Zhang et al. *Journal of Inequalities and Applications* (2018) 2018:264
https://doi.org/10.1186/s13660-018-1860-2

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Different types of quantum integral inequalities via (α, m)-convexity

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Abstract

In this paper, based on (α, m)-convexity, we establish different type inequalities via quantum integrals. These inequalities generalize some results given in the literature.

MSC: 34A08; 26A51; 26D10; 26D15

Keywords: Quantum integral inequalities; (α, m)-convex functions; Hermite–Hadamard's inequality; Simpson's inequality

1 Introduction and preliminaries

Throughout the paper, let $I := [a, b] \subseteq \mathbb{R}$ with $0 \le a < b$ be an interval, I° be the interior of I and let $0 < q < 1$ be a constant.

Let $f : I \to \mathbb{R}$ be convex on I, then the Hermite–Hadamard inequality holds:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

If $f : I \to \mathbb{R}$ is four times continuously differentiable on I° and $\|f^{(4)}\|_\infty = \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty$, then the Simpson inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{2880}\|f^{(4)}\|_\infty (b-a)^4. \tag{1.2}$$

Many researchers generalized the inequalities (1.1) and (1.2). For more details on these inequalities, see [5–8, 10–14, 16, 17, 22, 24, 25].

In 2014, Tariboon and Ntouyas defined the q-derivative and q-integral as follows.

Definition 1.1 ([28]) Let $f : I \to \mathbb{R}$ be a continuous function and let $x \in I$. Then the q-derivative on I of f at x is defined as

$$_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \ne a, \qquad _aD_q f(a) = \lim_{x\to a} {}_aD_q f(x).$$

Definition 1.2 ([28]) Let $f: I \to \mathbb{R}$ be a continuous function. Then the q-integral on I is defined as

$$\int_a^x f(t)\,{}_a\mathrm{d}_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f\bigl(q^n x + \bigl(1-q^n\bigr)a\bigr)$$

for $x \in I$. Moreover, if $c \in (a,x)$, then the q-integral on I is defined as

$$\int_c^x f(t)\,{}_a\mathrm{d}_q t = \int_a^x f(t)\,{}_a\mathrm{d}_q t - \int_a^c f(t)\,{}_a\mathrm{d}_q t.$$

In the same paper, they also proved the following q-Hölder inequality.

Theorem 1.1 ([28]) *Let $f, g: I \to \mathbb{R}$ be two continuous functions. Then the inequality*

$$\int_a^x \bigl|f(t)\bigr|\bigl|g(t)\bigr|\,{}_a\mathrm{d}_q t \le \biggl(\int_a^x \bigl|f(t)\bigr|^{r_1}\,{}_a\mathrm{d}_q t\biggr)^{\frac{1}{r_1}} \biggl(\int_a^x \bigl|g(t)\bigr|^{r_2}\,{}_a\mathrm{d}_q t\biggr)^{\frac{1}{r_2}}$$

holds for all $x \in I$ and $r_1, r_2 > 1$ with $r_1^{-1} + r_2^{-1} = 1$.

In 2018, Alp et al. generalized the Hermite–Hadamard inequality to the form of q-integrals as follows.

Theorem 1.2 ([2]) *Let $f: I \to \mathbb{R}$ be convex and differentiable on I with $0 < q < 1$. Then we have*

$$f\biggl(\frac{qa+b}{1+q}\biggr) \le \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x \le \frac{qf(a)+f(b)}{1+q}. \tag{1.3}$$

For more details on the inequality (1.3), see [15, 18, 20, 21, 23]. For other type quantum integral inequalities, the interested reader can refer to [3, 4, 27, 29, 31].

In 1993, Miheşan gave the definition of (α, m)-convex functions as follows.

Definition 1.3 ([19]) For $b^* > 0$, the function $f: [0, b^*] \to \mathbb{R}$ is named (α, m)-convex with $\alpha, m \in (0,1]$ if the inequality

$$f\bigl(tx + m(1-t)y\bigr) \le t^\alpha f(x) + m\bigl(1-t^\alpha\bigr)f(y)$$

holds for all $x, y \in [0, b^*]$ and $t \in [0,1]$.

