\phi}_j\Bigl(\max_{\tilde{\eta}\in[s-h,s]} g\bigl(u(\tilde{\eta},t)\bigr)\Bigr) \\ &+ \sum_{j=M+1}^{m}\int_{b_j(x_0)}^{b_j(x)}\int_{c_j(y_0)}^{c_j(y)} \hat{g}_j(x,y,s,t)\phi_j\bigl(u(s,t)\bigr)\hat{\phi}_j\Bigl(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Bigr), \\ &(x,y) \in [x_0,x_1) \times [y_0,y_1), \\ u(x,y) \le{}& \hat{\psi}(x,y), \quad (x,y) \in \bigl[b_*(x_0) - h, x_0\bigr] \times [y_0,y_1), \end{aligned} \tag{2.60}$$

where $\max_{s\in[b_*(x_0)-h,x_0]} \hat{\psi}(s,y) \le ((1+m)^{1-1/q}c(x_0,y))^{1/k}$ for all $y \in [y_0,y_1)$.

Then

$$u(x,y) \le \left(G_m^{-1}\bigl(G_m\bigl(e_m(x,y)\bigr)\bigr) + \int_{b_m(x_0)}^{b_m(x)} \int_{c_m(y_0)}^{c_m(y)} \tilde{g}_m(x,y,s,t)\,dt\,ds \right)^{1/(kq)} \tag{2.61}$$

for all $(x,y) \in [x_0,X_2) \times [y_0,Y_2)$, *where* G_j^{-1}*is the inverse of the function*

$$G_j(u) := \int_{t_j}^{t} \frac{ds}{\phi_j^q(s^{1/(kq)})\hat{\phi}_j^q(s^{1/(kq)})}, \quad t \ge t_j > 0, j = 1,\ldots,m. \tag{2.62}$$

In (2.61) *and* (2.62), $t_j > 0$ *is a given constant*, $\frac{1}{p} + \frac{1}{q} = 1$, $e_j(x,y)$ *is defined recursively by*

$$\begin{aligned} & e_1(x,y) = (1+m)^{q-1}\Bigl(\max_{(\iota,\xi)\in[x_0,x]\times[y_0,y]} c(\iota,\xi)\Bigr)^q, \quad \text{and} \\ & e_{j+1}(x,y) := G_j^{-1}\left[G_j\bigl(e_j(x,y)\bigr) + \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \tilde{g}_j(x,y,s,t)\,dt\,ds\right], \\ & \quad j = 1,\ldots,m-1, \end{aligned} \tag{2.63}$$

$$\begin{aligned} & \tilde{g}_j(x,y,s,t) := (1+m)^{q-1}\bigl(M_j x^{\theta_j}\bar{M}_j y^{\bar{\theta}_j}\bigr)^{q/p}\Bigl(\max_{(\iota,\xi)\in[x_0,x]\times[y_0,y]} \hat{g}_j(\iota,\xi,s,t)\Bigr)^q, \\ & \quad (x,y) \in [x_0,x_1) \times [y_0,y_1), \end{aligned} \tag{2.64}$$

$$\begin{aligned} & M_j = \alpha_j^{-1} B\left(\frac{p(\gamma_j-1)+1}{\alpha_j}, p(\beta_j-1)+1\right), \\ & \bar{M}_j = \bar{\alpha}_j^{-1} B\left(\frac{p(\bar{\gamma}_j-1)+1}{\bar{\alpha}_j}, p(\beta_j-1)+1\right), \\ & \theta_j = p\bigl(\alpha_j(\beta_j-1)+\gamma_j-1\bigr)+1, \\ & \bar{\theta}_j = p\bigl(\bar{\alpha}_j(\bar{\beta}_j-1)+\bar{\gamma}_j-1\bigr)+1, \quad j = 1,\ldots,M \\ & M_j = \bar{M}_j = 1, \qquad \theta_j = \bar{\theta}_j = 1, \quad j = M+1,\ldots,m, \end{aligned} \tag{2.65}$$

$X_2 \in [x_0,x_1)$, $Y_2 \in [y_0,y_1)$ *are chosen such that*

$$\begin{aligned} & G_j\bigl(r_j(X_2,Y_2)\bigr) + \int_{b_j(x_0)}^{b_j(X_1)} \int_{c_j(y_0)}^{c_j(Y_2)} \tilde{g}_j(X_2,Y_2,s,t)\,dt\,ds \\ & \quad \le \int_{t_j}^{\infty} \frac{ds}{\phi_j^q(s^{1/q})\hat{\phi}_j^q(s^{1/q})} \end{aligned} \tag{2.66}$$

for $j = 1,2,\ldots,m$.

