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Solutions to the nonlinear Schrödinger systems involving the fractional Laplacian

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Abstract

In this paper, we consider the following nonlinear Schrödinger system involving the fractional Laplacian operator:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u + au = f(v), \\ (-\Delta)^{\frac{\beta}{2}} v + bv = g(u), \end{cases} \quad \text{on } \Omega \subseteq \mathbb{R}^n,$$

where $a, b \geq 0$. When Ω is the unit ball or $\mathbb{R}^n$, we prove that the solutions (u, v) are radially symmetric and decreasing. When Ω is the parabolic domain on $\mathbb{R}^n$, we prove that the solutions (u, v) are increasing. Furthermore, if Ω is the $\mathbb{R}^n_+$, then we also derive the nonexistence of positive solutions to the system on the half-space. We assume that the nonlinear terms f, g and the solutions u, v satisfy some amenable conditions in different cases.

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Keywords: Fractional Laplacian; Nonlinear Schrödinger system; Moving plane method; Radial symmetry

1 Introduction

This paper is mainly devoted to investigating the properties of the solutions of the following system involving the fractional Laplacian operators:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u + au = f(v), \\ (-\Delta)^{\frac{\beta}{2}} v + bv = g(u), \end{cases} \quad \text{for some } a, b \geq 0, \tag{1.1}$$

with

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = C_{n,\alpha} \,\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+\alpha}} \, dy$$

and

$$(-\Delta)^{\frac{\beta}{2}} v(x) = C_{n,\beta} \,\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x-y|^{n+\beta}} \, dy,$$

where P. V. stands for the Cauchy principle value, $C_{n,\alpha}, C_{n,\beta} > 0$ and $0 < \alpha, \beta < 2$. To make sense for the integrals, we require $u \in C^{1,1}_{\mathrm{loc}} \cap L_\alpha$, $v \in C^{1,1}_{\mathrm{loc}} \cap L_\beta$, where

$$L_\alpha = \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \,\Big|\, \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}}\, dx < \infty \right\}$$

and

$$L_\beta = \left\{ v \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \,\Big|\, \int_{\mathbb{R}^n} \frac{|v(x)|}{1 + |x|^{n+\beta}}\, dx < \infty \right\}.$$

For more background on the fractional Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$, we refer to [1–4]. We mention that there are also several applications involving the fractional Laplacian in mathematical physics [5–8], finance [9], image processing [10], and so on.

Since the fractional Laplacian is nonlocal, that is, it does not act by pointwise differentiation but as a global integral with respect to a singular kernel, this is the main difficulty in studying problems involving it. To circumvent this difficulty, Caffarelli and Silvestre [11] introduced the *extension method* (CS extension) to overcome the difficulty of nonlocality. Their idea is to localize the fractional Laplacian by constructing a Dirichlet to Neumann operator of a degenerate elliptic equation. We can also use the *integral equation method*, the *method of moving planes in integral forms*, and *regularity lifting* to investigate equations involving the fractional Laplacian. Recently, Chen, Li, and Li [12] developed a new method that can handle directly these nonlocal operators. They used this property to develop some techniques needed in the direct method of moving planes in the whole space $\mathbb{R}^n$ and the upper half-space $\mathbb{R}^n_+$, such as the narrow region principle and decay at infinity. The direct method of moving planes is very useful, and a series of fruitful results have been obtained. For more articles concerning the method of moving planes for nonlocal equations and systems, mainly for integral equations, we refer to [13–21].

In this paper, following the ideas of [12], among others, we consider the properties of the solutions to system (1.1) for different domains Ω. More precisely, we get the following four theorems. Firstly, we consider the case where Ω is the unit ball. For simplicity, we denote $B = B_1(0)$. We have

Theorem 1.1 *Let $u \in C(\overline{B}) \cap C^{1,1}_{\mathrm{loc}}(B)$ and $v \in C(\overline{B}) \cap C^{1,1}_{\mathrm{loc}}(B)$ be positive solutions of the system*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u + au = f(v), & x \in B, \\ (-\Delta)^{\frac{\beta}{2}} v + bv = g(u), & x \in B, \quad \text{for some } a, b \geq 0. \\ u = v = 0, & x \notin B, \end{cases} \tag{1.2}$$

with $M > f'(\cdot), g'(\cdot) > 0$, where M is a positive constant. Then u is radially symmetric and decreasing about the origin.

Remark 1.1 Li [22] considered the similar problem with $f(v) = v^p$ and $g(u) = u^p$. So Theorem 1.1 can be regarded as an extension of the result in [22].

We denote by

$$\Omega = \{x = (x', x_n) \in \mathbb{R}^n \mid x_n > |x'|^2, x' = (x_1, x_2, \ldots, x_{n-1})\}$$

the parabolic domain on $\mathbb{R}^n$.

Theorem 1.2 *Let $u \in L_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\Omega)$ and $v \in L_\beta(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\Omega)$ be positive solutions of the system*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u + au = f(v), & x \in \Omega, \\ (-\Delta)^{\frac{\beta}{2}} v + bv = g(u), & x \in \Omega, \\ u \geq 0, \quad v \geq 0, & x \in \Omega, \\ u = v = 0, & x \notin \Omega, \end{cases} \quad \text{for some } a, b \geq 0, \tag{1.3}$$

with $M > f'(\cdot), g'(\cdot) > 0$, where M is a positive constant. Then u, v are increasing in x_n.

Now we consider the whole space case.

Theorem 1.3 *Letu $\in L_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n), v \in L_\beta(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ be positive solutions of the system*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u + au = f(v), & x \in \mathbb{R}^n, \\ (-\Delta)^{\frac{\beta}{2}} v + bv = g(u), & x \in \mathbb{R}^n, \\ u > 0, \quad v > 0, & x \in \mathbb{R}^n, \end{cases} \quad \text{for some } a, b \geq 0. \tag{1.4}$$

Suppose that, for $\gamma, \nu > 0$,

$$u(x) = o\left(\frac{1}{|x|^\nu}\right) \quad \text{and} \quad v(x) = o\left(\frac{1}{|x|^\gamma}\right) \quad \text{as } |x| \to \infty \tag{1.5}$$

and

$$0 < f'(s) \leq s^p \quad \text{and} \quad 0 < g'(s) \leq s^q \quad \text{with } p\gamma \geq \alpha \text{ and } q\nu \geq \beta. \tag{1.6}$$

Then $u(x)$ and $v(x)$ are radially symmetric and decreasing about some point x_0 in $\mathbb{R}^n$.

