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Majorization problems for two subclasses of analytic functions connected with the Liu–Owa integral operator and exponential function

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Abstract

In the present paper, we investigate majorization properties for the class $M^{\alpha}_{\beta}(p,\gamma)$ of uniformly starlike functions and the class $N^{\alpha}_{\beta}(p,\theta)$ of spiral-like functions related to an exponential function, which are defined through the Liu–Owa integral operator $Q^{\alpha}_{\beta,p}$ given by (1.5). Also, some special cases of our main results in a form of corollaries are shown.

MSC: Primary 30C45; secondary 30C80

Keywords: Analytic function; Subordination; Majorization; Exponential function; Liu–Owa integral operator

1 Introduction and definitions

Let $\mathbb{C}$ be a complex plane and assume that $\mathcal{A}_p$ denotes the class of analytic and p-valent functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad \bigl(p \in \mathbb{N} = \{1,2,\ldots\}\bigr) \tag{1.1}$$

in the open unit disk

$$\mathbb{U} = \bigl\{z : z \in \mathbb{C} \text{ and } |z| < 1\bigr\}.$$

Specially, for $p = 1$, we write $\mathcal{A} := \mathcal{A}_1$.

In 1967, MacGregor [22] introduced the notion of majorization as follows.

Definition 1.1 Let f and g be analytic in $\mathbb{U}$. We say that f is majorized by g in $\mathbb{U}$ and write

$$f(z) \ll g(z) \quad (z \in \mathbb{U}),$$

if there exists a function $\varphi(z)$, analytic in $\mathbb{U}$, satisfying

$$\bigl|\varphi(z)\bigr| \le 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in \mathbb{U}). \tag{1.2}$$

In 1970, Roberston [28] gave the concept of quasi-subordination as follows.

Definition 1.2 For two analytic functions f and g in $\mathbb{U}$, we say that f is quasi-subordinate to g in $\mathbb{U}$ and write

$$f(z) \prec_q g(z) \quad (z \in \mathbb{U}),$$

if there exist two analytic functions $\varphi(z)$ and $\omega(z)$ in $\mathbb{U}$ such that $\frac{f(z)}{\varphi(z)}$ is analytic in $\mathbb{U}$ and

$$\left|\varphi(z)\right| \leq 1, \qquad \omega(0) = 0 \quad \text{and} \quad \left|\omega(z)\right| \leq |z| < 1 \quad (z \in \mathbb{U}),$$

satisfying

$$f(z) = \varphi(z) g\bigl(\omega(z)\bigr) \quad (z \in \mathbb{U}). \tag{1.3}$$

Remark 1.3
(i) For $\varphi(z) \equiv 1$ in (1.3), we have

$$f(z) = g\bigl(\omega(z)\bigr) \quad (z \in \mathbb{U})$$

and say that f is subordinate to g in $\mathbb{U}$, denoted by (see [29])

$$f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

(ii) For $\omega(z) = z$ in (1.3), the quasi-subordination (1.3) becomes the majorization (1.2).

In 1991, Ma and Minda [21] introduced the following function class $S^*(\phi)$, which is defined by using the above subordination principle:

$$S^*(\phi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \; (z \in \mathbb{U}) \right\},$$

where $\phi(z)$ is analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0) = 1$ and $\Re(\phi(z)) > 0$ for $z \in \mathbb{U}$.

We notice that, for choosing a suitable function $\phi(z)$, the class $S^*(\phi)$ reduces to one of the well-known classes of functions. For instance:

(i) If we take

$$\phi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1; z \in \mathbb{U}),$$

then we obtain the class

$$S^*(A, B) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \; (-1 \leq B < A \leq 1; z \in \mathbb{U}) \right\},$$

which was introduced by Janowski [16]. As a special case, for $A = 1 - 2\alpha$ and $B = -1$, we have the class $S^*(1 - 2\alpha, -1) = S^*(\alpha)$ of starlike functions of order α $(0 \leq \alpha < 1)$. Further, for $A = 1$ and $B = -1$, we have the familiar class $S^*(1, -1) = S^*$ of starlike functions in $\mathbb{U}$.