Proof Let

$$\begin{aligned} & \hat{c}(x,y) := \max_{(\tau,\xi)\in[x_0,x]\times[y_0,y]} \bigl\{a(\tau,\xi)\bigr\}, \quad (x,y) \in [x_0,x_1) \times [y_0,y_1). \\ & \bar{g}_j(x,y,s,t) := \max_{(\iota,\xi)\in[x_0,x]\times[y_0,y]} g_j(\iota,\xi,s,t), \end{aligned} \tag{2.67}$$

which are increasing in x and y and satisfy $\bar{g}_j(x,y,s,t) \geq g_j(x,y,s,t) \geq 0$ for $j = 1, 2, \dots, m$. From (2.60), (2.67), and the definition of $\tilde{g}_j$, we obtain

$$\begin{aligned} u^k(x,y) \leq{}& \hat{c}(x,y) + \sum_{j=1}^{M} \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \left(x^{\alpha_j} - s^{\alpha_j}\right)^{\beta_j - 1} s^{\gamma_j - 1} \left(y^{\bar{\alpha}_j} - t^{\bar{\alpha}_j}\right)^{\bar{\beta}_j - 1} \\ &\times t^{\bar{\gamma}_j - 1} \bar{g}_j(x,y,s,t) \phi_j\big(u(s,t)\big) \hat{\phi}_j\Big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Big) \\ &+ \sum_{j=M+1}^{m} \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \bar{g}_j(x,y,s,t) \phi_j\big(u(s,t)\big) \hat{\phi}_j\Big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Big), \\ &(x,y) \in [x_0, x_1) \times [y_0, y_1), \\ u(x,y) \leq{}& \hat{\psi}(x,y), \quad (x,y) \in \big[b_*(x_0) - h, x_0\big] \times [y_0, y_1). \end{aligned} \tag{2.68}$$

Let $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then $q > 0$. By Lemma 1, Hölder's inequality, (S_4), and (2.68), we obtain, for all $(x,y) \in \Delta$,

$$\begin{aligned} &u^k(x,y) \\ &\leq \hat{c}(x,y) + \sum_{j=1}^{M} \left(\int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \left(x^{\alpha_j} - s^{\alpha_j}\right)^{p(\beta_j - 1)} s^{p(\gamma_j - 1)} \left(y^{\bar{\alpha}_j} - t^{\bar{\alpha}_j}\right)^{p(\bar{\beta}_j - 1)} t^{(\bar{\gamma}_j - 1)} \,dt\,ds \right)^{1/p} \\ &\quad\times \left(\int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \bar{g}_j^q(x,y,s,t) \phi_j^q\big(u(s,t)\big) \hat{\phi}_j^q\Big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Big) \,dt\,ds \right)^{1/q} \\ &\quad+ \sum_{j=M+1}^{m} \left(\int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} 1^p \,dt\,ds \right)^{1/p} \left(\int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \bar{g}_j^q(x,y,s,t) \phi_j^q\big(u(s,t)\big) \right. \\ &\quad\left. \times \hat{\phi}_j^q\Big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Big) \,dt\,ds \right)^{1/q} \\ &\leq \hat{c}(x,y) + \sum_{j=1}^{M} \left(\int_{b_j(0)}^{x} \int_{0}^{y} \left(x^{\alpha_j} - s^{\alpha_j}\right)^{p(\beta_j - 1)} s^{p(\gamma_j - 1)} \left(y^{\bar{\alpha}_j} - t^{\bar{\alpha}_j}\right)^{p(\bar{\beta}_j - 1)} t^{(\bar{\gamma}_j - 1)} \,dt\,ds \right)^{1/p} \\ &\quad\times \left(\int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \bar{g}_j^q(x,y,s,t) \phi_j^q\big(u(s,t)\big) \hat{\phi}_j^q\Big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Big) \,dt\,ds \right)^{1/q} \\ &\quad+ \sum_{j=M+1}^{m} \left(\int_{0}^{x} \int_{0)}^{y} 1^p \,dt\,ds \right)^{1/p} \left(\int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \bar{g}_j^q(x,y,s,t) \phi_j^q\big(u(s,t)\big) \right. \\ &\quad\left. \times \hat{\phi}_j^q\Big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Big) \,dt\,ds \right)^{1/q} \\ &\leq \hat{c}(x,y) + \sum_{j=1}^{m} \left(M_j x^{\theta_j} \bar{M}_j y^{\bar{\theta}_j}\right)^{1/p} \left(\int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \bar{g}_j^q(x,y,s,t) \right. \\ &\quad\left. \times \phi_j^q\big(u(s,t)\big) \hat{\phi}_j^q\Big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Big) \,dt\,ds \right)^{1/q}, \end{aligned} \tag{2.69}$$

where $0 \leq b_j(t) \leq t$, $0 \leq c_j(t) \leq t$, M_j, $\bar{M}_j$, θ_j, and $\bar{\theta}_j$ are given by (2.65) for $j = 1, \dots, m$.

By Jensen's inequality and (2.69), we get, for all $(x,y) \in \Delta$,

$$u^{kq}(x,y) \le (1+m)^{q-1}\big(\hat{c}^q(x,y) + \sum_{j=1}^{m} \big(M_j x^{\theta_j} \bar{M}_j y^{\bar{\theta}_j}\big)^{q/p} \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \bar{g}_j^q(x,y,s,t) \\ \times \phi_j^q\big(u(s,t)\big)\hat{\phi}_j^q\big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\big)\,dt\,ds. \tag{2.70}$$

By the definition of e_1 and $\tilde{g}_j$, from (2.70) we obtain

$$u^{kq}(x,y) \le e_1(x,y) + \sum_{j=1}^{m} \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \tilde{g}_j(x,y,s,t) \\ \times \phi_j^q\big(u(s,t)\big)\hat{\phi}_j^q\big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\big)\,dt\,ds, \quad (x,y)\in\Delta. \tag{2.71}$$

Concerning (2.71), we consider the auxiliary inequalities

$$u^{kq}(x,y) \le e_1(X,Y) + \sum_{j=1}^{m} \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \tilde{g}_j(X,Y,s,t) \\ \times \phi_j^q\big(u(s,t)\big)\hat{\phi}_j^q\big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\big)\,dt\,ds, \quad (x,y)\in[x_0,X]\times[y_0,Y], \tag{2.72}$$

$$u(x,y) \le \hat{\psi}(x,y), \quad (x,y)\in\big[b_*(x_0)-h,x_0\big]\times[y_0,Y],$$

where $x_0 \le X \le X_2$ and $y_0 \le Y \le Y_2$ are chosen arbitrarily.