Now we consider the nonexistence of positive solutions to system (1.1) in the half-space.

Theorem 1.4 *Let $u \in L_\alpha(\mathbb{R}^n_+) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n_+)$ and $v \in L_\beta(\mathbb{R}^n_+) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n_+)$ be nonnegative solutions of the system*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u + au = f(v), & x \in \mathbb{R}^n_+, \\ (-\Delta)^{\frac{\beta}{2}} v + bv = g(u), & x \in \mathbb{R}^n_+, \\ u \equiv 0, \quad v \equiv 0, & x \notin \mathbb{R}^n_+, \end{cases} \quad \text{for some } a, b \geq 0. \tag{1.7}$$

Suppose

$$\lim_{|x|\to\infty} u(x) = 0, \qquad \lim_{|x|\to\infty} v(x) = 0, \tag{1.8}$$

and $M > f'(\cdot), g'(\cdot) > 0$, *where* M *is a positive constant with* $f(0) = 0, g(0) = 0$. *Then* $u(x) \equiv 0$ *and* $v(x) \equiv 0$ *in* $\mathbb{R}^n$.

Remark 1.2 In Sect. 2, we introduce two maximum principles, namely, the narrow region principle and decay at infinity. This two maximum principles play a key role in the proof of Theorems 1.1–1.4. We give detailed proofs of our main theorems in Sect. 3.

2 Two maximum principles

Let T_λ be a hyperplane in $\mathbb{R}^n$. Without loss of generality, we assume that

$$T_\lambda = \{x = (x_1, x') \in \mathbb{R}^n \mid x_1 = \lambda, \lambda \in \mathbb{R}\},$$

where $x' = (x_2, x_3, \ldots, x_n)$. Let

$$x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$$

be the reflection of x about the plane T_λ. Set

$$\Sigma_\lambda = \{x \in \mathbb{R}^n : x_1 < \lambda\}, \qquad \Sigma_\lambda^c = \{\mathbb{R}^n \setminus \Sigma_\lambda\},$$
$$u_\lambda(x) = u(x^\lambda), \qquad U_\lambda(x) = u_\lambda(x) - u(x), \qquad V_\lambda(x) = u_\lambda(x) - u(x), \quad \forall x \in \mathbb{R}^n.$$

For simplicity of notations, we denote $U_\lambda(x)$ by $U(x)$ and $V_\lambda(x)$ by $V(x)$.

Lemma 2.1 (Narrow region principle) *Let* Ω *be a bounded narrow region in* Σ_λ *that is contained in*

$$\{x \mid \lambda - l < x_1 < \lambda\}$$

for small l. *Let* $U, V \in L_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\Omega)$ *and suppose that* U, V *are lower semicontinuous on* $\overline{\Omega}$. *Assume that* $C_1(x)$ *and* $C_4(x)$ *are bounded from below in* Ω, *whereas* $C_2(x), C_3(x) < 0$ *are bounded from below in* Ω. *If* U, V *satisfy the system*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} U(x) + C_1(x)U(x) + C_2(x)V(x) \geq 0, & x \in \Omega, \\ (-\Delta)^{\frac{\beta}{2}} V(x) + C_3(x)U(x) + C_4(x)V(x) \geq 0, & x \in \Omega, \\ U(x) \geq 0, \quad V(x) \geq 0, & x \in \Sigma_\lambda \setminus \Omega, \\ U(x^\lambda) = -U(x), & x \in \Sigma_\lambda, \\ V(x^\lambda) = -V(x), & x \in \Sigma_\lambda, \end{cases} \tag{2.1}$$

then, for sufficiently small l, *we have*

$$U(x) \geq 0, \qquad V(x) \geq 0, \quad x \in \Omega. \tag{2.2}$$

These conclusions hold for an unbounded domain Ω if we further assume that

$$\varliminf_{|x|\to\infty} U(x) \geq 0 \quad \textit{and} \quad \varliminf_{|x|\to\infty} V(x) \geq 0.$$

Furthermore, if there exists $x_0 \in \Omega$ such that

$$U(x_0) = 0 \quad \textit{or} \quad V(x_0) = 0,$$

then

$$U(x) \equiv V(x) \equiv 0, \quad x \in \mathbb{R}^n. \tag{2.3}$$

Remark 2.1 After finish this paper, we have found that this lemma was given by Niu and Wang [23]. For completeness, we give a full proof of the lemma with some changes.

Proof of Lemma 2.1 Suppose on the contrary, that (2.2) is false. Without loss of generality, we assume that there exists a point such that $U(x) < 0$. Since $U(x)$ is lower semicontinuous on $\overline{\Omega}$, there exists $x_0 \in \Omega$, such that

$$U(x_0) = \min_{\Omega} U(x) < 0. \tag{2.4}$$

Now let $\Sigma_\lambda^c = \mathbb{R}^n \setminus \Sigma_\lambda$. Then by the definition of $(-\Delta)^{\frac{\alpha}{2}}$ we have

$$\begin{aligned}
&(-\Delta)^{\frac{\alpha}{2}} U(x_0) \\
&\quad = C_{n,\alpha}\,\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{U(x_0) - U(y)}{|x_0 - y|^{n+\alpha}}\,dy \\
&\quad = C_{n,\alpha}\,\mathrm{P.V.} \int_{\Sigma_\lambda} \frac{U(x_0) - U(y)}{|x_0 - y|^{n+\alpha}}\,dy + C_{n,\alpha} \int_{\Sigma_\lambda^c} \frac{U(x_0) - U(y)}{|x_0 - y|^{n+\alpha}}\,dy \\
&\quad = C_{n,\alpha}\,\mathrm{P.V.} \int_{\Sigma_\lambda} \frac{U(x_0) - U(y)}{|x_0 - y|^{n+\alpha}}\,dy + C_{n,\alpha} \int_{\Sigma_\lambda} \frac{U(x_0) + U(y)}{|x_0 - y^\lambda|^{n+\alpha}}\,dy \\
&\quad = C_{n,\alpha}\,\mathrm{P.V.} \int_{\Sigma_\lambda} \left[\frac{1}{|x_0 - y|^{n+\alpha}} - \frac{1}{|x_0 - y^\lambda|^{n+\alpha}}\right]\bigl[U(x_0) - U(y)\bigr]\,dy \\
&\qquad + C_{n,\alpha} \int_{\Sigma_\lambda} \frac{2U(x_0)}{|x_0 - y^\lambda|^{n+\alpha}}\,dy \\
&\quad := I_1 + I_2.
\end{aligned} \tag{2.5}$$

