Journal of Inequalities and Applications
a SpringerOpen Journal

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A Suzuki-type multivalued contraction on weak partial metric spaces and applications

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Abstract

Based on a recent paper of Beg and Pathak (Vietnam J. Math. 46(3):693–706, 2018), we introduce the concept of $\mathcal{H}_q^+$-type Suzuki multivalued contraction mappings. We establish a fixed point theorem for this type of mappings in the setting of complete weak partial metric spaces. We also present an illustrated example. Moreover, we provide applications to a homotopy result and to an integral inclusion of Fredholm type. Finally, we suggest open problems for the class of 0-complete weak partial metric spaces, which is more general than complete weak partial metric spaces.

MSC: Primary 47H10; secondary 54H25; 05C40

Keywords: Weak partial metric; $\mathcal{H}^+$-type Pompeiu–Hausdorff metric; Suzuki-type fixed point result

1 Introduction

Throughout this paper, we use following notation: $\mathbb{N}$ is the set of all natural numbers, $\mathbb{R}$ is the set of all real numbers, and $\mathbb{R}^+$ is the set of all nonnegative real numbers.

Definition 1.1 ([2]) A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$:

(P_1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$;
(P_2) $p(x,x) \le p(x,y)$;
(P_3) $p(x,y) = p(y,x)$;
(P_4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

The pair (X, p) is called a partial metric space. Many fixed point results in partial metric spaces have been proved; see [3–17]. Recently, Beg and Pathak [1] introduced a weaker form of partial metrics called a weak partial metric.

Definition 1.2 ([1]) Let X be a nonempty set. A function $q : X \times X \to \mathbb{R}^+$ is called a weak partial metric on X if for all $x, y, z \in X$, the following conditions hold:

$(WP1)$ $q(x,x) = q(x,y)$ if and only if $x = y$;
$(WP2)$ $q(x,x) \le q(x,y)$;
$(WP3)$ $q(x,y) = q(y,x)$;
$(WP4)$ $q(x,y) \le q(x,z) + q(z,y)$.

The pair (X, q) is called a weak partial metric space.

Examples of weak partial metric spaces [1] are:

(1) $(\mathbb{R}^+, q)$, where $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $q(x,y) = |x-y| + 1$ for $x, y \in \mathbb{R}^+$.
(2) $(\mathbb{R}^+, q)$, where $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $q(x,y) = \frac{1}{4}|x-y| + \max\{x,y\}$ for $x, y \in \mathbb{R}^+$.
(3) $(\mathbb{R}^+, q)$, where $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $q(x,y) = \max\{x,y\} + e^{|x-y|} + 1$ for $x, y \in \mathbb{R}^+$.

Notice that

- If $q(x,y) = 0$, then $(WP1)$ and $(WP2)$ imply that $x = y$, but the converse need not be true.
- $(P1)$ implies $(WP1)$, but the converse need not be true.
- $(P4)$ implies $(WP4)$, but the converse need not be true.

Example 1.1 ([1]) If $X = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}$, then $q([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$ is a weak partial metric.

Each weak partial metric q on X generates a T_0 topology τ_q on X. Topology τ_q has as a base the family of open q-balls $\{B_q(x,\epsilon) : x \in X, \epsilon > 0\}$, where $B_q(x,\epsilon) = \{y \in X : q(x,y) < q(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

If q is a weak partial metric on X, then the function $q^s : X \times X \to [0,\infty)$ given by $q^s(x,y) = q(x,y) - \frac{1}{2}[q(x,x) + q(y,y)]$ defines a metric on X.

Definition 1.3 Let (X,q) be a weak partial metric space.

(i) A sequence $\{x_n\}$ in (X,q) converges to a point $x \in X$, with respect to τ_q if $q(x,x) = \lim_{n\to\infty} q(x,x_n)$;
(ii) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty} q(x_n,x_m)$ exists and is finite;
(iii) (X,q) is called complete if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$ with respect to topology τ_q.

Clearly, we also have the following:

Lemma 1.1 *Let (X,q) be a weak partial metric space. Then*

(a) *A sequence $\{x_n\}$ in X is Cauchy sequence in (X,q) if and only if it is a Cauchy sequence in the metric space (X,q^s);*
(b) *(X,q) is complete if and only if the metric space (X,q^s) is complete. Furthermore, a sequence $\{x_n\}$ converges in (X,q^s) to a point $x \in X$ if and only if*

$$\lim_{n,m\to\infty} q(x_n,x_m) = \lim_{n\to\infty} q(x_n,x) = q(x,x). \tag{1.1}$$

Let (X,q) be a weak partial metric space. Let $CB^q(X)$ be the family of all nonempty closed bounded subsets of (X,q). Here, the boundedness is given as follows: E is a bounded subset in (X,q) if there exist $x_0 \in X$ and $M \geq 0$ such that, for all $a \in E$, we have $a \in B_q(x_0,M)$, that is, $q(x_0,a) < q(a,a) + M$.