(ii) If we put

$$\phi(z) = e^z \quad (z \in \mathbb{U}),$$

then we get the class

$$S_e^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec e^z \ (z \in \mathbb{U}) \right\},$$

which was introduced and investigated by Mendiratta et al. [23] and implies that

$$f \in S_e^* \quad \Longleftrightarrow \quad \left| \log \frac{zf'(z)}{f(z)} \right| < 1 \quad (z \in \mathbb{U}). \tag{1.4}$$

In 2004, Liu and Owa [20] (see also [4–9, 32]) introduced the integral operator $Q_{\beta,p}^{\alpha} : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ as follows:

$$Q_{\beta,p}^{\alpha} f(z) = \binom{p+\alpha+\beta-1}{p+\beta-1} \frac{\alpha}{z^{\beta}} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\, dt \quad (\alpha > 0; \beta > -1; p \in \mathbb{N}) \tag{1.5}$$

and

$$Q_{\beta,p}^{0} f(z) = f(z) \quad (\alpha = 0; \beta > -1).$$

If the function $f \in \mathcal{A}_p$ given by (1.1), then from (1.5) we show that

$$Q_{\beta,p}^{\alpha} f(z) = z^p + \frac{\Gamma(\alpha+\beta+p)}{\Gamma(\beta+p)} \sum_{k=1}^{\infty} \frac{\Gamma(\beta+p+k)}{\Gamma(\alpha+\beta+p+k)} a_{k+p} z^{k+p}$$
$$(\alpha \geq 0; \beta > -1; p \in \mathbb{N}). \tag{1.6}$$

Also, we easily find the relationship, from (1.6), that (see [20])

$$z\bigl(Q_{\beta,p}^{\alpha} f(z)\bigr)' = (\alpha+\beta+p-1) Q_{\beta,p}^{\alpha-1} f(z) - (\alpha+\beta-1) Q_{\beta,p}^{\alpha} f(z). \tag{1.7}$$

On the other hand, we observe that

(i) for $p = 1$, we get the Jung–Kim–Srivastava integral operator $Q_{\beta}^{\alpha} := Q_{\beta,1}^{\alpha}$ (see [17]; also see [3, 11]);

(ii) for $\alpha = 1$ and $\beta = \delta$, we obtain the generalized Libera operator $J_{\delta,p} := Q_{\delta,p}^{1}$, which is presented as follows (see [10]; see also [19, 25]):

$$J_{\delta,p}(f)(z) := Q_{\delta,p}^{1} f(z) = \frac{\delta+p}{z^{\delta}} \int_0^z t^{\delta-1} f(t)\, dt \quad (\delta > -p; p \in \mathbb{N}). \tag{1.8}$$

Inspired by the above class S_e^*, we now use the Liu–Owa integral operator $Q_{\beta,p}^{\alpha}$ to define the following two subclasses $M_{\beta}^{\alpha}(p,\gamma)$ and $N_{\beta}^{\alpha}(p,\theta)$ of functions $f \in \mathcal{A}_p$.

Definition 1.4 Let $p \in \mathbb{N}$; $\alpha \geq 0$; $\beta > -1$ and $\gamma \geq 0$. A function $f \in \mathcal{A}_p$ belongs to the class $M_\beta^\alpha(p,\gamma)$ of uniformly starlike functions, related to exponential function, if and only if

$$\left[\left(\frac{z(Q_{\beta,p}^\alpha f(z))'}{Q_{\beta,p}^\alpha f(z)} + 1 - p\right) - \gamma\left|\frac{z(Q_{\beta,p}^\alpha f(z))'}{Q_{\beta,p}^\alpha f(z)} - p\right|\right] \prec e^z. \tag{1.9}$$

Remark 1.5

(i) For $p = 1$ in (1.9), we have the function class

$$M_\beta^\alpha(\gamma) := M_\beta^\alpha(1,\gamma) = \left\{f \in \mathcal{A} : \left[\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} - \gamma\left|\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} - 1\right|\right] \prec e^z \ (\gamma \geq 0)\right\}.$$