To estimate I_1, we notice that

$$\frac{1}{|x_0 - y|^{n+\alpha}} > \frac{1}{|x_0 - y^\lambda|^{n+\alpha}} \quad \text{for } y \in \Sigma_\lambda. \tag{2.6}$$

Since $U(y) \geq 0$ and $y \in \Sigma_\lambda \setminus \Omega$, by (2.4) we have

$$U(x_0) - U(y) \leq 0 \quad \text{for } y \in \Sigma_\lambda.$$

We can see that

$$I_1 \le 0,$$

which implies

$$(-\Delta)^{\frac{\alpha}{2}} U(x_0) \le I_2 = 2C_{n,\alpha} U(x_0) \int_{\Sigma_\lambda} \frac{dy}{|x_0 - y^\lambda|^{n+\alpha}}. \tag{2.7}$$

Choose $x_0^* = (3l + (x_0)_1, x')$ in Σ_λ^c, where $x_0 = ((x_0)_1, x')$. It is easy to see that $B_l(x_0^*) \subset \Sigma_\lambda^c$. Moreover, there exists $C > 0$ such that

$$\begin{aligned}
\int_{\Sigma_\lambda} \frac{dy}{|x_0 - y^\lambda|^{n+\alpha}} &= \int_{\Sigma_\lambda^c} \frac{dy}{|x_0 - y|^{n+\alpha}} \\
&\ge \int_{B_l(x_0^*)} \frac{dy}{|x_0 - y|^{n+\alpha}} \\
&\ge \int_{B_l(x_0^*)} \frac{dy}{4^{n+\alpha} l^{n+\alpha}} \\
&= \frac{C}{l^\alpha}.
\end{aligned}$$

Combining the previous estimate with (2.7), we have

$$(-\Delta)^{\frac{\alpha}{2}} U(x_0) \le \frac{CU(x_0)}{l^\alpha}. \tag{2.8}$$

Combining (2.8) with (2.1), we have

$$\frac{CU(x_0)}{l^\alpha} + C_1(x_0)U(x_0) + C_2(x_0)V(x_0) \ge 0,$$

which is equivalent to

$$U(x_0)\left[\frac{C}{l^\alpha} + C_1(x_0)\right] \ge -C_2(x_0)V(x_0). \tag{2.9}$$

Since we can choose l small enough and C_1 is bounded from below, we have $\frac{C}{l^\alpha} + C_1(x_0) > 0$. Since

$$U(x_0) \ge -\frac{C_2(x_0)}{\frac{C}{l^\alpha} + C_1(x_0)} V(x_0), \tag{2.10}$$

by the condition $C_2(x) < 0$ and (2.4) we get that

$$V(x_0) < 0.$$

On the other hand, V is lower semicontinuous on $\overline{\Omega}$; hence there exists $\bar{x}_0$ such that

$$V(\bar{x}_0) = \min_{\Omega} V(x) < 0. \tag{2.11}$$

Similarly to (2.8), we derive that

$$(-\Delta)^{\frac{\beta}{2}} V(\bar{x}_0) \le \frac{CV(\bar{x}_0)}{l^\beta}. \tag{2.12}$$

Combining (2.8) and (2.12) with (2.1), we have

$$\begin{aligned}
0 &\le (-\Delta)^{\frac{\beta}{2}} V(\bar{x}_0) + C_3(\overline{x_0})U(\bar{x}_0) + C_4(\overline{x_0})V(\bar{x}_0) \\
&\le \frac{CV(\bar{x}_0)}{l^\beta} + C_3(\bar{x}_0)U(x_0) + C_4(\bar{x}_0)V(\bar{x}_0) \\
&\le \left[\frac{C}{l^\beta} + C_4(\bar{x}_0)\right]V(\bar{x}_0) - C_3(\bar{x}_0)\frac{C_2(x_0)}{\frac{C}{l^\alpha} + C_1(x_0)}V(x_0) \\
&\le \left[\frac{C}{l^\beta} + C_4(\bar{x}_0)\right]V(\bar{x}_0) - C_3(\bar{x}_0)\frac{C_2(x_0)}{\frac{C}{l^\alpha} + C_1(x_0)}V(\bar{x}_0) \\
&= V(\bar{x}_0)\left[\frac{C}{l^\beta} + C_4(\bar{x}_0) - C_3(\overline{x_0})\frac{C_2(x_0)}{\frac{C}{l^\alpha} + C_1(x_0)}\right].
\end{aligned}$$

We notice that $C_4(x)$ is bounded from below in Ω and $C_2(x), C_3(x) < 0$ are bounded from below in Ω. Choosing l small enough, we can derive that $V(\bar{x}_0) \ge 0$. This yields a contradiction with (2.11). So (2.2) holds.

Furthermore, if Ω is an unbounded domain, then by the decay condition of U, V it is easy to see that the negative minimum of U, V cannot be taken at infinity.