Since $\max_{s\in[b_*(x_0)-h,x_0]} \hat{\psi}(s,y) \le ((1+m)^{1-1/q}a(x_0,y))^{\frac{1}{k}}$ for $y\in[y_0,y_1)$, $a(x_0,y)\le\hat{c}(x_0,y)$, we have $\max_{s\in[b_*(x_0)-h,x_0]} \psi(s,y) \le (e_1^{1/q}(X,Y))^{\frac{1}{k}}$, $y\in[y_0,Y]$. Define a function $z(x,y):[b_*(x_0)-h,X)\times[y_0,Y)\to\mathbb{R}_+$ by

$$z(x,y) = \begin{cases} e_1(X,Y) + \sum_{j=1}^{m} \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \tilde{g}_j(X,Y,s,t) \\ \quad \times \phi_j^q(u(s,t))\hat{\phi}_j^q\big(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\big)\,dt\,ds, & (x,y)\in[x_0,X]\times[y_0,Y], \\ e_1(X,Y), \quad (x,y)\in[b_*(x_0)-h,x_0]\times[y_0,Y]. \end{cases}$$

Clearly, $z(x,y)$ is increasing in x. From (2.72) and the definition of z, we have

$$u(x,y) \le z^{1/(kq)}(x,y), \quad (x,y)\in\big[b_*(x_0)-h,X\big]\times[y_0,Y]. \tag{2.73}$$

Then, noting that z is increasing, from (2.51) we get for $(s,y)\in[b_*(x_0),X]\times[y_0,Y]$

$$\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},y) \le \max_{\tilde{\eta}\in[s-h,s]} z^{1/(kq)}(\tilde{\eta},y) \le \big(\max_{\tilde{\eta}\in[s-h,s]} z(\tilde{\eta},y)\big)^{1/(kq)}. \tag{2.74}$$

From (2.42), (2.73), (2.74), and the definition of z, we have

$$z(x,y) \le e_1(X,Y) + \sum_{j=1}^{m} \int_{b_j(x_0)}^{b_j(x)} \int_{b_j(y_0)}^{b_j(y)} \tilde{g}_j(X,Y,s,t)\phi_j^q\big(z^{1/(kq)}(s,t)\big) \\ \times \hat{\phi}_j^q\Big(\max_{\tilde{\eta}\in[s-h,s]} \big(z^{1/(kq)}(\tilde{\eta},t)\big)\Big)\,dt\,ds$$

$$\begin{aligned} &\le e_1(X,Y) + \sum_{j=1}^{m} \int_{b_j(x_0)}^{b_j(x)} \int_{b_j(y_0)}^{b_j(y)} \tilde{g}_j(X,Y,s,t)\phi_j^q\bigl(z^{1/(kq)}(s,t)\bigr) \\ &\quad \times \hat{\phi}_j^q\Bigl(\bigl(\max_{\tilde{\eta}\in[s-h,s]} z(\tilde{\eta},t)\bigr)^{1/(kq)}\Bigr)\,dt\,ds, \quad (x,y)\in[x_0,X]\times[y_0,Y], \\ & z(x,y)\le e_1(X,Y), \quad (x,y)\in\bigl[b_*(x_0)-h,x_0\bigr]\times[y_0,Y]. \end{aligned} \tag{2.75}$$

Let $\nu(t) := t^{1/(kq)}$, which is a continuous and increasing function on $\mathbb{R}_+$. Thus $\phi_j^q(\nu(t))$ and $\hat{\phi}_j^q(\nu(t))$ $(j=1,\dots,m)$ are continuous and increasing on $\mathbb{R}_+$ and positive on $(0,\infty)$. Moreover, since $\phi_j\hat{\phi}_j \propto \phi_{j+1}\hat{\phi}_{j+1}$, we have $(\phi_{j+1}\circ\nu)^q(\hat{\phi}_{j+1}\circ\nu)^q \propto (\phi_j\circ\nu)^q(\hat{\phi}_j\circ\nu)^q$ $(j=1,\dots,m-1)$. Taking $g_j(x,y,s,t)=\tilde{g}_j(X,Y,s,t)$ and $h_j(t)=\phi_j^q(\nu(t))$, $\bar{h}_j(t)=\hat{\phi}_j^q(\nu(t))$, $j=1,2,\dots,m$, in Lemma 2 and (2.75),we obtain

$$\begin{aligned} z(x,y) &\le G_m^{-1}\Bigl(G_m\bigl(\tilde{e}_m(X,Y,x,y)\bigr) \\ &\quad + \int_{b_m(x_0)}^{b_m(x)} \int_{c_m(y_0)}^{c_m(y)} \tilde{g}_m(X,Y,s,t)\,dt\,ds\Bigr) \end{aligned} \tag{2.76}$$