(ii) For $\gamma = 0$ in (1.9), we get the function class

$$M_\beta^\alpha(p) := M_\beta^\alpha(p,0) = \left\{f \in \mathcal{A}_p : \frac{z(Q_{\beta,p}^\alpha f(z))'}{Q_{\beta,p}^\alpha f(z)} \prec \left(e^z + p - 1\right) \ (p \in \mathbb{N})\right\}.$$

(iii) Further, for $\gamma = p - 1 = 0$ in (1.9), we obtain the function class

$$M_\beta^\alpha := M_\beta^\alpha(1,0) = \left\{f \in \mathcal{A} : \frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} \prec e^z\right\}.$$

Definition 1.6 Let $p \in \mathbb{N}$; $\alpha \geq 0$; $\beta > -1$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. A function $f \in \mathcal{A}_p$ belongs to the class $N_\beta^\alpha(p,\theta)$ of spiral-like functions, related to an exponential function, if and only if

$$e^{i\theta}\left(\frac{z(Q_{\beta,p}^\alpha f(z))'}{Q_{\beta,p}^\alpha f(z)}\right) \prec e^z \cos\theta + i\sin\theta. \tag{1.10}$$

Remark 1.7

(i) For $p = 1$ in (1.10), we obtain the function class

$$\begin{aligned} N_\beta^\alpha(\theta) &:= N_\beta^\alpha(1,\theta) \\ &= \left\{f \in \mathcal{A} : e^{i\theta}\left(\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)}\right) \prec e^z \cos\theta + i\sin\theta \ \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)\right\}. \end{aligned}$$

(ii) For $\theta = 0$ in (1.10), we have the function class

$$N_\beta^\alpha(p) := N_\beta^\alpha(p,0) = \left\{f \in \mathcal{A}_p : \frac{z(Q_{\beta,p}^\alpha f(z))'}{Q_{\beta,p}^\alpha f(z)} \prec e^z \ (p \in \mathbb{N})\right\}.$$

(iii) Further, for $\theta = p - 1 = 0$ in (1.10), we get the function class $M_\beta^\alpha = N_\beta^\alpha := N_\beta^\alpha(1,0)$.

A majorization problem for the normalized class of starlike functions has been investigated by MacGregor [22] and Altintas et al. [1] (see also [2]). Recently, many researchers have studied several majorization problems for univalent and multivalent functions or meromorphic and multivalent meromorphic functions, which are all subordinate to certain function $\phi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, involving various different operators; the interested reader can, for example, see [13–15, 18, 26, 27, 30, 31, 33]. However, we note that there is no article dealing with the above-mentioned problems for functions which are subordinate to $\phi(z) = e^z$. Hence, in the present paper, we investigate the problems of majorization of the classes $M_\beta^\alpha(p,\gamma)$ and $N_\beta^\alpha(p,\theta)$ defined by the Liu–Owa integral operator $Q_{\beta,p}^\alpha$ given by (1.5), which are related to an exponential function.

2 Majorization problem for the class $M_\beta^\alpha(p,\gamma)$

Firstly, we give and prove majorization property for the class $M_\beta^\alpha(p,\gamma)$.

Theorem 2.1 *Let the function $f \in \mathcal{A}_p$ and suppose that $g \in M_\beta^\alpha(p,\gamma)$ with $|\alpha+\beta+p-2| \geq \gamma(\alpha+\beta+p-1)+e$. If $Q_{\beta,p}^\alpha f(z)$ is majorized by $Q_{\beta,p}^\alpha g(z)$ in $\mathbb{U}$, that is,*

$$Q_{\beta,p}^\alpha f(z) \ll Q_{\beta,p}^\alpha g(z) \quad (z \in \mathbb{U}),$$

then, for $|z| \leq r_1$, we have

$$\left|Q_{\beta,p}^{\alpha-1} f(z)\right| \leq \left|Q_{\beta,p}^{\alpha-1} g(z)\right|,$$

where $r_1 = r_1(p,\alpha,\beta,\gamma)$ is the smallest positive root of the equation

$$\begin{aligned} &r^2 e^r - \bigl[|\alpha+\beta+p-2| - \gamma(\alpha+\beta+p-1)\bigr] r^2 - e^r - 2(1+\gamma)r \\ &\quad + |\alpha+\beta+p-2| - \gamma(\alpha+\beta+p-1) = 0 \quad (p \in \mathbb{N}; \alpha \geq 0; \beta > -1; \gamma \geq 0). \end{aligned} \tag{2.1}$$