Now we prove (2.3). Without loss of generality, we assume that there exists $x_0 \in \Omega$ such that $U(x_0) = 0$. Then, due to (2.2) and the fact that $C_2(x_0)V(x_0) \le 0$, combining (2.5) with the first equation of (2.1), we have

$$\begin{aligned}
0 &\le (-\Delta)^{\frac{\alpha}{2}} U(x_0) + C_2(x_0)V(x_0) \\
&\le C_{n,\alpha}\,\mathrm{P.V.}\int_{\Sigma_\lambda}\left[\frac{1}{|x_0 - y|^{n+\alpha}} - \frac{1}{|x_0 - y^\lambda|^{n+\alpha}}\right]\left[-U(y)\right]dy.
\end{aligned}$$

If $U(y) \not\equiv 0$, $y \in \Sigma_\lambda$, then noticing that $U(y) \ge 0$, $y \in \Sigma_\lambda$, we have

$$(-\Delta)^{\frac{\alpha}{2}} U(x_0) + C_2(x_0)V(x_0) < 0.$$

This yields a contradiction. So

$$U(y) \equiv 0, \quad y \in \Sigma_\lambda.$$

By (2.10) we immediately get $V(x) \equiv 0, x \in \Sigma_\lambda$. So since U and V are antisymmetric functions, we have (2.3). □

Lemma 2.2 (Decay at infinity) *Let Ω be an unbounded domain in Σ_λ. Let $U, V \in L_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\Omega)$ be lower semicontinuous on $\overline{\Omega}$. Assume that*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} U(x) + C_1(x)U(x) + C_2(x)V(x) \geq 0, & x \in \Omega, \\ (-\Delta)^{\frac{\beta}{2}} V(x) + C_3(x)U(x) + C_4(x)V(x) \geq 0, & x \in \Omega, \\ U(x) \geq 0, \qquad V(x) \geq 0, & x \in \Sigma_\lambda \setminus \Omega, \\ U(x^\lambda) = -U(x), & x \in \Sigma_\lambda, \\ V(x^\lambda) = -V(x), & x \in \Sigma_\lambda, \end{cases} \tag{2.13}$$

where $C_1(x)$, $C_4(x)$ are nonnegative on Ω, whereas $C_2(x), C_3(x) < 0$ on Ω, and furthermore,

$$\varliminf_{|x|\to\infty} C_2(x)|x|^\alpha = 0 \quad \text{and} \quad \varliminf_{|x|\to\infty} C_3(x)|x|^\beta = 0. \tag{2.14}$$

Then there exists a constant $R_0 > 0$ (depending on $C_i(x)$ but independent of U, V) such that if

$$U(\widetilde{x}) = \min_{\overline{\Omega}} U(x) < 0 \quad \text{and} \quad V(\overline{x}) = \min_{\overline{\Omega}} V(x) < 0,$$

then

$$\widetilde{x} \leq R_0 \quad \text{or} \quad \overline{x} \leq R_0.$$

Remark 2.2 This lemma is quite the same as that of Niu and Wang [23] but with some difference. In [23], C_1, C_4 should satisfying

$$\varliminf_{|x|\to\infty} C_1(x)|x|^\alpha = 0 \quad \text{and} \quad \varliminf_{|x|\to\infty} C_4(x)|x|^\beta = 0. \tag{2.15}$$

So we conclude that the condition of our lemma is different with [23], even they both have the same result.

Proof of Lemma 2.2 Without loss of generality, we assume that there exist a point $x_0 \in \Omega$ such that

$$U(\widetilde{x}) = \min_{\Omega} U(x) < 0.$$

Then as in the proof as (2.7), we have

$$(-\Delta)^{\frac{\alpha}{2}} U(\widetilde{x}) \leq I_2 = C_{n,\alpha} \int_{\Sigma_{\lambda_0}} \frac{2U(\widetilde{x})}{|\widetilde{x} - y^\lambda|^{n+\alpha}}\, dy. \tag{2.16}$$

Choose a point in $\Sigma_\lambda^c : \widetilde{x}^* = (3|\widetilde{x}| + x_1, x')$, where $\widetilde{x} = (x_1, x')$. Then $B_{|\widetilde{x}|}(\widetilde{x}^*) \subset \Sigma_\lambda^c$, and there exists $C > 0$ such that

$$\int_{\Sigma_\lambda} \frac{1}{|\widetilde{x} - y^\lambda|^{n+\alpha}}\, dy = \int_{\Sigma_\lambda^c} \frac{1}{|\widetilde{x} - y|^{n+\alpha}}\, dy \geq \int_{B_{|\widetilde{x}|}(\widetilde{x}^*)} \frac{1}{|\widetilde{x} - y|^{n+\alpha}}\, dy \geq \frac{C}{|\widetilde{x}|^\alpha}. \tag{2.17}$$

So combining (2.17) with (2.16), we have

$$(-\Delta)^{\frac{\alpha}{2}} U(\widetilde{x}) \le \frac{CU(\widetilde{x})}{|\widetilde{x}|^{\alpha}}. \tag{2.18}$$

By (2.13) we have that

$$(-\Delta)^{\frac{\alpha}{2}} U(\widetilde{x}) + C_1(\widetilde{x})U(\widetilde{x}) + C_2(\widetilde{x})V(\widetilde{x}) \ge 0.$$

Combining this with (2.18), we get that

$$\frac{CU(\widetilde{x})}{|\widetilde{x}|^{\alpha}} + C_1(\widetilde{x})U(\widetilde{x}) \ge -C_2(\widetilde{x})V(\widetilde{x}).$$

Now by the conditions $C_1(\widetilde{x}) \ge 0$ and $C_2(\widetilde{x}) < 0$ we easily calculate that

$$U(\widetilde{x}) \ge -\frac{C_2(\widetilde{x})}{\frac{C}{|\widetilde{x}|^{\alpha}+C_1(\widetilde{x})}} V(\widetilde{x}). \tag{2.19}$$

Noticing that $U(\widetilde{x}) < 0$ and $C_2(\widetilde{x}) < 0$, we get $V(\widetilde{x}) < 0$. Then since V are lower semicontinuous on $\overline{\Omega}$. there exists $\overline{x_0}$ such that

$$V(\overline{x}) = \min_{\Omega} V(x) < 0. \tag{2.20}$$

Similarly to (2.18), we derive that

$$(-\Delta)^{\frac{\beta}{2}} V(\overline{x}) \le \frac{CV(\overline{x})}{|\overline{x}|^{\beta}}. \tag{2.21}$$