This paper aims to establish different types of quantum integral inequalities via (α, m)-convexity. Some relevant connections of the results obtained in this paper with previous ones are also pointed out.

2 Auxiliary results

For proving main results, we need the following lemma.

Lemma 2.1 *Let $f : I \to \mathbb{R}$ be a continuous and q-differentiable function on I° with $0 < q < 1$. Then the identity*

$$\begin{aligned}
&\lambda\bigl[\mu f(b) + (1-\mu)f(a)\bigr] + (1-\lambda)f\bigl(\mu b + (1-\mu)a\bigr) - \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x \\
&= (b-a)\biggl\{\int_0^\mu (qt + \lambda\mu - \lambda)\,{}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t \\
&\quad + \int_\mu^1 (qt + \lambda\mu - 1)\,{}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t\biggr\}
\end{aligned}$$

holds for all $\lambda, \mu \in [0,1]$ if ${}_aD_q f$ is integrable on I.

Proof By an identical transformation, we get

$$\begin{aligned}
&(b-a)\biggl\{\int_0^\mu (qt + \lambda\mu - \lambda)\,{}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t \\
&\quad + \int_\mu^1 (qt + \lambda\mu - 1)\,{}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t\biggr\} \\
&= (b-a)\biggl\{\int_0^1 (qt + \lambda\mu - 1)\,{}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t \\
&\quad + \int_0^\mu (1-\lambda)\,{}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t\biggr\}.
\end{aligned} \tag{2.1}$$

From Definition 1.1, we get

$$\begin{aligned}
{}_aD_q f\bigl(tb + (1-t)a\bigr) &= \frac{f(tb + (1-t)a) - f(q[tb + (1-t)a] + (1-q)a)}{(1-q)(tb + (1-t)a - a)} \\
&= \frac{f(tb + (1-t)a) - f(qtb + (1-qt)a)}{t(1-q)(b-a)}.
\end{aligned}$$

Utilizing the above calculation and Definition 1.2, we have

$$\begin{aligned}
&\int_0^1 t\,{}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t \\
&= \int_0^1 \frac{f(tb + (1-t)a) - f(qtb + (1-qt)a)}{(1-q)(b-a)}\,{}_0\mathrm{d}_q t \\
&= \frac{1}{b-a}\Biggl\{\sum_{n=0}^\infty q^n f\bigl(q^n b + \bigl(1-q^n\bigr)a\bigr) \\
&\quad - \sum_{n=0}^\infty q^n f\bigl(q^{n+1} b + \bigl(1-q^{n+1}\bigr)a\bigr)\Biggr\} \\
&= \frac{1}{b-a}\Biggl\{\sum_{n=0}^\infty q^n f\bigl(q^n b + \bigl(1-q^n\bigr)a\bigr) \\
&\quad - \frac{1}{q}\sum_{n=0}^\infty q^{n+1} f\bigl(q^{n+1} b + \bigl(1-q^{n+1}\bigr)a\bigr)\Biggr\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{b-a}\left\{f(b) + \left(1-\frac{1}{q}\right)\sum_{n=1}^{\infty} q^n f\bigl(q^n b + \bigl(1-q^n\bigr)a\bigr)\right\} \\
&= \frac{1}{b-a}\left\{\frac{1}{q}f(b) - \frac{1-q}{q}\sum_{n=0}^{\infty} q^n f\bigl(q^n b + \bigl(1-q^n\bigr)a\bigr)\right\} \\
&= \frac{f(b)}{q(b-a)} - \frac{1}{q(b-a)^2}\int_a^b f(x)\,{}_a\mathrm{d}_q x, \qquad (2.2)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 {}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t \\
&= \int_0^1 \frac{f(tb + (1-t)a) - f(qtb + (1-qt)a)}{t(1-q)(b-a)}\,{}_0\mathrm{d}_q t \\
&= \frac{1}{b-a}\left\{\sum_{n=0}^{\infty} f\bigl(q^n b + \bigl(1-q^n\bigr)a\bigr) - \sum_{n=0}^{\infty} f\bigl(q^{n+1} b + \bigl(1-q^{n+1}\bigr)a\bigr)\right\} \\
&= \frac{f(b) - f(a)}{b-a} \qquad (2.3)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^\mu {}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t \\
&= \int_0^\mu \frac{f(tb + (1-t)a) - f(qtb + (1-qt)a)}{t(1-q)(b-a)}\,{}_0\mathrm{d}_q t \\
&= \frac{1}{b-a}\left\{\sum_{n=0}^{\infty} f\bigl(q^n \mu b + \bigl(1-q^n\mu\bigr)a\bigr) - \sum_{n=0}^{\infty} f\bigl(q^{n+1}\mu b + \bigl(1-q^{n+1}\mu\bigr)a\bigr)\right\} \\
&= \frac{f(\mu b + (1-\mu)a) - f(a)}{b-a}. \qquad (2.4)
\end{aligned}
$$