for $x_0\le x\le \min\{X,X_2^*\}$, $y_0\le y\le\min\{Y,Y_2^*\}$, where $\tilde{e}_j(X,Y,x,y)$ is defined inductively by $\tilde{e}_1(X,Y,x,y) := e_1(X,Y)$ and

$$\tilde{e}_j(X,Y,x,y) := G_{j-1}^{-1}\Bigl(G_{j-1}\bigl(\tilde{e}_{j-1}(X,Y,x,y)\bigr) + \int_{b_{j-1}(x_0)}^{b_{j-1}(x)} \int_{c_{j-1}(y_0)}^{c_{j-1}(y)} \tilde{g}_{j-1}(X,Y,s,t)\,dt\,ds\Bigr)$$

for $j=2,\dots,m$, and X_2^*, Y_2^* are chosen such that

$$\begin{aligned} & G_j\bigl(\tilde{e}_j(X,Y,\bar{X}_1,\bar{Y}_1)\bigr) + \int_{b_j(x_0)}^{b_j(X_2)} \int_{c_j(y_0)}^{c_j(\bar{Y}_1)} \tilde{g}_j(X,Y,s,t) \\ &\quad \le \int_{t_j}^{\infty} \frac{ds}{\tilde{\omega}_j^q(\varphi^{-1}(s^{1/q}))} \end{aligned} \tag{2.77}$$

for $j=1,\dots,m$.

Note that $X_2^* = X_2$ and $Y_2^* = Y_2$. It follows from (2.73) and (2.76) that

$$\begin{aligned} u(x,y) &\le \Bigl(G_m^{-1}\Bigl(G_n\bigl(\tilde{g}_m(X,Y,x,y)\bigr) \\ &\quad + \int_{b_m(x_0)}^{b_m(x)} \int_{c_m(y_0)}^{c_m(y)} \tilde{g}_m(X,Y,s,t)\,dt\,ds\Bigr)\Bigr)^{1/(kq)} \end{aligned} \tag{2.78}$$

for $x_0\le x\le\min\{X,X_2^*\}$, $y_0\le y\le\min\{Y,Y_2^*\}$.

Taking $x=X$, $y=Y$ in (2.56), we have

$$\begin{aligned} u(X,Y) &\le \Bigl(G_m^{-1}\Bigl(G_m\bigl(\tilde{e}_m(X,Y,X,Y)\bigr) \\ &\quad + \int_{b_m(x_0)}^{b_m(X)} \int_{c_m(y_0)}^{c_m(Y)} \tilde{g}_m(X,Y,s,t)\,dt\,ds\Bigr)\Bigr)^{1/(kq)} \end{aligned} \tag{2.79}$$

for $x_0 \leq X \leq X_2$, $y_0 \leq Y \leq Y_2$. It is easy to verify $\tilde{e}_m(X,Y,X,Y) = e_m(X,Y)$. Thus, (2.57) can be written as

$$u(X,Y) \leq \biggl(G_m^{-1}\biggl(G_n\bigl(r_m(X,Y)\bigr) + \int_{b_m(x_0)}^{b_m(X)} \int_{c_m(y_0)}^{c_m(Y)} \tilde{g}_m(X,Y,s,t)\,dt\,ds \biggr)\biggr)^{1/(kq)}. \tag{2.80}$$

Since X, Y are arbitrary, replacing X and Y with x and y, respectively, we get

$$u(x,y) \leq \biggl(G_m^{-1}\biggl(G_n\bigl(e_m(x,y)\bigr) + \int_{b_m(x_0)}^{b_m(x)} \int_{c_m(y_0)}^{c_m(y)} \tilde{g}_m(x,y,s,t)\,dt\,ds \biggr)\biggr)^{1/(kq)} \tag{2.81}$$

for all $(x,y) \in [x_0, X_2] \times [y_0, Y_2]$. This completes the proof. □

Corollary 2.3 *Let the following conditions be fulfilled*:

(B_1) *all* $b_j \in C^1([x_0,x_1),\mathbb{R}_+)$ *and* $c_j \in C^1([y_0,y_1),[y_0,y_1))$ *are nondecreasing with* $b_j(x) \leq x$ *on* $[x_0,x_1)$, $c_j(y) \leq y$ *on* $[y_0,y_1)$, *and* $c_j(y_0) = y_0$ *for all* $j = 1,\ldots,m$;

(B_2) $a \in C(\Delta,\mathbb{R}_+)$ *and* $\hat{\psi} \in C(\Xi,\mathbb{R}_+)$, $\varphi_1 \in C(\mathbb{R}_+,\mathbb{R}_+)$, *and* φ_1 *is strictly increasing such that* $\lim_{t\to\infty}\varphi(t) = \infty$,*and* $f_j \in C(\Delta \times [b_*(x_0),x_1) \times [y_0,y_1),\mathbb{R}_+)$ *for all* $j = 1,\ldots,m$;

(B_3) *all* ψ_j $(j = 1,\ldots,m)$ *are continuous and increasing functions on* $\mathbb{R}_+$ *and positive on* $(0,+\infty)$ *such that* $\psi_1 \propto \psi_2 \propto \ldots \propto \psi_m$;