Proof Since $g \in M_\beta^\alpha(p,\gamma)$, then, from (1.9) and the subordination relationship, we get

$$\left[\left(\frac{z(Q_{\beta,p}^\alpha g(z))'}{Q_{\beta,p}^\alpha g(z)} + 1 - p\right) - \gamma \left|\frac{z(Q_{\beta,p}^\alpha g(z))'}{Q_{\beta,p}^\alpha g(z)} - p\right|\right] = e^{\omega(z)}, \tag{2.2}$$

where $\omega(z) = c_1 z + c_2 z^2 + \cdots$ is bounded and analytic in $\mathbb{U}$, satisfying (see, for details, Goodman [12])

$$\omega(0) = 0 \quad \text{and} \quad \left|\omega(z)\right| \leq |z| \quad (z \in \mathbb{U}). \tag{2.3}$$

Letting

$$\varpi = \frac{z(Q_{\beta,p}^\alpha g(z))'}{Q_{\beta,p}^\alpha g(z)} + 1 - p \tag{2.4}$$

in (2.2), we have

$$\varpi - \gamma|\varpi - 1| = e^{\omega(z)},$$

which implies that

$$\varpi = \frac{e^{\omega(z)} - \gamma e^{-i\phi}}{1 - \gamma e^{-i\phi}}. \tag{2.5}$$

From (2.4) and (2.5), we easily obtain

$$\frac{z(Q^{\alpha}_{\beta,p}g(z))'}{Q^{\alpha}_{\beta,p}g(z)} = \frac{p-1-p\gamma e^{-i\phi}+e^{\omega(z)}}{1-\gamma e^{-i\phi}}. \tag{2.6}$$

Now, using (1.7) in (2.6) and making simple computations, we have

$$\frac{Q^{\alpha-1}_{\beta,p}g(z)}{Q^{\alpha}_{\beta,p}g(z)} = \frac{(\alpha+\beta+p-2)-\gamma(\alpha+\beta+p-1)e^{-i\phi}+e^{\omega(z)}}{(\alpha+\beta+p-1)(1-\gamma e^{-i\phi})}, \tag{2.7}$$

which, by virtue of (2.3), yields the inequality

$$\left|Q^{\alpha}_{\beta,p}g(z)\right| \le \frac{(1+\gamma)(\alpha+\beta+p-1)}{|\alpha+\beta+p-2|-\gamma(\alpha+\beta+p-1)-e^{|z|}}\left|Q^{\alpha-1}_{\beta,p}g(z)\right|. \tag{2.8}$$

Again, because $Q^{\alpha}_{\beta,p}f(z)$ is majorized by $Q^{\alpha}_{\beta,p}g(z)$ in $\mathbb{U}$, so we find from (1.2) that

$$Q^{\alpha}_{\beta,p}f(z) = \varphi(z)Q^{\alpha}_{\beta,p}g(z). \tag{2.9}$$

Differentiating (2.9) on both sides with respect to z and multiplying by z, we obtain

$$z\bigl(Q^{\alpha}_{\beta,p}f(z)\bigr)' = z\varphi'(z)Q^{\alpha}_{\beta,p}g(z) + z\varphi(z)\bigl(Q^{\alpha}_{\beta,p}g(z)\bigr)'. \tag{2.10}$$

By using (1.7) in (2.10), together with (2.9), we have

$$Q^{\alpha-1}_{\beta,p}f(z) = \frac{1}{\alpha+\beta+p-1}z\varphi'(z)Q^{\alpha}_{\beta,p}g(z) + \varphi(z)Q^{\alpha-1}_{\beta,p}g(z). \tag{2.11}$$