For $E, F \in CB^q(X)$ and $x \in X$, define

$$q(x,E) = \inf\bigl\{q(x,a), a \in E\bigr\}, \qquad \delta_q(E,F) = \sup\bigl\{q(a,F) : a \in E\bigr\}$$

and

$$\delta_q(F,E) = \sup\{q(b,E) : b \in F\}.$$

Now, $q(x,E) = 0$ implies $q^s(x,E) = 0$, where $q^s(x,E) = \inf\{q^s(x,a), a \in E\}$.

Remark 1.1 ([1]) Let (X,q) be a weak partial metric space, and let E be a nonempty set in (X,q). Then

$$a \in \overline{E} \quad \text{if and only if} \quad q(a,E) = q(a,a), \tag{1.2}$$

where $\overline{E}$ denotes the closure of E with respect to the weak partial metric q.

Note that E is closed in (X,q) if and only if $E = \overline{E}$.

First, we study properties of the mapping $\delta_q : CB^q(X) \times CB^q(X) \to [0,\infty)$.

Proposition 1.1 ([1]) *Let* (X,q) *be a weak partial metric space,We have the following*:

(i) $\delta_q(E,E) = \sup\{q(a,a) : a \in E\}$;

(ii) $\delta_q(E,E) \le \delta_q(E,F)$;

(iii) $\delta_q(E,F) = 0$ *implies* $E \subseteq F$;

(iv) $\delta_q(E,F) \le \delta_q(E,H) + \delta_q(H,F)$ *for all* $E,F,H \in CB^q(X)$.

Definition 1.4 ([1]) Let (X,q) be a weak partial metric space. For $E,F \in CB^q(X)$, define

$$\mathcal{H}_q^+(E,F) = \frac{1}{2}\{\delta_q(E,F) + \delta_q(F,E)\}. \tag{1.3}$$

The following proposition is a consequence of Proposition 1.1.

Proposition 1.2 ([1]) *Let* (X,q) *be a weak partial metric space. Then, for all* $E,F,H \in CB^q(X)$, *we have*

(wh1) $\mathcal{H}_q^+(E,E) \le \mathcal{H}_q^+(E,F)$;

(wh2) $\mathcal{H}_q^+(E,F) = \mathcal{H}_q^+(F,E)$;

(wh3) $\mathcal{H}_q^+(E,F) \le \mathcal{H}_q^+(E,H) + \mathcal{H}_q^+(H,F)$.

The mapping $\mathcal{H}_q^+ : CB^q(X) \times CB^q(X) \to [0,+\infty)$, is called the $\mathcal{H}^+$-type Pompeiu–Hausdorff metric induced by q.

Definition 1.5 ([1]) Let (X,q) be a complete weak partial metric space. A multivalued map $T : X \to CB^q(X)$ is called an $\mathcal{H}_q^+$-contraction if

(1°) there exists k in $(0,1)$ such that

$$\mathcal{H}_q^+(Tx \setminus \{x\}, Ty \setminus \{y\}) \le kq(x,y) \quad \text{for every } x,y \in X, \tag{1.4}$$

(2°) for all x in X, y in Tx, and $\epsilon > 0$, there exists z in Ty such that

$$q(y,z) \le \mathcal{H}_q^+(Ty,Tx) + \epsilon. \tag{1.5}$$

Beg and Pathak [1] proved the following fixed point theorem.

Theorem 1.1 ([1]) *Let (X,q) be a complete weak partial metric space. Every $\mathcal{H}_q^+$-type multivalued contraction mapping $T : X \to CB^q(X)$ with Lipschitz constant $k < 1$ has a fixed point.*

In this paper, we generalize the concept of $\mathcal{H}_q^+$-type multivalued contractions by introducing $\mathcal{H}_q^+$-type Suzuki mult-valued contraction mappings.

2 Fixed point results

First, let $\psi : [0,1) \to (0,1]$ be the nonincreasing function

$$\psi(r) = \begin{cases} 1 & \text{if } 0 \le r < \frac{1}{2}, \\ 1 - r & \text{if } \frac{1}{2} \le r < 1. \end{cases} \tag{2.1}$$

Now, we state a fixed point result for $\mathcal{H}_q^+$-type Suzuki multivalued contraction mappings.

Theorem 2.1 *Let (X,q) be a complete weak partial metric space, and let $F : X \to \mathcal{CB}^q(X)$ be a multivalued mapping. Let $\psi : [0,1) \to (0,1]$ be the nonincreasing function defined by (2.1). Suppose that there exists $0 \le s < 1$ such that T satisfies the condition*

$$\psi(s)q(x,Fx) \le q(x,y) \quad \textit{implies } \mathcal{H}_q^+\bigl(Fx \setminus \{x\}, Fy \setminus \{y\}\bigr) \le sq(x,y) \tag{2.2}$$

for all $x, y \in X$. Suppose also that, for all x in X, y in Fx, and $t > 1$, there exists z in Fy such that

$$q(y,z) \le t\mathcal{H}_q^+(Fy,Fx). \tag{2.3}$$

Then F has a fixed point.