(B_4) $\alpha_j, \bar{\alpha}_j \in (0,1]$, $\beta_j, \bar{\beta}_j \in (0,1)$, $\gamma_j > 1 - \frac{1}{p}$, $\bar{\gamma}_j > 1 - \frac{1}{p}$ *such that* $\frac{1}{p} + \alpha_j(\beta_j - 1) + \gamma_j - 1 \geq 0$, $\frac{1}{p} + \bar{\alpha}_j(\bar{\beta}_j - 1) + \bar{\gamma}_j - 1 \geq 0$, $p(\beta_j - 1) + 1 > 0$, $p(\bar{\beta}_j - 1) + 1 > 0$, $p > 1$, $j = 1,2,\ldots,m$;

(B_5) $u \in C([b_*(x_0) - h, x_1) \times [y_0,y_1),\mathbb{R}_+)$ *satisfies the integral inequality*

$$\begin{aligned} \varphi_1\bigl(u(x,y)\bigr) \leq{}& a(x,y) + \sum_{j=1}^{M} \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \bigl(x^{\alpha_j} - s^{\alpha_j}\bigr)^{\beta_j - 1} s^{\gamma_j - 1} \bigl(y^{\bar{\alpha}_j} - t^{\bar{\alpha}_j}\bigr)^{\bar{\beta}_j - 1} \\ &\times t^{\bar{\gamma}_j - 1} f_j(x,y,s,t)\psi_j\bigl(u(s,t)\bigr)\,dt\,ds \\ &+ \sum_{j=M+1}^{m} \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \bigl(x^{\alpha_j} - s^{\alpha_j}\bigr)^{\beta_j - 1} s^{\gamma_j - 1} \bigl(y^{\bar{\alpha}_j} - t^{\bar{\alpha}_j}\bigr)^{\bar{\beta}_j - 1} \\ &\times t^{\bar{\gamma}_j - 1} f_j(x,y,s,t)\psi_j\Bigl(\max_{\tilde{\eta}\in[s-h,s]} u(\tilde{\eta},t)\Bigr)\,dt\,ds, \\ &(x,y) \in [x_0,x_1) \times [y_0,y_1), \\ u(x,y) \leq{}& \hat{\psi}(x,y), \quad (x,y) \in \bigl[b_*(x_0) - h, x_0\bigr] \times [y_0,y_1), \end{aligned} \tag{2.82}$$

where $\max_{s\in[b_*(x_0)-h,x_0]} \hat{\psi}(s,y) \leq \varphi_1^{-1}((1+m)^{1-1/q} a(x_0,y))$ *for all* $y \in [y_0,y_1)$.

Then

$$u(x,y) \leq \varphi_1^{-1}\biggl(\check{G}_m^{-1}\biggl(\check{G}_m\bigl(r_m(x,y)\bigr) + \int_{b_m(x_0)}^{b_m(x)} \int_{c_m(y_0)}^{c_m(y)} \tilde{f}_m(x,y,s,t)\,dt\,ds \biggr)^{1/q}\biggr) \tag{2.83}$$

for all $(x,y) \in [x_0, X_2) \times [y_0, Y_2)$, *where* G_j^{-1} *is the inverse of the function*

$$\check{G}_j(t) := \int_{t_j}^{t} \frac{ds}{\psi_j^q(\varphi_1^{-1}(s^{1/q}))}, \quad t \ge t_j > 0, j = 1, 2, \ldots, m, \tag{2.84}$$

t_j *is a given constant*, $r_j(x,y)$ *is defined recursively by*

$$\begin{aligned} & r_1(x,y) = (1+m)^{q-1}\Bigl(\max_{(\iota,\xi)\in[x_0,x]\times[y_0,y]} a(\iota,\xi)\Bigr)^q, \quad \textit{and} \\ & r_{j+1}(x,y) := \check{G}_j^{-1}\biggl[\check{G}_j\bigl(r_j(x,y)\bigr) + \int_{b_j(x_0)}^{b_j(x)} \int_{c_j(y_0)}^{c_j(y)} \tilde{f}_j(x,y,s,t)\,dt\,ds\biggr], \\ & \quad j = 1, \ldots, m-1, \end{aligned} \tag{2.85}$$

$$\begin{aligned} & \tilde{f}_j(x,y,s,t) := (1+m)^{q-1}\bigl(M_j x^{\theta_j} \bar{M}_j y^{\bar{\theta}_j}\bigr)^{q/p}\Bigl(\max_{(\iota,\xi)\in[x_0,x]\times[y_0,y]} \check{f}_j(\iota,\xi,s,t)\Bigr)^q, \\ & \quad (x,y) \in [x_0, x_1) \times [y_0, y_1), \end{aligned} \tag{2.86}$$

$M_j := \alpha_j^{-1} B(\frac{p(\gamma_j-1)+1}{\alpha_j}, p(\beta_j-1)+1)$, $\bar{M}_j := \bar{\alpha}_j^{-1} B(\frac{p(\bar{\gamma}_j-1)+1}{\bar{\alpha}_j}, p(\bar{\beta}_j-1)+1)$, $\theta_j := p(\alpha_j(\beta_j-1)+\gamma_j-1)+1$, $\bar{\theta}_j := p(\bar{\alpha}_j(\bar{\beta}_j-1)+\bar{\gamma}_j-1)+1$, $\frac{1}{p}+\frac{1}{q}=1$, $X_2 \in [x_0, x_1)$, $Y_2 \in [y_0, y_1)$ *are chosen such that*

$$\check{G}_j\bigl(r_j(X_2,Y_2)\bigr) + \int_{b_j(x_0)}^{b_j(X_1)} \int_{c_j(y_0)}^{c_{i-1}(Y_2)} \tilde{f}_j(X_2,Y_2,s,t)\,dt\,ds \le \int_{t_j}^{\infty} \frac{ds}{\tilde{\omega}_j^q(\varphi^{-1}(s^{1/q}))} \tag{2.87}$$

for $j = 1, 2, \ldots, m$.