Substituting (2.2), (2.3) and (2.4) into (2.1), we can obtain the desired result. This ends the proof. □

Remark 2.1 In Lemma 2.1, if one takes $q \to 1^-$, one has [9, Lemma 2].

Remark 2.2 Consider Lemma 2.1.

(i) Putting $\mu = 0$, we have

$$
\begin{aligned}
&f(a) - \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x \\
&\quad = (b-a)\int_0^1 (qt-1)\,{}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t. \qquad (2.5)
\end{aligned}
$$

(ii) Putting $\mu = 1$, we have

$$
f(b) - \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x = (b-a)\int_0^1 qt\,{}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t. \qquad (2.6)
$$

(iii) Putting $\mu = \frac{1}{1+q}$, we have

$$\lambda \frac{qf(a)+f(b)}{1+q} + (1-\lambda)f\left(\frac{qa+b}{1+q}\right) - \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x$$
$$= (b-a)\left\{\int_0^{\frac{1}{1+q}} \left(qt - \frac{\lambda q}{1+q}\right) {}_aD_q f\big(tb+(1-t)a\big)\,{}_0\mathrm{d}_q t\right.$$
$$\left. + \int_{\frac{1}{1+q}}^1 \left(qt + \frac{\lambda}{1+q} - 1\right) {}_aD_q f\big(tb+(1-t)a\big)\,{}_0\mathrm{d}_q t\right\}. \tag{2.7}$$

Remark 2.3 Consider Lemma 2.1.

(i) Putting $\lambda = 0$, we get

$$f\big(\mu b + (1-\mu)a\big) - \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x$$
$$= (b-a)\left\{\int_0^{\mu} qt\,{}_aD_q f\big(tb+(1-t)a\big)\,{}_0\mathrm{d}_q t\right.$$
$$\left. + \int_{\mu}^1 (qt-1)\,{}_aD_q f\big(tb+(1-t)a\big)\,{}_0\mathrm{d}_q t\right\}. \tag{2.8}$$

Specially, taking $\mu = \frac{1}{1+q}$, we obtain the midpoint-like integral identity

$$f\left(\frac{qa+b}{1+q}\right) - \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x$$
$$= (b-a)\left\{\int_0^{\frac{1}{1+q}} qt\,{}_aD_q f\big(tb+(1-t)a\big)\,{}_0\mathrm{d}_q t\right.$$
$$\left. + \int_{\frac{1}{1+q}}^1 (qt-1)\,{}_aD_q f\big(tb+(1-t)a\big)\,{}_0\mathrm{d}_q t\right\},$$

which is presented by Alp et al. in [2, Lemma 11].