On the other hand, noticing that the Schwarz function φ satisfies the inequality (see, e.g., Nehari [24])

$$\left|\varphi'(z)\right| \le \frac{1-|\varphi(z)|^2}{1-|z|^2} \quad (z\in\mathbb{U}), \tag{2.12}$$

and in terms of (2.8) and (2.12) in (2.11), we get

$$\left|Q^{\alpha-1}_{\beta,p}f(z)\right| \le \left[\left|\varphi(z)\right| + \frac{|z|(1+\gamma)(1-|\varphi(z)|^2)}{(1-|z|^2)(|\alpha+\beta+p-2|-\gamma(\alpha+\beta+p-1)-e^{|z|})}\right]\left|Q^{\alpha-1}_{\beta,p}g(z)\right|,$$

which, by taking

$$|z| = r, \qquad \left|\varphi(z)\right| = \rho \quad (0\le\rho\le 1),$$

reduces to the inequality

$$\left|Q^{\alpha-1}_{\beta,p}f(z)\right| \le \Phi_1(r,\rho)\left|Q^{\alpha-1}_{\beta,p}g(z)\right|,$$

where

$$\Phi_1(r,\rho) = \frac{r(1+\gamma)(1-\rho^2)}{(1-r^2)[|\alpha+\beta+p-2|-\gamma(\alpha+\beta+p-1)-e^{r}]} + \rho.$$

In order to determine r_1, we must choose

$$\begin{aligned} r_1 &= \max\{r \in [0,1) : \Phi_1(r,\rho) \le 1, \forall \rho \in [0,1]\} \\ &= \max\{r \in [0,1) : \Psi_1(r,\rho) \ge 0, \forall \rho \in [0,1]\}, \end{aligned}$$

where

$$\Psi_1(r,\rho) = (1-r^2)\big[|\alpha+\beta+p-2| - \gamma(\alpha+\beta+p-1) - e^r\big] - r(1+\gamma)(1+\rho).$$

Obviously, for $\rho = 1$, the function $\Psi_1(r,\rho)$ takes its minimum value, namely

$$\min\{\Psi_1(r,\rho) : \rho \in [0,1]\} = \Psi_1(r,1) := \psi_1(r),$$

where

$$\psi_1(r) = (1-r^2)\big[|\alpha+\beta+p-2| - \gamma(\alpha+\beta+p-1) - e^r\big] - 2r(1+\gamma).$$

Further, because $\psi_1(0) = |\alpha+\beta+p-2| > \gamma(\alpha+\beta+p-1) + e$ and $\psi_1(1) = -2(1+\gamma) < 0$, so there exists r_1 such that $\psi_1(r) \ge 0$ for all $r \in [0, r_1]$, where $r_1 = r_1(p,\alpha,\beta,\gamma)$ is the smallest positive root of equation (2.1). This completes the proof of Theorem 2.1. □

3 Majorization problem for the class $N_\beta^\alpha(p,\theta)$

Next, we discuss majorization property for the class $N_\beta^\alpha(p,\theta)$.

Theorem 3.1 *Let the function $f \in \mathcal{A}_p$ and assume that $g \in N_\beta^\alpha(p,\theta)$ with $|\alpha+\beta-1| \ge |\tan\theta||\alpha+\beta| + e$. If $Q_{\beta,p}^\alpha f(z)$ is majorized by $Q_{\beta,p}^\alpha g(z)$ in $\mathbb{U}$, that is,*

$$Q_{\beta,p}^\alpha f(z) \ll Q_{\beta,p}^\alpha g(z) \quad (z \in \mathbb{U}),$$

then, for $|z| \le r_2$, we have

$$\big|Q_{\beta,p}^{\alpha-1} f(z)\big| \le \big|Q_{\beta,p}^{\alpha-1} g(z)\big|, \tag{3.1}$$

where $r_2 = r_2(\alpha,\beta,\theta)$ is the smallest positive root of the equation

$$\begin{aligned} &r^2 e^r - \big[|\alpha+\beta-1| - |\tan\theta||\alpha+\beta|\big]r^2 - e^r - 2|\sec\theta| r + |\alpha+\beta-1| - |\tan\theta||\alpha+\beta| = 0 \\ &\left(\alpha \ge 0; \beta > -1; -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right). \end{aligned} \tag{3.2}$$