Combining (2.19) and (2.21) with (2.13), we have

$$\begin{aligned}
0 &\le (-\Delta)^{\frac{\beta}{2}} V(\overline{x}) + C_3(\overline{x})U(\overline{x}) + C_4(\overline{x})V(\overline{x}) \\
&\le \frac{CV(\overline{x})}{|\overline{x}|^{\beta}} + C_3(\overline{x})U(\widetilde{x}) + C_4(\overline{x})V(\overline{x}) \\
&\le \left[\frac{C}{|\overline{x}|^{\beta}} + C_4(\overline{x})\right]V(\overline{x}) - C_3(\overline{x})\frac{C_2(\widetilde{x})}{\frac{C}{|\widetilde{x}|^{\alpha}} + C_1(\widetilde{x})}V(\widetilde{x}) \\
&\le \left[\frac{C}{|\overline{x}|^{\beta}} + C_4(\overline{x})\right]V(\overline{x}) - C_3(\overline{x})\frac{C_2(\widetilde{x})}{\frac{C}{|\widetilde{x}|^{\alpha}} + C_1(\widetilde{x})}V(\overline{x}) \\
&= V(\overline{x})\left[\frac{C}{|\overline{x}|^{\beta}} + C_4(\overline{x}) - C_3(\overline{x})\frac{C_2(\widetilde{x})}{\frac{C}{|\widetilde{x}|^{\alpha}} + C_1(\widetilde{x})}\right].
\end{aligned}$$

Choosing $|\widetilde{x}|$, $|\overline{x}|$ large enough, by (2.14) we can derive $V(\overline{x}) \ge 0$. This yields a contradiction with (2.20). So the lemma is proved. □

3 Proof of Theorems 1.1-1.4

Proof of Theorem 1.1 Let T_λ, x_λ, u_λ, Σ_λ, and U_λ, V_λ be defined as in the previous section.

Step 1: We will show that for $\lambda > -1$ and sufficiently close to -1, we have

$$U_\lambda(x) \geq 0, \qquad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda \cap B. \tag{3.1}$$

By the first equation in system (1.2) and the mean value theorem it is easy to see that

$$(-\Delta)^{\frac{\alpha}{2}} U_\lambda(x) + aU_\lambda(x) - f'\big(\xi(x)\big) V_\lambda(x) = 0, \tag{3.2}$$

where $\xi(x)$ is between $v(x)$ and $v_\lambda(x)$. Similarly, we also have

$$(-\Delta)^{\frac{\beta}{2}} V_\lambda(x) + bV_\lambda(x) - g'\big(\eta(x)\big) U_\lambda(x) = 0,$$

where $\eta(x)$ is between $u(x)$ and $u_\lambda(x)$. Choosing $C_1 = a$, $C_2 = -f'(\xi(x))$, $C_3 = b$, and $C_4 = -g'(\eta(x))$, by the narrow region principle (Lemma 2.1) we get (3.1).

Step 2: Define

$$\lambda_0 = \sup\big\{\lambda \leq 0 \mid U_\mu(x) \geq 0, V_\mu(x) \geq 0, \forall x \in \Sigma_\mu, \forall \mu \leq \lambda\big\}.$$

Then we claim that

$$\lambda_0 = 0. \tag{3.3}$$

Suppose the claim is not true. If $\lambda_0 < 0$, then we will show that the plane can be moved to the right a little more so that inequality (3.1) will still valid. More precisely, there exists small $\epsilon > 0$ such that, for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$, inequality (3.1) holds, which contradicts the definition of λ_0.

First, since U_{λ_0} and V_{λ_0} are not identically zero, from the proof of the narrow region principle (Lemma 2.1) we have

$$U_{\lambda_0} > 0, \qquad V_{\lambda_0} > 0, \quad \forall x \in \Sigma_{\lambda_0} \cap B.$$

Thus U_{λ_0} and V_{λ_0} can take the minimum values if $x \in \Sigma_{\lambda_0 - \delta} \cap B$. More precisely, for any $\delta > 0$,

$$U_{\lambda_0} \geq c_\delta > 0, \qquad V_{\lambda_0} \geq c_\delta > 0, \quad \forall x \in \Sigma_{\lambda_0 - \delta} \cap B.$$

By the continuity of U_λ and V_λ with respect to λ there exists $\epsilon > 0$ such that

$$U_\lambda \geq 0, \qquad V_\lambda \geq 0, \quad \forall x \in \Sigma_{\lambda_0 - \delta} \cap B, \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon).$$

We can see that $(\Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta}) \cap B$ is a narrow region if ϵ and δ are small enough. Then by the narrow region principle (Lemma 2.1) we have

$$U_\lambda \geq 0, \qquad V_\lambda \geq 0, \quad \forall x \in \Sigma_\lambda \cap B, \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon).$$

This contradicts the definition of λ_0. Therefore we prove the claim (3.3). It follows that

$$U_0 \geq 0, \qquad V_0 \geq 0, \quad x \in \Sigma_0 \cap B,$$

or, more apparently,

$$u(-x_1,x') \le u(x_1,x'), \qquad v(-x_1,x') \le v(x_1,x'), \quad 0 < x_1 < 1.$$

Since the x_1-direction can be chosen arbitrarily, it follows that u, v are radially symmetric about the origin. The monotonicity is a consequence of the fact that

$$U_\lambda \ge 0, \qquad V_\lambda \ge 0, \quad x \in \Sigma_\lambda,$$

for all $-1 < \lambda \le 0$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 Let T_λ, x_λ, u_λ, and U_λ, V_λ be defined as in the previous section. Let

$$\Sigma_\lambda = \{x = (x', x_n) \mid x_n < \lambda\}.$$

Step 1: Similarly to (3.1), we can get that, for $\lambda > 0$ sufficiently close to 0, we have

$$U_\lambda(x) \ge 0, \qquad V_\lambda(x) \ge 0, \quad x \in \Sigma_\lambda \cap \Omega. \tag{3.4}$$