Proof Let $s_1 \in (0,1)$ be such that $0 \le s \le s_1 < 1$ and $w_0 \in X$. Since Fw_0 is nonempty, it follows that if $w_0 \in Fw_0$, then the proof is completed. Let $w_0 \notin Fw_0$. Then there exists $w_1 \in Fw_0$ such that $w_1 \ne w_0$.

Similarly, there exists $w_2 \in Fw_1$ such that $w_1 \ne w_2$, and from (2.3) we have

$$q(w_1,w_2) \le \frac{1}{\sqrt{s_1}} H_q^+(Fw_0,Fw_1). \tag{2.4}$$

Since

$$\psi(s)q(w_1,Fw_1) \le q(w_1,Fw_1) \le q(w_1,w_2),$$

from (2.2) and (2.4) we get

$$\begin{aligned} q(w_1,w_2) &\le \frac{1}{\sqrt{s_1}} H_q^+(Fw_0,Fw_1) \le \frac{1}{\sqrt{s_1}} H_q^+\bigl(Fw_0 \setminus \{w_0\}, Fw_1 \setminus \{w_1\}\bigr) \\ &\le \frac{1}{\sqrt{s_1}} .s.q(w_0,w_1) < \sqrt{s_1}.q(w_0,w_1). \end{aligned}$$

By repeating this process n times we obtain

$$q(w_n, w_{n+1}) \leq (\sqrt{s_1})^n \cdot q(w_0, w_1). \tag{2.5}$$

Hence

$$\lim_{n\to\infty} q(w_n, w_{n+1}) = 0. \tag{2.6}$$

Now we prove that $\{w_n\}$ is a Cauchy sequence in (X, q^s). For all $m \in N$, we have

$$\begin{aligned} q^s(w_n, w_{n+m}) &= q(w_n, w_{n+m}) - \frac{1}{2}\big[q(w_n, w_n) + q(w_{n+m}, w_{n+m})\big] \\ &\leq q(w_n, w_{n+m}) \\ &\leq q(w_n, w_{n+1}) + q(w_{n+1}, w_{n+2}) + \cdots + q(w_{n+m-1}, w_{n+m}) \\ &\leq \big[(\sqrt{s_1})^n + (\sqrt{s_1})^{n+1} + \cdots + (\sqrt{s_1})^{n+m-1}\big] q(w_0, w_1) \\ &\leq (\sqrt{s_1})^n \frac{1}{1-\sqrt{s_1}} q(w_0, w_1). \end{aligned}$$

Hence

$$\lim_{n\to\infty} q^s(w_n, w_{n+m}) = 0. \tag{2.7}$$

This implies that $\{w_n\}$ is a Cauchy sequence in the complete metric space (X, q^s). It follows that there exists $u \in X$ such that

$$\lim_{n\to\infty} q(w_n, u) = \lim_{n,m\to\infty} q(w_n, w_m) = q(u, u). \tag{2.8}$$

From $(WP2)$ we obtain

$$\frac{1}{2}\big[q(w_n, w_n) + q(w_{n+1}, w_{n+1})\big] \leq q(w_n, w_{n+1}). \tag{2.9}$$

By taking the limit as $n \to \infty$ from (2.6) we get

$$\lim_{n\to\infty} q(w_n, w_n) = \lim_{n\to\infty} q(w_{n+1}, w_{n+1}) = \lim_{n\to\infty} q(w_n, w_{n+1}) = 0. \tag{2.10}$$

Also, from (2.7) and (2.10) we find

$$\lim_{n\to\infty} q^s(w_n, w_{n+m}) = 0 = \lim_{n\to\infty} q(w_n, w_{n+m}) - \frac{1}{2}\lim_{n\to\infty}\big[q(w_n, w_n) + q(w_{n+m}, w_{n+m})\big]. \tag{2.11}$$

Therefore

$$\lim_{n\to\infty} q(w_n, w_{n+m}) = 0 = \lim_{n\to\infty} q(w_n, u) = q(u, u). \tag{2.12}$$

Now, we prove that

$$q(u, Fx) \leq 2sq(u, x) \quad \text{for all } x \in X \setminus \{u\}. \tag{2.13}$$

Since $\lim_{n\to\infty} q(w_n,u) = 0$, there exists $n_0 \in \mathbb{N}$ such that

$$q(w_n,u) \le \frac{1}{3}q(x,u) \quad \text{for all } n \ge n_0.$$

Then

$$\begin{aligned}
\psi(s)q(w_n,Fw_n) &\le q(w_n,Fw_n) \\
&\le q(w_n,w_{n+1}) \\
&\le q(w_n,u) + q(u,w_{n+1}) \\
&\le \frac{1}{3}q(u,x) + \frac{1}{3}q(u,x) \\
&\le q(u,x) - \frac{1}{3}q(u,x) \\
&\le q(u,x) - q(u,w_n) \le q(x,w_n).
\end{aligned}$$