Proof Because $g \in N_\beta^\alpha(p,\theta)$, so from (1.10) we show that

$$e^{i\theta}\left(\frac{z(Q_{\beta,p}^\alpha g(z))'}{Q_{\beta,p}^\alpha g(z)}\right) = e^{\omega(z)}\cos\theta + i\sin\theta, \tag{3.3}$$

where $\omega(z)$ is defined as (2.3).

From (3.3) it follows that

$$\frac{z(Q^{\alpha}_{\beta,p}g(z))'}{Q^{\alpha}_{\beta,p}g(z)} = \frac{e^{\omega(z)} + i\tan\theta}{1 + i\tan\theta}. \tag{3.4}$$

Now, putting (1.7) in (3.4) and making some calculations, we get

$$\frac{Q^{\alpha-1}_{\beta,p}g(z)}{Q^{\alpha}_{\beta,p}g(z)} = \frac{(\alpha+\beta-1) + i\tan\theta(\alpha+\beta) + e^{\omega(z)}}{(1+i\tan\theta)(\alpha+\beta+p-1)},$$

which, using (2.3), becomes the inequality

$$\left|Q^{\alpha}_{\beta,p}g(z)\right| \le \frac{|\sec\theta|(\alpha+\beta+p-1)}{|\alpha+\beta-1| - |\tan\theta||\alpha+\beta| - e^{|z|}}\left|Q^{\alpha-1}_{\beta,p}g(z)\right|. \tag{3.5}$$

Next, in view of (2.12) as well as (3.5) in (2.11), and just as the proof of Theorem 2.1, we have

$$\left|Q^{\alpha-1}_{\beta,p}f(z)\right| \le \left[\left|\varphi(z)\right| + \frac{|z||\sec\theta|(1-|\varphi(z)|^2)}{(1-|z|^2)(|\alpha+\beta-1| - |\tan\theta||\alpha+\beta| - e^{|z|})}\right]\left|Q^{\alpha-1}_{\beta,p}g(z)\right|,$$

which, by setting

$$|z| = r, \qquad \left|\varphi(z)\right| = \rho \quad (0 \le \rho \le 1),$$

reduces to the inequality

$$\left|Q^{\alpha-1}_{\beta,p}f(z)\right| \le \frac{\Phi_2(\rho)}{(1-r^2)[|\alpha+\beta-1| - |\tan\theta||\alpha+\beta| - e^{r}]}\left|Q^{\alpha-1}_{\beta,p}g(z)\right|, \tag{3.6}$$

where the function $\Phi_2(\rho)$ given by

$$\Phi_2(\rho) = -r|\sec\theta|\rho^2 + \left(1-r^2\right)\left[|\alpha+\beta-1| - |\tan\theta||\alpha+\beta| - e^{r}\right]\rho + r|\sec\theta|$$

takes its maximum value at $\rho = 1$ with $r_2 = r_2(p,\alpha,\beta,\theta)$ defined by (3.2). Furthermore, if $0 \le \sigma \le r_2(p,\alpha,\beta,\theta)$, then the function

$$\Psi_2(\rho) = -\sigma|\sec\theta|\rho^2 + \left(1-\sigma^2\right)\left[|\alpha+\beta-1| - |\tan\theta||\alpha+\beta| - e^{\sigma}\right]\rho + \sigma|\sec\theta|$$

increases on the interval $0 \le \rho \le 1$, therefore

$$\Psi_2(\rho) \le \Psi_2(1) = \left(1-\sigma^2\right)\left[|\alpha+\beta-1| - |\tan\theta||\alpha+\beta| - e^{\sigma}\right] \quad \left(0 \le \sigma \le r_2(p,\alpha,\beta,\theta)\right).$$

Hence, from this fact and (3.6), we conclude that inequality (3.1) holds true for $|z| \le r_2$, where $r_2 = r_2(p,\alpha,\beta,\theta)$ is given by (3.2). We complete the proof of Theorem 3.1. □

4 Some corollaries

As a special case of Theorem 2.1, when $p = 1$, we get the following result.