Step 2: Define

$$\lambda_0 = \sup\{\lambda > 0 \mid U_\mu(x) \ge 0, V_\mu(x) \ge 0, \forall x \in \Sigma_\mu, \forall \mu \le \lambda\}.$$

Then we must have

$$\lambda_0 = +\infty.$$

Otherwise, suppose that $\lambda_0 < +\infty$, Then we claim that

$$U_{\lambda_0}(x) \equiv 0, \qquad V_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0} \cap \Omega. \tag{3.5}$$

If (3.5) were not true, then by the narrow region principle (Lemma 2.1) we would have

$$U_{\lambda_0}(x) > 0, \qquad V_{\lambda_0}(x) > 0, \quad x \in \Sigma_{\lambda_0} \cap \Omega. \tag{3.6}$$

We will show that the plane T_λ can be moved further to the right. More precisely, there exists small $\epsilon > 0$ such that, for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$,

$$U_\lambda(x) \ge 0, \qquad V_\lambda(x) \ge 0, \quad x \in \Sigma_\lambda \cap \Omega. \tag{3.7}$$

This is a contraction with the definition of λ_0.

If (3.6) holds, then for any $\delta > 0$,

$$U_{\lambda_0} \ge c_\delta > 0, \qquad V_{\lambda_0} \ge c_\delta > 0, \quad \forall x \in \Sigma_{\lambda_0 - \delta} \cap \Omega.$$

By the continuity of U_λ, V_λ with respect to λ there exists $\epsilon > 0$ such that

$$U_\lambda \ge 0, \qquad V_\lambda \ge 0, \quad \forall x \in \Sigma_{\lambda_0 - \delta} \cap \Omega, \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon).$$

We can see that $\Sigma_\lambda \setminus \Sigma_{\lambda_0-\delta}$ is a narrow region since ϵ and δ are small enough. Then by the narrow region principle (Lemma 2.1) we have

$$U_\lambda \geq 0, \qquad V_\lambda \geq 0, \quad \forall x \in \Sigma_\lambda \cap \Omega, \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon).$$

This contradicts the definition of λ_0. Then the claim (3.5) holds, which implies

$$u(x', 2\lambda_0) = u(x', 0) = 0, \qquad v(x', 2\lambda_0) = v(x', 0) = 0,$$

which contradicts the fact that $u, v > 0$ on Ω. We have shown that $\lambda_0 = \infty$ and $U \geq 0$, $V \geq 0$. This shows that $u(x)$ and $v(x)$ are increasing in x_n, which completes the proof of Theorem 1.2. □

Proof of Theorem 1.3 Start moving the plane T_λ from $-\infty$ to that right along the x_1-direction.

Step 1: We will show that for λ sufficiently negative,

$$U_\lambda(x) \geq 0, \qquad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \tag{3.8}$$

By the first equation in system (1.4) and the mean value theorem it is easy to see that

$$(-\Delta)^{\frac{\alpha}{2}} U_\lambda(x) + aU_\lambda(x) - f'\big(\xi(x)\big) V_\lambda(x) = 0, \tag{3.9}$$

where $\xi(x)$ is between $v(x)$ and $v_\lambda(x)$. Similarly, we also have

$$(-\Delta)^{\frac{\beta}{2}} V_\lambda(x) + bV_\lambda(x) - g'\big(\eta(x)\big) U_\lambda(x) = 0,$$

where $\eta(x)$ is between $u(x)$ and $u_\lambda(x)$. In fact, $0 < v_\lambda(x) \leq \xi(x) \leq v(x)$ and $0 < u_\lambda(x) \leq \eta(x) \leq u(x)$.

At those points, for $|x|$ sufficiently large, the decay assumptions (1.5) and (1.6) immediately yield

$$\begin{aligned} &\underline{\lim}_{|x|\to\infty} f'\big(\xi(x)\big)|x|^\alpha \\ &\quad \leq \underline{\lim}_{|x|\to\infty} \xi^p(x)|x|^\alpha \\ &\quad \leq \underline{\lim}_{|x|\to\infty} u^p(x)|x|^\alpha \\ &\quad = 0, \end{aligned}$$

and going through a similar proof, we have

$$\underline{\lim}_{|x|\to\infty} g'\big(\eta(x)\big)|x|^\beta = 0.$$

Then by the decay at infinity (Lemma 2.2) there exists a constant $R_0 > 0$ such that for $\lambda < -R_0$ in Lemma 2.2, one of $U_\lambda(x)$ and $V_\lambda(x)$ must be positive in Σ_λ. Without loss of generality, we assume that

$$V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda.$$

Now we can also prove $U_\lambda(x) \geq 0$, $x \in \Sigma_\lambda$. If not, then by the decay condition of $u(x)$ there must exist a point $x_0 \in \Sigma_\lambda$ such that

$$U_\lambda(x_0) = \min_{x \in \Sigma_\lambda} U_\lambda(x) < 0. \tag{3.10}$$

From previous arguments (2.18) and (3.9) we have

$$\frac{CU_\lambda(x_0)}{|x_0|^\alpha} + aU_\lambda(x) - f'\bigl(\xi(x_0)\bigr)V_\lambda(x_0) \geq 0,$$

and then

$$\left(\frac{C}{|x_0|^\alpha} + a\right)U_\lambda(x_0) \geq f'\bigl(\xi(x_0)\bigr)V_\lambda(x_0) \geq 0,$$

since $f'(\cdot) > 0$, $V_\lambda(x_0) \geq 0$. We can derive that $U_\lambda(x_0) \geq 0$, which contradicts with (3.10). So (3.8) holds.

Step 2: Keep moving the planes to the right to the limiting positive T_{λ_0} as long as (3.8) holds.

Let

$$\lambda_0 = \sup\bigl\{\lambda \mid U_\mu(x) \geq 0, V_\mu(x) \geq 0, \forall x \in \Sigma_\mu, \forall \mu \leq \lambda\bigr\}.$$

We have that

$$\lambda_0 < \infty.$$

Otherwise, if $\lambda_0 = \infty$, then the solution $u(x)$ is increasing with respect to x_1. This contradicts condition (1.5), so that $\lambda_0 < \infty$.