This implies that

$$H_q^+(Fw_n,Fx) \le sq(w_n,x).$$

Since $w_{n+1} \in Fw_n$, we have

$$\begin{aligned}
q(w_{n+1},Fx) &\le \delta_q(Fw_n,Fx) \\
&\le 2H_q^+(Fw_n,Fx) \\
&\le 2sq(w_n,x) \\
&\le 2s\bigl[q(w_n,u) + q(u,x)\bigr].
\end{aligned}$$

By taking the limit as $n \to \infty$ we get

$$\lim_{n\to\infty} q(w_{n+1},Fx) \le 2sq(u,x). \tag{2.14}$$

Also, since

$$q(u,Fx) \le q(u,w_{n+1}) + q(w_{n+1},Fx)$$

and

$$q(w_{n+1},Fx) \le q(w_{n+1},w_n) + q(w_n,u) + q(u,Fx),$$

we have

$$\lim_{n\to\infty} q(w_{n+1},Fx) = q(u,Fx). \tag{2.15}$$

From (2.14) and (2.15) we find that

$$q(u,Fx) \le 2sq(u,x) \quad \text{for all } x \in X \setminus \{u\}. \tag{2.16}$$

We claim that

$$H_q^+(Fx, Fu) \leq sq(u, u) \quad \text{for all } x \in X.$$

If $x = u$, then at that point, this clearly holds. So, let $x \neq u$. Then for every positive integer $n \in \mathbb{N}$, there exists $y_n \in Fx$ such that

$$q(u, y_n) \leq q(u, Fx) + \frac{1}{n} q(u, x).$$

Therefore

$$\begin{aligned} q(x, Fx) &\leq q(x, y_n) \\ &\leq q(x, u) + q(u, y_n) \\ &\leq q(x, u) + q(u, Fx) + \frac{1}{n} q(x, u). \end{aligned} \tag{2.17}$$

From (2.16) and (2.17) we get

$$q(x, Fx) \leq q(u, x) + 2sq(u, x) + \frac{1}{n} q(x, u) \tag{2.18}$$

$$= \left[1 + 2s + \frac{1}{n}\right] q(x, u). \tag{2.19}$$

Hence

$$\frac{1}{1 + 2s + \frac{1}{n}} q(x, Fx) \leq q(u, x).$$

This implies that

$$H_q^+(Fu, Fx) \leq sq(u, x).$$

Finally, we show that $u \in Fu$. For this,

$$\begin{aligned} q(u, Fu) &= \lim_{n\to\infty} q(w_{n+1}, Fu) \\ &\leq \lim_{n\to\infty} \delta_q(Fw_n, Fu) \\ &\leq 2 \lim_{n\to\infty} H_q^+(Fw_n, Fu) \\ &\leq 2s \lim_{n\to\infty} q(w_n, u) = 0. \end{aligned}$$

We deduce that $q(u, u) = q(u, Fu) = 0$. Since Fu is closed, $u \in \overline{Fu} = Fu$. □

We provide the following example.

Example 2.1 Let $X = \{0, \frac{1}{2}, 1\}$ and define a weak partial metric $q : X \times X \to [0, \infty)$ as follows: $q(0, 0) = 0$, $q(\frac{1}{2}, \frac{1}{2}) = \frac{1}{3}$, $q(1, 1) = \frac{1}{4}$, $q(0, \frac{1}{2}) = q(\frac{1}{2}, 0) = \frac{1}{2}$, $q(\frac{1}{2}, 1) = q(1, \frac{1}{2}) = \frac{3}{4}$, and

$q(1,0) = q(0,1) = 1$. It is clear that (X,q) is a weak partial metric space. Note that

$$q(1,0) = 1 \nleq q\left(1,\frac{1}{2}\right) + q\left(\frac{1}{2},0\right) - q\left(\frac{1}{2},\frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2} - \frac{1}{3}.$$

Then (X,q) is not a partial metric space. Define the mapping $F : X \to CB^q(X)$ by $F(0) = F(\frac{1}{2}) = \{0\}$ and $F(1) = \{0, \frac{1}{3}\}$. Choose $s = 0.5$. From the definition of ψ we have $\psi(s) = 1$.