(ii) Putting $\lambda = \frac{1}{3}$, we get

$$\frac{1}{3}\big[\mu f(b) + (1-\mu)f(a) + 2f\big(\mu b+(1-\mu)a\big)\big] - \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x$$
$$= (b-a)\left\{\int_0^{\mu} \left(qt + \frac{1}{3}\mu - \frac{1}{3}\right) {}_aD_q f\big(tb+(1-t)a\big)\,{}_0\mathrm{d}_q t\right.$$
$$\left. + \int_{\mu}^1 \left(qt + \frac{1}{3}\mu - 1\right) {}_aD_q f\big(tb+(1-t)a\big)\,{}_0\mathrm{d}_q t\right\}. \tag{2.9}$$

Specially, taking $\mu = \frac{1}{1+q}$, we obtain the Simpson-like integral identity

$$\frac{1}{3}\left[\frac{qf(a)+f(b)}{1+q} + 2f\left(\frac{qa+b}{1+q}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x$$
$$= (b-a)\left\{\int_0^{\frac{1}{1+q}} \left(qt - \frac{q}{3+3q}\right) {}_aD_q f\big(tb+(1-t)a\big)\,{}_0\mathrm{d}_q t\right.$$
$$\left. + \int_{\frac{1}{1+q}}^1 \left(qt + \frac{1}{3+3q} - 1\right) {}_aD_q f\big(tb+(1-t)a\big)\,{}_0\mathrm{d}_q t\right\}. \tag{2.10}$$

(iii) Putting $\lambda = \frac{1}{2}$, we get

$$\begin{aligned} &\frac{1}{2}\left[\mu f(b) + (1-\mu)f(a) + f\bigl(\mu b + (1-\mu)a\bigr)\right] - \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x \\ &\quad = (b-a)\left\{\int_0^{\mu}\left(qt + \frac{1}{2}\mu - \frac{1}{2}\right){}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t \right. \\ &\qquad \left. + \int_{\mu}^1 \left(qt + \frac{1}{2}\mu - 1\right){}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t\right\}. \end{aligned} \tag{2.11}$$

Specially, taking $\mu = \frac{1}{1+q}$, we obtain the averaged midpoint-trapezoid-like integral identity

$$\begin{aligned} &\frac{1}{2}\left[\frac{qf(a) + f(b)}{1+q} + f\left(\frac{qa+b}{1+q}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x \\ &\quad = (b-a)\left\{\int_0^{\frac{1}{1+q}}\left(qt - \frac{q}{2+2q}\right){}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t \right. \\ &\qquad \left. + \int_{\frac{1}{1+q}}^1 \left(qt + \frac{1}{2+2q} - 1\right){}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t\right\}. \end{aligned} \tag{2.12}$$

(iv) Putting $\lambda = 1$, we get

$$\begin{aligned} &\mu f(b) + (1-\mu)f(a) - \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x \\ &\quad = (b-a)\int_0^1 (qt + \mu - 1)\,{}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t. \end{aligned} \tag{2.13}$$

Specially, taking $\mu = \frac{1}{1+q}$, we obtain the trapezoid-like integral identity

$$\begin{aligned} &\frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x \\ &\quad = (b-a)\int_0^1 \left(qt + \frac{1}{1+q} - 1\right){}_aD_q f\bigl(tb + (1-t)a\bigr)\,{}_0\mathrm{d}_q t, \end{aligned}$$

which is presented by Sudsutad et al. in [26, Lemma 3.1].

It is worth to mention here that to the best of our knowledge the obtained identities (2.5)–(2.13) are new in the literature.

Next we provide some calculations which will be used in this paper.