Corollary 4.1 *Let the function $f \in \mathcal{A}$ and assume that $g \in M_\beta^\alpha(\gamma)$ with $|\alpha+\beta-1| \geq \gamma(\alpha+\beta)+e$. If $Q_\beta^\alpha f(z)$ is majorized by $Q_\beta^\alpha g(z)$ in $\mathbb{U}$, then, for $|z| \leq r_3$, we have*

$$\left|Q_\beta^{\alpha-1} f(z)\right| \leq \left|Q_\beta^{\alpha-1} g(z)\right|,$$

where $r_3 := r_1(1,\alpha,\beta,\gamma)$ is the smallest positive root of the equation

$$r^2 e^r - \bigl[|\alpha+\beta-1| - \gamma(\alpha+\beta)\bigr] r^2 - e^r - 2(1+\gamma) r + |\alpha+\beta-1| - \gamma(\alpha+\beta) = 0$$
$$(\alpha \geq 0; \beta > -1; \gamma \geq 0).$$

Setting $\gamma = 0$ in Theorem 2.1, we obtain the following corollary.

Corollary 4.2 *Let the function $f \in \mathcal{A}_p$ and assume that $g \in M_\beta^\alpha(p)$ with $|\alpha+\beta+p-2| \geq e$. If $Q_{\beta,p}^\alpha f(z)$ is majorized by $Q_{\beta,p}^\alpha g(z)$ in $\mathbb{U}$, then, for $|z| \leq r_4$, we have*

$$\left|Q_{\beta,p}^{\alpha-1} f(z)\right| \leq \left|Q_{\beta,p}^{\alpha-1} g(z)\right|,$$

where $r_4 := r_1(p,\alpha,\beta,0)$ is the smallest positive root of the equation

$$r^2 e^r - |\alpha+\beta+p-2| r^2 - e^r - 2r + |\alpha+\beta+p-2| = 0 \quad (p \in \mathbb{N}; \alpha \geq 0; \beta > -1).$$

Taking $\theta = 0$ in Theorem 3.1, we state the following corollary.

Corollary 4.3 *Let the function $f \in \mathcal{A}_p$ and suppose that $g \in N_\beta^\alpha(p)$ with $|\alpha+\beta-1| \geq e$. If $Q_{\beta,p}^\alpha f(z)$ is majorized by $Q_{\beta,p}^\alpha g(z)$ in $\mathbb{U}$, then, for $|z| \leq r_5$, we have*

$$\left|Q_{\beta,p}^{\alpha-1} f(z)\right| \leq \left|Q_{\beta,p}^{\alpha-1} g(z)\right|,$$

where $r_5 := r_2(\alpha,\beta,0)$ is the smallest positive root of the equation

$$r^2 e^r - |\alpha+\beta-1| r^2 - e^r - 2r + |\alpha+\beta-1| = 0 \quad (\alpha \geq 0; \beta > -1). \tag{4.1}$$

5 Conclusions

In this paper, we investigate the problems of majorization of the classes $M_\beta^\alpha(p,\gamma)$ and $N_\beta^\alpha(p,\theta)$ defined by the Liu–Owa integral operator $Q_{\beta,p}^\alpha$ given by (1.5), which are also related to an exponential function. The results obtained generalize and unify the theory of majorization in geometric function theory. In addition, we notice that, if we put $p = 1$ and $\alpha = 1$, $\beta = \delta$ in Theorems 2.1 and 3.1, as well as Corollaries 4.2 and 4.3 of this paper, respectively, then we easily get the corresponding majorization results for the Jung–Kim–Srivastava integral operator Q_β^α and the generalized Libera operator $J_{\delta,p}$ ($\delta > -p$; $p \in \mathbb{N}$), which are mentioned in the Introduction.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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