Proof Applying Theorem 2.1 to specified $\omega_j(u) \equiv \psi_j(u)$ $(j = 1, \ldots, M)$, $\mu_j(u) \equiv 1$ $(j = 1, \ldots, M)$, $\omega_j(u) \equiv 1$ $(j = M+1, \ldots, m)$, $\mu_j(u) \equiv \psi_j(u)$ $(j = M+1, \ldots, m)$, $f_j(x,y,s,t) = \check{f}_j(x,y,s,t)$, $g(t) = t$, from (2.82) we obtain estimate (2.83). The proof is complete. □

3 Applications

Consider a nonlinear weakly singular integral equation with maxima

$$\begin{cases} z(x,y) = a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\theta_1-1}s^{\gamma_1-1}(y-t)^{\theta_2-1}t^{\gamma_2-1} \\ \qquad \times F(x,y,s,t,z(s,t),\max_{\tilde{\eta}\in[s-h,s]} z(\tilde{\eta},t))\,ds\,dt, \quad (x,y) \in \Delta, \\ z(x,y) = \psi(x,y), \quad (x,y) \in [x_0-h, x_0] \times [y_0, y_1), \end{cases} \tag{3.1}$$

where $F \in C(\Delta \times \mathbb{R}^4, \mathbb{R})$, h is a positive constant, $\psi \in C([x_0-h, x_0] \times [y_0, y_1), \mathbb{R})$, $a \in C(\Delta, \mathbb{R})$, $\theta_j \in (0,1)$, and $p(\gamma_j-1)+1 > 0$ such that $\frac{1}{p}+\theta_j+\gamma_j-2 \ge 0$ and $p(\theta_j-1)+1 > 0$, $p > 1$, $j = 1, 2$.

The following result gives an estimate for its solutions.

Corollary 3.1 *Suppose that functions* F *in* (3.1) *satisfy*

$$\bigl|F(x,y,s,t,u,v)\bigr| \le h_1(x,y,s,t)\mu_1\bigl(|u|\bigr) + h_2(x,y,s,t)\mu_2\bigl(|v|\bigr), \tag{3.2}$$

where $h_j \in C([x_0, x_1) \times [y_0, y_1) \times \mathbb{R}^2, \mathbb{R}_+)$, *and* $h_j(x,y,s,t)$ *is nondecreasing in* x *and* y *for each fixed* s *and* t, *and* $\mu_j \in C(\mathbb{R}_+, (0,\infty))$ $(j = 1, 2)$ *such that* $\mu_1 \propto \mu_2$, $\max_{s\in[x_0-h,x_0]} \psi(s,y) \le 3^{1-1/q}|a(x_0,y)|$ *for all* $y \in [y_0, y_1)$.

Then any solution $z(x,y)$ of (3.1) has the estimate

$$\begin{aligned} &\left|z(x,y)\right| \\ &\quad\le \left[Q_2^{-1}\left(Q_2\bigl(\gamma(x,y)\bigr) + 3^{q-1}\bigl(M_1x^{\delta_1}M_2y^{\delta_2}\bigr)^{q/p}\int_{x_0}^{x}\int_{y_0}^{y}h_2(x,y,s,t)dt\,ds\right)\right]^{1/q} \end{aligned} \tag{3.3}$$

for all $(x,y)\in[x_0,X_1)\times[y_0,Y_1)$, where

$$\begin{aligned} &\gamma(x,y) := Q_1^{-1}\left(Q_1\bigl(\eta_1(x,y)\bigr) + 3^{q-1}\bigl(M_1x^{\delta_1}M_2y^{\delta_2}\bigr)^{q/p}\int_{x_0}^{x}\int_{y_0}^{y}h_1^q(x,y,s,t)\,dt\,ds\right), \\ &\eta_1(x,y) := 3^{q-1}\Bigl(\max_{(s,t)\in[x_0,x]\times[y_0,y]}\left|a(s,t)\right|\Bigr)^q, \qquad Q_1(u) := \int_{u_1}^{u}\frac{ds}{\mu_1^q(s^{\frac{1}{q}})}, \quad u\ge u_1>0, \\ &Q_2(u) := \int_{u_1}^{u}\frac{ds}{\mu_2^q(s^{\frac{1}{q}})}, \quad u\ge u_2>0, \end{aligned}$$