Lemma 2.2 *Let $\mu \in [0,1]$ and $\tau \in [0,\infty)$. From Definition 1.2, we have*

$$\int_0^{\mu} t^{\tau}\,{}_0\mathrm{d}_q t = (1-q)\sum_{n=0}^{\infty}\mu^{\tau+1}q^{(\tau+1)n} = \frac{\mu^{\tau+1}(1-q)}{1-q^{\tau+1}}$$

and

$$\int_0^{\mu}(1-t)^{\tau}\,{}_0\mathrm{d}_q t = (1-q)\mu\sum_{n=0}^{\infty}q^n\bigl(1-q^n\mu\bigr)^{\tau}.$$

Lemma 2.3 *Let* $\lambda, \mu \in [0,1]$ *and* $\tau \in [0,\infty)$. *Then we have*

$$\int_0^{\mu} t^{\tau} \left| qt - (\lambda - \lambda\mu) \right| {}_0\mathrm{d}_q t = \begin{cases} \frac{\mu^{\tau+1}(1-q)(\lambda-\lambda\mu)}{1-q^{\tau+1}} - \frac{q\mu^{\tau+2}(1-q)}{1-q^{\tau+2}}, & (\lambda+q)\mu \le \lambda, \\ \frac{2(1-q)^2(\lambda-\lambda\mu)^{\tau+2}}{(1-q^{\tau+1})(1-q^{\tau+2})} + \frac{q\mu^{\tau+2}(1-q)}{1-q^{\tau+2}} - \frac{\mu^{\tau+1}(1-q)(\lambda-\lambda\mu)}{1-q^{\tau+1}}, & (\lambda+q)\mu > \lambda, \end{cases}$$

and

$$\int_0^{\mu} (1-t)^{\tau} \left| qt - (\lambda - \lambda\mu) \right| {}_0\mathrm{d}_q t = \begin{cases} (1-q)\mu \sum_{n=0}^{\infty} q^n (\lambda - \lambda\mu - q^{n+1}\mu)(1-q^n\mu)^{\tau}, & (\lambda+q)\mu \le \lambda, \\ \begin{bmatrix} 2(1-q)(\lambda-\lambda\mu)^2 \sum_{n=0}^{\infty} q^{n-1}(1-q^n)[1-q^{n-1}(\lambda-\lambda\mu)]^{\tau} \\ -(1-q)\mu \sum_{n=0}^{\infty} q^n(\lambda-\lambda\mu-q^{n+1}\mu)(1-q^n\mu)^{\tau} \end{bmatrix}, & (\lambda+q)\mu > \lambda. \end{cases}$$

Proof When $(\lambda+q)\mu \le \lambda$, making use of Lemma 2.2, we get

$$\begin{aligned} \int_0^{\mu} t^{\tau} \left| qt - (\lambda - \lambda\mu) \right| {}_0\mathrm{d}_q t &= \int_0^{\mu} \left[(\lambda - \lambda\mu)t^{\tau} - qt^{\tau+1} \right] {}_0\mathrm{d}_q t \\ &= \frac{\mu^{\tau+1}(1-q)(\lambda-\lambda\mu)}{1-q^{\tau+1}} - \frac{q\mu^{\tau+2}(1-q)}{1-q^{\tau+2}}. \end{aligned}$$

When $(\lambda+q)\mu > \lambda$, making use of Lemma 2.2 again, we get

$$\begin{aligned} &\int_0^{\mu} t^{\tau} \left| qt - (\lambda - \lambda\mu) \right| {}_0\mathrm{d}_q t \\ &= \int_0^{\frac{\lambda-\lambda\mu}{q}} \left[(\lambda-\lambda\mu)t^{\tau} - qt^{\tau+1} \right] {}_0\mathrm{d}_q t + \int_{\frac{\lambda-\lambda\mu}{q}}^{\mu} \left[qt^{\tau+1} - (\lambda-\lambda\mu)t^{\tau} \right] {}_0\mathrm{d}_q t \\ &= 2\int_0^{\frac{\lambda-\lambda\mu}{q}} \left[(\lambda-\lambda\mu)t^{\tau} - qt^{\tau+1} \right] {}_0\mathrm{d}_q t + \int_0^{\mu} \left[qt^{\tau+1} - (\lambda-\lambda\mu)t^{\tau} \right] {}_0\mathrm{d}_q t \\ &= \frac{2(1-q)^2(\lambda-\lambda\mu)^{\tau+2}}{(1-q^{\tau+1})(1-q^{\tau+2})} + \frac{q\mu^{\tau+2}(1-q)}{1-q^{\tau+2}} - \frac{\mu^{\tau+1}(1-q)(\lambda-\lambda\mu)}{1-q^{\tau+1}}. \end{aligned}$$