To prove the contraction condition (2.2), we need the following cases:

Case 1. At $x = 0$, we have

$$\psi(s)q\bigl(0,F(0)\bigr) = q(0,0) = 0 \leq q(0,y) \quad \text{for all } x \in X.$$

For $y = 0$, we have

$$H_q^+\bigl(F(0)\setminus\{0\},F(0)\setminus\{0\}\bigr) = H_q^+(\phi,\phi) = 0 \leq sq(0,0).$$

For $y = \frac{1}{2}$, we get

$$H_q^+\left(F(0)\setminus\{0\},F\left(\frac{1}{2}\setminus\left\{\frac{1}{2}\right\}\right)\right) = H_q^+\bigl(\phi,\{0\}\bigr) = 0 \leq sq\left(0,\frac{1}{2}\right).$$

If f $y = 1$, then

$$H_q^+\bigl(F(0)\setminus\{0\},F(1)\setminus\{1\}\bigr) = H_q^+\left(\phi,\left\{0,\frac{1}{2}\right\}\right) = 0 \leq sq(0,1).$$

Case 2. At $x = \frac{1}{2}$, we have

$$\psi(s)q\left(\frac{1}{2},F\left(\frac{1}{2}\right)\right) = q\left(\frac{1}{2},0\right) = \frac{1}{2} \leq q\left(\frac{1}{2},y\right) \quad \text{for all } y \in X\setminus\left\{\frac{1}{2}\right\}.$$

Similarly, if $y = 0$, *then*

$$H_q^+\left(F\left(\frac{1}{2}\setminus\left\{\frac{1}{2}\right\},F(0)\setminus\{0\}\right) = H_q^+\bigl(\{0\},\phi\bigr) = 0 \leq sq\left(\frac{1}{2},0\right),$$

If $y = 1$, then

$$H_q^+\left(F\left(\frac{1}{2}\right)\setminus\left\{\frac{1}{2}\right\},F(1)\setminus\{1\}\right) = H_q^+\left(\{0\},\left\{0,\frac{1}{2}\right\}\right) = \frac{1}{4} < sq\left(\frac{1}{2},1\right) = \frac{3}{8}.$$

Case 3. At $x = 1$, we have

$$\psi(s)q\bigl(1,F(1)\bigr) = q\left(1,\frac{1}{2}\right) = \frac{3}{4} \leq q(1,y) \quad \text{for all } y \in X\setminus\{1\}.$$

Again, if $y = 0$, then

$$H_q^+\bigl(F(1)\setminus\{1\},F(0)\setminus\{0\}\bigr) = H_q^+\left(\left\{0,\frac{1}{2}\right\},\phi\right) = 0 \leq sq(1,0).$$

If $y = \frac{1}{2}$, then

$$H_q^+\left(F(1)\setminus\{1\}, F\left(\frac{1}{2}\setminus\left\{\frac{1}{2}\right\}\right) = H_q^+\left(\left\{0,\frac{1}{2}\right\},\{0\}\right) = \frac{1}{4} < sq\left(1,\frac{1}{2}\right) = \frac{3}{8}.$$

Finally, we will enquire the condition (2.3) with $t = 2$. For this, we discuss the following situations:

(i) If $x = 0$ or $x = \frac{1}{2}$, then $y \in F(0) = F(\frac{1}{2}) = \{0\}$. This yields that $y = 0$, so there exists $z \in F(y)$ such that

$$0 = q(y,z) \leq 2H_q^+\big(F(x),F(y)\big).$$

(ii) If $x = 1$, then $y \in F(1) = \{0, \frac{1}{2}\}$. If $y = 0$, then $z = 0$, and condition (2.3) is satisfied. Also, If $y = \frac{1}{2}$, then $z = 0$, so that

$$\frac{1}{2} = q(y,z) = 2H_q^+\left(F(1),F\left(\frac{1}{2}\right)\right) = \frac{1}{2}.$$

Therefore all conditions of Theorem 2.1 are satisfied, and the function F has a fixed point $u = 0$.

On the other hand, the result of Beg and Pathak [1] is not applicable. Indeed,

$$H_q^+\big(F(1)\setminus\{1\},F(1)\setminus\{1\}\big) = \frac{1}{3} > \frac{1}{2}q(1,1) = \frac{1}{8}.$$

3 Applications

First, we present an application concerning a homotopy result for complete weak partial metric spaces.

Theorem 3.1 *Let (X,q) be a complete weak partial metric space, let D be an open subset of X, and let W be a closed subset of X with $D \subset W$. Let $F : W \times [0,1] \to CB^q(X)$ be an operator satisfying*:

(i) *$x \notin F(x,t)$ for each $x \in W \setminus D$ and each $t \in [0,1]$;*

(ii) *there exists $s \in (0,\frac{1}{2})$ such that, for each $t \in [0,1]$ and each $x,y \in W$, we have*

$$\psi(s)q\big(x,F(x,t)\big) \leq q(x,y) \Rightarrow H_q^+\big(F(x,t)\setminus\{x\},F(y,t)\setminus\{y\}\big) \leq sq(x,y);$$