Then we claim that

$$U_{\lambda_0}(x) \equiv 0, \qquad V_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0}. \tag{3.11}$$

If (3.11) were not true, then by the proof of the narrow region principle (Lemma 2.1) we would have

$$U_{\lambda_0}(x) > 0, \qquad V_{\lambda_0}(x) > 0, \quad x \in \Sigma_{\lambda_0}. \tag{3.12}$$

We will show that the plane T_λ can be moved further to the right. More precisely, there exists small $\epsilon > 0$ such that, for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$,

$$U_\lambda(x) \geq 0, \qquad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \tag{3.13}$$

This is a contraction with the definition of λ_0.

If (3.12) is true, then let R_0 be determined in the decay at infinity (Lemma 2.2). It follows that, for any $\delta > 0$,

$$U_{\lambda_0} \geq C_0 > 0, \qquad V_{\lambda_0}(x) \geq C_0 > 0, \quad x \in \overline{\Sigma_{\lambda_0-\delta} \cap B_{R_0}(0)}.$$

Since we have the continuity of $U_\lambda(x)$ and $V_\lambda(x)$ with respect to λ, there exists $\epsilon > 0$ such that, for any $\lambda \in [\lambda_0, \lambda + \epsilon)$,

$$U_\lambda(x) \geq 0, \qquad V_\lambda(x) \geq 0, \quad x \in \overline{\Sigma_{\lambda_0-\delta} \cap B_{R_0}(0)}. \tag{3.14}$$

Suppose (3.13) is not true. If x_0 and $\overline{x_0}$ are the negative minima of $U_\lambda(x)$ and $V_\lambda(x)$ in Σ_λ, then by decay at infinity (Lemma 2.2) and (3.14) we can get that they are all in the bounded narrow region $(\Sigma_{\lambda_0+\epsilon} \setminus \Sigma_{\lambda_0-\delta}) \cap B_{R_0}(0)$ for δ and ϵ small enough, which contradicts the narrow region principle (Lemma 2.1). So (3.13) has to be true, which is a contradiction with the definition of λ_0.

Now we have proved the claim (3.11). Since the x_1-direction can be chosen arbitrarily, we get that $u(x)$ and $v(x)$ are radially symmetric and decreasing about some point x_0. This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4 First, we claim that

$$u(x) > 0, \qquad v(x) > 0, \quad x \in \mathbb{R}^n_+ \quad \text{or} \quad u(x) \equiv 0, \qquad v(x) \equiv 0, \quad x \in \mathbb{R}^n_+. \tag{3.15}$$

To prove (3.15), we assume that $u(x) \not\equiv 0$. If there exists $x_0 \in \mathbb{R}^n_+$ such that $u(x_0) = 0$, then we have that

$$\begin{aligned}
(-\Delta)^{\frac{\alpha}{2}} u(x_0) &= C_{n,\alpha}\,\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x_0) - u(y)}{|x_0 - y|^{n+\alpha}}\,dy \\
&= -C_{n,\alpha}\,\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(y)}{|x_0 - y|^{n+\alpha}}\,dy \\
&< 0.
\end{aligned}$$

On the other hand, by the function (1.7) and the condition on $f(x)$ and $g(x)$ we have that $(-\Delta)^{\frac{\alpha}{2}} u = f(v) - mv \geq 0$. This yields a contradiction, so we have that

$$\text{either} \quad u \equiv 0 \quad \text{or} \quad u > 0 \quad \text{in } \mathbb{R}^n_+.$$

If $u \equiv 0$ in $\mathbb{R}^n_+$, by (1.7) we have $f(v(x)) \equiv mv$ in $\mathbb{R}^n_+$. Together with the condition $f'(s) > m$ and $f(0) = 0$, we can obtain that $v \equiv 0$ in $\mathbb{R}^n_+$. Hence if $u(x)$ attains 0 somewhere in $\mathbb{R}^n_+$, then $u(x) = v(x) \equiv 0$. Similarly, we can also derive that if $v(x)$ attains 0 somewhere in $\mathbb{R}^n_+$, then $u(x) = v(x) \equiv 0$, so (3.15) holds. Now we assume that

$$u(x) > 0, \qquad v(x) > 0, \quad x \in \mathbb{R}^n_+. \tag{3.16}$$

Denote $T_\lambda = \{x \in \mathbb{R}^n_+ \mid x_n = \lambda, \lambda > 0\}$, $\Sigma_\lambda = \{x \in \mathbb{R}^n_+ \mid 0 < x_n < \lambda\}$. Let $x^\lambda = (x_1, \dots, x_{n-1}, 2\lambda - x_n)$ be the reflection of x about the plane T_λ, and let $U_\lambda(x) = u_\lambda(x) - u(x)$ and $V_\lambda(x) = v_\lambda(x) - v(x)$.

Step 1. For $\lambda > 0$ sufficiently close to 0, set $\Sigma = \Sigma_\lambda \cup \mathbb{R}^n_-$, where $\mathbb{R}^n_- = \{x \in \mathbb{R}^n \mid x_n \leq 0\}$.

First, we know that $U(x) = u_\lambda(x) - u(x) \geq -u(x)$ and $V(x) = v_\lambda(x) - v(x) \geq -v(x)$, so by condition (1.8) we have

$$\lim_{|x|\to\infty} U_\lambda(x) \geq \lim_{|x|\to\infty} -u(x) = 0; \qquad \lim_{|x|\to\infty} V_\lambda(x) \geq \lim_{|x|\to\infty} -v(x) = 0.$$

Then by the narrow region principle (Lemma 2.1) it is easy to see that

$$U_\lambda(x) \geq 0, \qquad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda, \tag{3.17}$$

since Σ_λ is a narrow region.