$M_j := B(p(\gamma_j-1)+1, p(\theta_j-1)+1)$ $(j=1,2)$, $\delta_j := p(\theta_j+\gamma_j-2)+1$, $j=1,2$, $\frac{1}{p}+\frac{1}{q}=1$, *and constants u_1, u_2 are given arbitrarily, $X_1\in[x_0,x_1)$, $Y_1\in[y_0,y_1)$ are chosen such that*

$$\begin{aligned} &Q_1\bigl(\gamma_1(X_1,Y_1)\bigr) + 3^{q-1}\bigl(M_1X_1^{\delta_1}M_2Y_1^{\delta_2}\bigr)^{q/p}\int_{x_0}^{X_1}\int_{y_0}^{Y_1}h_1^q(X_1,Y_1,s,t)\,dt\,ds \le \int_{u_1}^{\infty}\frac{ds}{\mu_1^q(s^{\frac{1}{q}})}, \\ &Q_2\bigl(\gamma_2(X_1,Y_1)\bigr) + 3^{q-1}\bigl(M_1X_1^{\delta_1}M_2Y_1^{\delta_2}\bigr)^{q/p}\int_{x_0}^{x}\int_{y_0}^{y}h_2^q(X_1,Y_1,s,t)\,dt\,ds \le \int_{u_2}^{\infty}\frac{ds}{\mu_2^q(s^{\frac{1}{q}})}. \end{aligned}$$

Proof From (3.1) we obtain

$$\begin{aligned} \left|z(x,y)\right| &\le \left|a(x,y)\right| + \int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\theta_1-1}s^{\gamma_1-1}(y-t)^{\theta_2-1}t^{\gamma_2-1} \\ &\quad\cdot\left|F\Bigl(x,y,s,t,z(s,t),\max_{\tilde\eta\in[s-h,s]}z(\tilde\eta,t)\Bigr)\right|dt\,ds \\ &\le \left|a(x,y)\right| + \int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\theta_1-1}s^{\gamma_1-1}(y-t)^{\theta_2-1}t^{\gamma_2-1} \\ &\quad\cdot h_1(x,y,s,t)\mu_1\bigl(\left|z(s,t)\right|\bigr)\,dt\,ds \\ &\quad+\int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\theta_1-1}s^{\gamma_1-1}(y-t)^{\theta_2-1}t^{\gamma_2-1}h_2(x,y,s,t) \\ &\quad\cdot\mu_2\Bigl(\Bigl|\max_{\tilde\eta\in[s-h,s]}z(\tilde\eta,t)\Bigr)\Bigr|\Bigr)\,dt\,ds, \quad (x,y)\in\Delta, \\ \left|z(x,y)\right| &\le \left|\psi(x,y)\right|, \quad (x,y)\in[x_0-h,x_0]\times[y_0,y_1). \end{aligned} \tag{3.4}$$

Set $v(x,y)=|z(x,y)|$ for all $(x,y)\in[x_0-h,x_1)\times[y_0,y_1)$. From (3.4) we get

$$\begin{aligned} v(x,y) &\le \left|a(x,y)\right| + \int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\theta_1-1}s^{\gamma_1-1}(y-t)^{\theta_2-1}t^{\gamma_2-1} \\ &\quad\cdot h_1(x,y,s,t)\mu_1\bigl(v(s,t)\bigr)\,dt\,ds \end{aligned}$$

$$
\begin{aligned}
&+\int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\theta_1-1}s^{\gamma_1-1}(y-t)^{\theta_2-1}t^{\gamma_2-1}h_2(x,y,s,t) \\
&\cdot\mu_2\Bigl(\max_{\tilde\eta\in[s-h,s]}\bigl|v(\tilde\eta,t)\bigr|\Bigr)\,dt\,ds,\quad (x,y)\in\Delta, \\
v(x,y)&\le\bigl|\psi(x,y)\bigr|,\quad (x,y)\in[x_0-h,x_0]\times[y_0,y_1).
\end{aligned}
\tag{3.5}
$$

Applying Corollary 2.3 to the specified $M=1$, $m=2$, $\varphi_1(u)=u$, $f_j(x,y,s,t)=h_j(x,y,s,t)$, $b_j(t)=t$, $c_j(t)=t$, $\alpha_j=\bar\alpha_j=1$, $g(t)=t$, we obtain (3.3) from (3.5). □

Corollary 3.2 *Suppose that functions F and ψ in* (3.1) *satisfy*

$$
\bigl|F(x,y,s_1,t_1)-F(x,y,s_2,t_2)\bigr|\le h_1(x,y)|s_1-s_2|+h_2(x,y)|t_1-t_2|
\tag{3.6}
$$

for all $(x,y)\in\Delta$ and $s_j,t_j\in\mathbb{R}$ $(i=1,2)$, where $h_j\in C(\Delta,\mathbb{R}_+)$. Then system (3.1) *has at most one solution on Δ.*

Proof Assume that equation (3.1) has two solutions $u(x,y)$, $v(x,y)$. By the equivalent integral equation (3.1), we have

$$
\begin{aligned}
\bigl|u(x,y)-v(x,y)\bigr|\le{}&\int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\theta_1-1}s^{\gamma_1-1}(y-t)^{\theta_2-1}t^{\gamma_2-1}h_1(s,t)\bigl|u(s,t)-v(s,t)\bigr|\,dt\,ds \\
&+\int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\theta_1-1}s^{\gamma_1-1}(y-t)^{\theta_2-1}t^{\gamma_2-1}h_2(s,t) \\
&\cdot\Bigl|\max_{\tilde\eta\in[s-h,s]}u(\tilde\eta,t)-\max_{\tilde\eta\in[s-h,s]}v(\tilde\eta,t)\Bigr|\,dt\,ds
\end{aligned}
\tag{3.7}
$$