(iii) *for all $x \in W$, $y \in F(x,t)$, and $h > 1$, there exists $z \in F(y,t)$ such that*

$$q(y,z) \leq hH_q^+\big(F(y,t),F(x,t)\big);$$

(iv) *there exists a continuous function $\eta : [0,1] \to \mathbb{R}$ such that*

$$H_q^+\big(F(x,t_1)\setminus\{x\},F(x,t_2)\setminus\{x\}\big) \leq s\big|\eta(t_1)-\eta(t_2)\big|$$

for all $t_1,t_2 \in [0,1]$ and $x \in W$;

(v) *if $x \in F(x,t)$, then $F(x,t) = \{x\}$. Then $F(\cdot,0)$ has a fixed point if and only if $F(\cdot,1)$ has a fixed point.*

Proof Define the set

$$\Delta := \{t \in [0,1]; x \in F(x,t) \text{ for some } x \in D\}.$$

Since $F(\cdot,0)$ has a fixed point, from condition (i), we get $0 \in \Delta$, so $\Delta \neq \phi$. First, we want to show that Δ is an open set. Let $t_1 \in \Delta$ and $x_1 \in D$ be such that $x_1 \in F(x_1,t_1)$. Since D is open in (X,q), there exists $r > 0$ such that $B(x_1,r) \subset D$. Consider $\epsilon = (\frac{1-2s}{2})(q(x_1,x_1)+r) > 0$. Since η is continuous at t_1, there exists $\delta(\epsilon) > 0$ such that $|\eta(t) - \eta(t_1)| < \epsilon$ for all $t \in (t_1 - \delta(\epsilon), t_1 + \delta(\epsilon))$.

Let $t \in (t_1 - \delta(\epsilon), t_1 + \delta(\epsilon))$ and $x \in B(x_1,r) = \{x \in X; q(x_1,x) \leq q(x_1,x_1) + r\}$. Since $x_1 \in F(x_1,t_1)$, from $(WP2)$ we have

$$\psi(s)q(x_1,F(x_1,t_1)) \leq q(x_1,x_1) \leq q(x_1,x) \quad \text{for all } x \in X.$$

Thus

$$\begin{aligned} q(x_1,F(x,t)) &\leq 2H_q^+(F(x,t),F(x_1,t_1)) \\ &\leq 2[H_q^+(F(x,t),F(x,t_1)) + H_q^+(F(x,t_1),F(x_1,t_1))] \\ &= 2[H_q^+(F(x,t)\setminus\{x\},F(x,t_1)\setminus\{x\}) + H_q^+(F(x,t_1)\setminus\{x\},F(x_1,t_1)\setminus\{x_1\})] \\ &\leq 2[|\eta(t)-\eta(t_1)| + sq(x,x_1)] \\ &\leq 2[\epsilon + s(q(x_1,x_1)+r)] \\ &\leq 2\left[\left(\frac{1-2s}{2}\right)(q(x_1,x_1)+r) + s(q(x_1,x_1)+r)\right] \\ &\leq q(x_1,x_1)+r. \end{aligned}$$

Therefore $F(x,t) \subset B(x_1,r)$. Since $F(\cdot,t) : B(x_1,r) \to CB^q(X)$ for each fixed $t \in (t_1 - \delta(\epsilon), t - 1 + \delta(\epsilon))$ and (ii) holds, all the hypotheses of Theorem 2.1 are satisfied. We conclude that $F(\cdot,t)$ has a fixed point in $B(x_1,r) \subset W$. This fixed point must be in D due to (i). Hence $(t_1 - \delta(\epsilon), t - 1 + \delta(\epsilon)) \subset \Delta$, and therefore Δ is open in $[0,1]$.

Second, we prove that Δ is closed in $[0,1]$. To show this, choose a sequence $\{t_n\}$ in Δ such that $t_n \to t^* \in [0,1]$ as $n \to \infty$. We must show that $t^* \in \Delta$. By the definition of Δ there exists $x_n \in D$ with $x_n \in F(x_n,t_n)$. Then

$$\psi(s)q(x_n,F(x_n,t_n)) \leq q(x_n,x_n) \leq q(x_n,x) \quad \text{for all } x \in X.$$

This implies that, for all positive integers $m,n \in \mathbb{N}$, using (v) and $(Wh3)$, we have

$$\begin{aligned} q(x_n,x_m) &\leq 2H_q^+(F(x_n,t_n),F(x_m,t_m)) \\ &\leq 2H_q^+(F(x_n,t_n),F(x_n,t_m)) + 2H_q^+(F(x_n,t_m),F(x_m,t_m)) \\ &= 2H_q^+(F(x_n,t_n)\setminus\{x_n\},F(x_n,t_m)\setminus\{x_n\}) \\ &\quad + 2H_q^+(F(x_n,t_m)\setminus\{x_n\},F(x_m,t_m)\setminus\{x_m\}) \\ &\leq 2s|\eta(t_n)-\eta(t_m)| + 2sq(x_n,x_m). \end{aligned}$$