Step 2. Next, we move the plane T_λ along the x_n-axis to the right as long as (3.17) holds and set

$$\lambda_0 = \sup\bigl\{\lambda > 0 \mid U_\mu(x) \geq 0, V_\mu(x) \geq 0, \forall x \in \Sigma_\mu, \forall \mu \leq \lambda\bigr\}.$$

We claim that

$$\lambda_0 = \infty.$$

Otherwise, if $\lambda_0 < \infty$, then by the proof of the narrow region principle (Lemma 2.1) we have

$$U_{\lambda_0} > 0, \qquad V_{\lambda_0} > 0, \quad x \in \Sigma_{\lambda_0} \quad \text{or} \quad U_{\lambda_0} \equiv 0, \qquad V_{\lambda_0} \equiv 0, \quad x \in \Sigma_{\lambda_0}. \tag{3.18}$$

Then going through similar arguments as in (3.11), we can see that

$$U_{\lambda_0} \equiv 0, \qquad V_{\lambda_0} \equiv 0, \quad x \in \Sigma_{\lambda_0}.$$

If we choose the point $\bar{x} = (x_1, x_2, \ldots, x_{n-1}, 0)$ in the hyperplane $\{x_n = 0\}$, then $\bar{x}^{\lambda_0} \in \mathbb{R}^n_+$, which implies

$$u(x_1, x_2, \ldots, x_{n-1}, 2\lambda_0) = u(x_1, x_2, \ldots, x_{n-1}, 0) = 0$$

and

$$v(x_1, x_2, \ldots, x_{n-1}, 2\lambda_0) = v(x_1, x_2, \ldots, x_{n-1}, 0) = 0.$$

This contradicts with (3.16).

Therefore we have proved the claim $\lambda_0 = \infty$, and consequently the solutions $u(x)$ and $v(x)$ are increasing with respect to x_n. We recall that condition (1.8) tells us that $\underline{\lim}_{|x|\to\infty} u(x) = 0$ and $\underline{\lim}_{|x|\to\infty} v(x) = 0$. So the claim (3.16) is not true, and thus $u(x) \equiv 0$ and $v(x) \equiv 0$, $x \in \mathbb{R}^n$. □

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References

1. Berestycki, H., Nirenberg, L.: On the method of moving planes and the sliding method. Bol. Soc. Bras. Mat. **22**(1), 1–37 (1991)
2. Caffarelli, L.A., Salsa, S., Silvestre, L.: Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. Invent. Math. **171**(2), 425–461 (2008). https://doi.org/10.1007/s00222-007-0086-6
3. Caffarelli, L.A., Salsa, S., Silvestre, L.: Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. Invent. Math. **171**(2), 425–461 (2008)
4. Silvestre, L.: Regularity of the obstacle problem for a fractional power of the Laplace operator. Commun. Pure Appl. Math. **60**(1), 67–112 (2007)
5. Shi, S.: Some notes on supersolutions of fractional p-Laplace equation. J. Math. Anal. Appl. **463**(2), 1052–1074 (2018). https://doi.org/10.1016/j.jmaa.2018.03.064
6. Shi, S., Xiao, J.: Fractional capacities relative to bounded open Lipschitz sets complemented. Calc. Var. Partial Differ. Equ. **56**(1), 3–22 (2017). https://doi.org/10.1007/s00526-016-1105-5
7. Shi, S., Xiao, J.: On fractional capacities relative to bounded open Lipschitz sets. Potential Anal. **45**(2), 261–298 (2016). https://doi.org/10.1007/s11118-016-9545-2
8. Shi, S., Xiao, J.: A tracing of the fractional temperature field. Sci. China Math. **60**(11), 2303–2320 (2017). https://doi.org/10.1007/s11425-016-0494-6
9. Cont, R., Tankov, P.: Financial Modelling with Jump Processes. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton (2004)
10. Gilboa, G., Osher, S.: Nonlocal operators with applications to image processing. Multiscale Model. Simul. **7**(3), 1005–1028 (2008). https://doi.org/10.1137/070698592
11. Caffarelli, L., Silvestre, L.: An extension problem related to the fractional Laplacian. Commun. Partial Differ. Equ. **32**(7–9), 1245–1260 (2007)
12. Chen, W., Li, C., Li, Y.: A direct method of moving planes for the fractional Laplacian. Adv. Math. **308**, 404–437 (2017)
13. Chen, W., Fang, Y., Yang, R.: Liouville theorems involving the fractional Laplacian on a half space. Adv. Math. **274**, 167–198 (2015)
14. Chen, W., Li, C., Li, Y.: A direct blowing-up and rescaling argument on nonlocal elliptic equations. Int. J. Math. **27**(8), 1650064 (2016)
15. Chen, W., Zhu, J.: Indefinite fractional elliptic problem and Liouville theorems. J. Differ. Equ. **260**(5), 4758–4785 (2016)
16. Cheng, T., Huang, G., Li, C.: The maximum principles for fractional Laplacian equations and their applications. Commun. Contemp. Math. **19**(6), 1750018 (2017)
17. Fang, Y., Chen, W.: A Liouville type theorem for poly-harmonic Dirichlet problems in a half space. Adv. Math. **229**(5), 2835–2867 (2012)
18. Frank, R.L., Lieb, E.H.: Inversion positivity and the sharp Hardy–Littlewood–Sobolev inequality. Calc. Var. Partial Differ. Equ. **39**(1–2), 85–99 (2010)
19. Han, X., Lu, G., Zhu, J.: Characterization of balls in terms of Bessel-potential integral equation. J. Differ. Equ. **252**(2), 1589–1602 (2012)
20. Teng, K., Wu, X.: Existence and multiplicity of solutions for nonlocal systems involving fractional Laplacian with non-differentiable terms. Appl. Anal. **96**(3), 528–548 (2017)
21. Wang, L., Zhang, B., Zhang, H.: Fractional Laplacian system involving doubly critical nonlinearities in $\mathbb{R}^N$. Electron. J. Qual. Theory Differ. Equ. **2017**, 57 (2017)
22. Li, J.: Monotonicity and radial symmetry results for Schrödinger systems with fractional diffusion. Pac. J. Math. **294**(1), 107–121 (2018). https://doi.org/10.2140/pjm.2018.294.107
23. Niu, P., Wang, P.: A direct method of moving planes for a fully nonlinear nonlocal system. Commun. Pure Appl. Anal. **16**(5), 1707–1718 (2017)