for all $(x,y)\in[x_0,x_1)\times[y_0,y_1)$. Since $u(x,y)$ is a continuous function, it implies that, for any fixed $t\in[y_0,y]$ and $s\in[x_0,x]$, there exists $\tau\in[s-h,s]$ such that $\max_{\tilde\eta\in[s-h,s]}u(\tilde\eta,t)=u(\tau,t)$ holds. Now we suppose $\max_{\tilde\eta\in[s-h,s]}u(\tilde\eta,t)\ge\max_{\tilde\eta\in[s-h,s]}v(\tilde\eta,t)$ and have

$$
\begin{aligned}
\Bigl|\max_{\tilde\eta\in[s-h,s]}u(\tilde\eta,t)-\max_{\tilde\eta\in[s-h,s]}v(\tilde\eta,t)\Bigr|&=\Bigl|u(\tau,t)-\max_{\tilde\eta\in[s-h,s]}v(\tilde\eta,t)\Bigr| \\
&\le\bigl|u(\tau,t)-v(\tau,t)\bigr|\le\max_{\tilde\eta\in[s-h,s]}\bigl|u(\tilde\eta,t)-v(\tilde\eta,t)\bigr|.
\end{aligned}
\tag{3.8}
$$

It follows from (3.7) and (3.8) that

$$
\begin{aligned}
\bigl|u(x,y)-v(x,y)\bigr|\le{}&\int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\theta_1-1}s^{\gamma_1-1}(y-t)^{\theta_2-1}t^{\gamma_2-1}h_1(s,t)\bigl|u(s,t)-v(s,t)\bigr|\,dt\,ds \\
&+\int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\theta_1-1}s^{\gamma_1-1}(y-t)^{\theta_2-1}t^{\gamma_2-1}h_2(s,t) \\
&\cdot\max_{\tilde\eta\in[s-h,s]}\bigl|u(\tilde\eta,t)-v(\tilde\eta,t)\bigr|\,dt\,ds.
\end{aligned}
\tag{3.9}
$$

Let

$$
\phi(x,y):=\bigl|u(x,y)-v(x,y)\bigr|,\quad (x,y)\in\bigl[\alpha(x_0)-h,x_0\bigr]\times[y_0,y_1).
$$

From (3.7) we obtain

$$\begin{aligned}\phi(x,y) \le{}& \int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\theta_1-1}s^{\gamma_1-1}(y-t)^{\theta_2-1}t^{\gamma_2-1}h_1(s,t)\phi(s,t)\,dt\,ds \\ &+ \int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\theta_1-1}s^{\gamma_1-1}(y-t)^{\theta_2-1}t^{\gamma_2-1}(x-s)^{\theta_1-1}s^{\gamma_1-1} \\ &\cdot (y-t)^{\theta_2-1}t^{\gamma_2-1}h_2(s,t)\max_{\tilde{\eta}\in[s-h,s]}\phi(\tilde{\eta},t)\,dt\,d\eta, \\ &(x,y)\in[x_0,x_1)\times[y_0,y_1), \\ \phi(x,y) \le{}& 0, \quad (x,y)\in[x_0-h,x_0]\times[y_0,y_1).\end{aligned} \tag{3.10}$$

Let $\varepsilon > 0$ be an arbitrary number. Then from (3.10) we have

$$\begin{aligned}\phi(x,y) \le{}& \varepsilon + \int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\theta_1-1}s^{\gamma_1-1}(y-t)^{\theta_2-1}t^{\gamma_2-1}h_1(s,t)\phi(s,t)\,dt\,ds \\ &+ \int_{x_0}^{x}\int_{y_0}^{y}(x-s)^{\theta_1-1}s^{\gamma_1-1}(y-t)^{\theta_2-1}t^{\gamma_2-1} \\ &\cdot h_2(s,t)\max_{\tilde{\eta}\in[\alpha(s)-h,\alpha(s)]}\phi(\tilde{\eta},t)\,dt\,d\eta, \\ &(x,y)\in[x_0,x_1)\times[y_0,y_1), \\ \phi(x,y) \le{}& 0, \quad (x,y)\in[x_0-h,x_0]\times[y_0,y_1).\end{aligned} \tag{3.11}$$

Applying Corollary 2.3 to specified $N = 1$, $m = 2$, $\varphi_1(u) = u$, $g(t) = t$, $b_j(t) = c_j(t) = t$, $f_j(x,y,s,t) = h_2(s,t)$, $j = 12$, $a(x,y) = \epsilon$, from (3.11) we obtain, for all $(x,y) \in \Delta$,

$$\begin{aligned}&\phi(x,y) \\ &\quad\le 3^{\frac{q-1}{q}}\varepsilon\exp\biggl(q^{-1}\biggl(3^{\frac{q-1}{q}}\bigl(M_1x^{\delta_1}\bar{M}_1y^{\delta_2}\bigr)^{\frac{q}{p}}\int_{x_0}^{x}\int_{y_0}^{y}\bigl(h_1^q(s,t)+h_2^q(s,t)\bigr)\,dt\,ds\biggr)\biggr),\end{aligned} \tag{3.12}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, M_j and δ_j $(j = 1,2)$ are defined as in Corollary 3.1. Letting $\varepsilon \to 0$, we obtain the uniqueness of the solution of equation (3.1). The uniqueness is proved. □

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