This implies that

$$q(x_n, x_m) \leq \frac{2s}{1-2s}\big(\big|\eta(t_n) - \eta(t_m)\big|\big).$$

Hence $\lim_{n,m\to\infty} q(x_n, x_m) = 0$. Therefore $\{x_n\}$ is a Cauchy sequence in (X, q). Since (X, q) is complete, there exists $x^* \in W$ such that

$$q\big(x^*, x^*\big) = \lim_{n\to\infty} q\big(x^*, x_n\big) = \lim_{n,m\to\infty} q(x_n, x_m) = 0.$$

On the other hand, we have

$$\begin{aligned} q\big(x_n, F\big(x^*, t^*\big)\big) &\leq 2H_q^+\big(F(x_n, t_n), F\big(x^*, t^*\big)\big) \\ &\leq 2H_q^+\big(F(x_n, t_n), F\big(x_n, t^*\big) + 2H_q^+\big(F\big(x_n, t^*\big)\big), F\big(x^*, t^*\big)\big) \\ &= 2H_q^+\big(F(x_n, t_n) \setminus \{x_n\}, F\big(x_n, t^*\big) \setminus \{x_n\}\big) \\ &\quad + 2H_q^+\big(F\big(x_n, t^*\big) \setminus \{x_n\}, F\big(x^*, t^*\big) \setminus \big\{x^*\big\}\big) \\ &\leq 2s\big|\eta(t_n) - \eta\big(t^*\big)\big| + 2sq\big(x_n, x^*\big). \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we get

$$q\big(x^*, F\big(x^*, t^*\big)\big) = \lim_{n\to\infty} q\big(x_n, F\big(x^*, t^*\big)\big) = 0.$$

It follows that $x^* \in F(x^*, t^*)$. Thus $t^* \in \Delta$, and hence Δ is closed in $[0, 1]$. By the connectedness of $[0, 1]$ we have $\Delta = [0, 1]$.

The reverse implication easily follows by applying the same strategy. This completes the proof. □

Now, we give another application to the solvability of integral inclusions of Fredholm type. Let $I = [0, 1]$, and let $C(I, \mathbb{R})$ be the space of all continuous functions $f : I \to R$. Consider the weak partial metric on X given by

$$q(x, y) = \sup_{t\in I}\big|x(t) - y(t)\big| + \alpha$$

for all $x, y \in C(I, R)$ and $\alpha > 0$. We have $q^s(x, y) = \sup_{t\in I} |x(t) - y(t)|$, so by Lemma 1.1 $(C(I, \mathbb{R}), q)$ is a complete weak partial metric space. Denote by $P_{cv}(\mathbb{R})$ the family of all nonempty compact and convex subsets of $\mathbb{R}$ and by $P_{cl}(\mathbb{R})$ the family of all nonempty closed subsets of $\mathbb{R}$.

Theorem 3.2 *Consider the integral inclusion of Fredholm type*

$$h(t) \in f(t) + \int_0^1 K\big(t, u, h(u)\big)\, du, \quad t \in [0, 1]. \tag{3.1}$$

Suppose that:

(i) $K : I \times I \times R \to P_{cv}(\mathbb{R})$ *is such that* $K_h(t, u) := K(t, u, h(u))$ *is a lower semicontinuous for all* $(t, u) \in I \times I$ *and* $h \in C(I, \mathbb{R})$,

(ii) $f \in C(I,R)$;

(iii) *for each* $t \in I$, *there exists* $l(t,\cdot) \in L^1(I)$ *such that* $\sup_{t\in I}\int_0^1 l(t,u)\,du = \frac{s}{2}$ *with* $s \in [0,1)$ *and*

$$H_q^+\big(K\big(t,u,h(u)\big),K\big(t,u,r(u)\big)\big) \le l(t,u)\Big(\sup_{u\in I}\big|h(u)-r(u)\big| + \alpha\Big)$$

for all $t,u \in I$ *and all* $h,r \in C(I,\mathbb{R})$.

Then the integral inclusion (3.1) *has at least one solution in* $C(I,\mathbb{R})$.

Proof Consider the multivalued operator $T : C(I,R) \to P_{CL}(C(I,R))$ defined by

$$Tx(t) = \left\{h \in C(I,\mathbb{R}) \text{ such that } h(t) \in f(t) + \int_0^1 K\big(t,u,x(u)\big)\,du, t \in I\right\}$$

for $x \in C(I,\mathbb{R})$. For each $K_x(t,u) : I \times I \to P_{cv}(\mathbb{R})$, by the Michael selection theorem there exists a continuous operator $k_x : I \times I \to \mathbb{R}$ such that $k_x(t,u) \in K_x(t,u)$ for all $t,u \in I$. This implies that $f(t) + \int_0^1 k_x(t,u)\,du \in Tx$, and so $Tx \ne \emptyset$. It is easy to prove that Tx is closed, and so we omit the details (see also [18]). This implies that Tx is closed in $(C(I,\mathbb{R}),q)$.

Now, we will show that T is H_q^+-type Suzuki multivalued contraction mapping. Let $x_1,x_2 \in C(I,\mathbb{R})$ and $h \in Tx$. Then there exists $k_{x_1}(t,u) \in K_{x_1}(t,u)$ with $t,u \in I$ such that $h(t) = f(t) + \int_0^1 k_x(t,u)\,du, t \in I$. Also, by hypothesis (iii),

$$H_q^+\big(K\big(t,u,x_1(u)\big),K\big(t,u,x_2(u)\big)\big) \le l(t,u)\Big(\sup_{u\in I}\big|x_1(u)-x_2(u)\big| + \alpha\Big) \quad \forall t,u \in I.$$

Then there exists $z(t,u) \in K_{x_2}(t,u)$ such that

$$\big|k_{x_1}(t,u) - z(t,u)\big| + n \le l(t,u)\big[\big|x_1(u)-x_2(u)\big| + \alpha\big]$$

for all $t,u \in I$. Now, we define the multivalued operator $M(t,u)$ by

$$M(t,u) = K_{x_2}(t,u) \cap \big\{m \in \mathbb{R}, \big|k_{x_1}(t,u) - m\big| + \alpha \le l(t,u)\big(\big|x_1(u)-x_2(u)\big| + \alpha\big)\big\}$$

for $t,u \in I$. Since M is a lower semicontinuous operator, there exists a continuous operator $k_{x_2} : I \times I \to \mathbb{R}$ such that $k_{x_2}(t,u) \in M(t,u)$ for all $t,u \in I$ and

$$w(t) = f(t) + \int_0^1 k_{x_2}(t,u)\,du \in f(t) + \int_0^1 K\big(t,u,x_2(u)\big)\,du.$$

Therefore

$$\begin{aligned} q\big(h(t),Tx_2(t)\big) &\le q\big(h(t),w(t)\big) \\ &= \sup_{t\in I}\big|h(t)-w(t)\big| + \alpha \\ &= \sup_{t\in I}\left|\int_0^1 \big[k_{x_1}(t,u) - k_{x_2}(t,u)\big]\,du\right| + \alpha \\ &\le \sup_{t\in I}\int_0^1 \big(\big|k_{x_1}(t,u) - k_{x_2}(t,u)\big| + \alpha - \alpha\big)\,du + \alpha \end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{t\in I} \int_0^1 l(t,u)\bigl[\bigl|x_1(u)-x_2(u)\bigr|+\alpha\bigr]\,du - \int_0^1 \alpha\,du + \alpha \\
&= \Bigl(\sup_{t\in I}\bigl|x_1(u)-x_2(u)\bigr|+\alpha\Bigr)\int_0^1 l(t,u)\,du \\
&\leq sq\bigl(x_1(t),x_2(t)\bigr).
\end{aligned}
$$

Since $h(t) \in Tx_1$ is arbitrary, we have

$$\delta_q(Tx_1, Tx_2) \leq sq(x_1,x_2). \tag{3.2}$$

Similarly, we can get

$$\delta_q(Tx_2, Tx_1) \leq sq(x_1,x_2). \tag{3.3}$$

From (3.2) and (3.3) we have

$$H_q^+(Tx_1, Tx_2) = \frac{\delta_q(Tx_1, Tx_2) + \delta_q(Tx_2, Tx_1)}{2} \leq sq(x_1,x_2).$$

In particular, the previous inequality holds for any $t \in I$, so that

$$\psi(s)q(x_1, Tx_1) \leq q(x_1,x_2).$$

Thus all conditions of Theorem 2.1 are satisfied, and hence a solution of (3.1) exists. □

4 Perspectives

In 2010, Romaguera [19] introduced the notions of 0-Cauchy sequences and 0-complete partial metric spaces and proved some characterizations of partial metric spaces in terms of completeness and 0-completeness. Adapting the same concepts, we introduce the concepts of 0-Cauchy sequences and 0-complete weak partial metric spaces.

Definition 4.1 Let (X,q) be a weak partial metric space.

(i) A sequence $\{x_n\}$in X is said to be 0-Cauchy if $\lim_{n,m\to\infty} q(x_n,x_m) = 0$;

(iii) (X,q) is called 0-complete if every 0-Cauchy sequence $\{x_n\}$ in X converges to $x \in X$ such that $q(x,x) = 0$.

Open problems: Since 0-completeness is more general than completeness, we would like to prove

(i) Theorem 1.1 and Theorem 2.1, and

(ii) a Hardy–Rogers-type result

in the class of 0-complete weak partial metric spaces.

Funding
The first author is funded by China Medical University.

Availability of data and materials
No data were used to support this study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the manuscript.

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Publisher's Note

Received: 5 September 2018 Accepted: 28 September 2018 Published online: 05 October 2018

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