Similarly, we also get

$$\int_0^{\mu} (1-t)^{\tau} \left| qt - (\lambda - \lambda\mu) \right| {}_0\mathrm{d}_q t = \begin{cases} (1-q)\mu \sum_{n=0}^{\infty} q^n (\lambda - \lambda\mu - q^{n+1}\mu)(1-q^n\mu)^{\tau}, & (\lambda+q)\mu \le \lambda, \\ \begin{bmatrix} 2(1-q)(\lambda-\lambda\mu)^2 \sum_{n=0}^{\infty} q^{n-1}(1-q^n)[1-q^{n-1}(\lambda-\lambda\mu)]^{\tau} \\ -(1-q)\mu \sum_{n=0}^{\infty} q^n(\lambda-\lambda\mu-q^{n+1}\mu)(1-q^n\mu)^{\tau} \end{bmatrix}, & (\lambda+q)\mu > \lambda. \end{cases}$$

This completes the proof. □

The following results of Lemma 2.4, Lemma 2.5 and Lemma 2.6 are stated without proof.

Lemma 2.4 *Let* $\lambda, \mu \in [0,1]$ *and* $\tau \in [0,\infty)$. *Then we have*

$$\int_0^1 t^\tau \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t = \begin{cases} \frac{(1-q)(1-\lambda\mu)}{1-q^{\tau+1}} - \frac{q(1-q)}{1-q^{\tau+2}}, & \lambda\mu+q\le 1, \\ \frac{2(1-q)^2(1-\lambda\mu)^{\tau+2}}{(1-q^{\tau+1})(1-q^{\tau+2})} + \frac{q(1-q)}{1-q^{\tau+2}} - \frac{(1-q)(1-\lambda\mu)}{1-q^{\tau+1}}, & \lambda\mu+q>1, \end{cases}$$

and

$$\int_0^1 (1-t)^\tau \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t = \begin{cases} (1-q)\sum_{n=0}^\infty q^n(1-\lambda\mu-q^{n+1})(1-q^n)^\tau, & \lambda\mu+q\le 1, \\ \left[\begin{array}{l} 2(1-q)(1-\lambda\mu)^2\sum_{n=0}^\infty q^{n-1}(1-q^n)[1-q^{n-1}(1-\lambda\mu)]^\tau \\ \quad -(1-q)\sum_{n=0}^\infty q^n(1-\lambda\mu-q^{n+1})(1-q^n)^\tau \end{array}\right], & \lambda\mu+q>1. \end{cases}$$

Lemma 2.5 *Let* $\lambda, \mu \in [0,1]$ *and* $\tau \in [0,\infty)$. *Then we have*

$$\int_0^\mu t^\tau \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t = \begin{cases} \frac{\mu^{\tau+1}(1-\lambda\mu)(1-q)}{1-q^{\tau+1}} - \frac{q\mu^{\tau+2}(1-q)}{1-q^{\tau+2}}, & (\lambda+q)\mu\le 1, \\ \frac{2(1-q)^2(1-\lambda\mu)^{\tau+2}}{(1-q^{\tau+1})(1-q^{\tau+2})} + \frac{q\mu^{\tau+2}(1-q)}{1-q^{\tau+2}} - \frac{\mu^{\tau+1}(1-\lambda\mu)(1-q)}{1-q^{\tau+1}}, & (\lambda+q)\mu>1, \end{cases}$$

and

$$\int_0^\mu (1-t)^\tau \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t = \begin{cases} (1-q)\mu\sum_{n=0}^\infty q^n(1-\lambda\mu-q^{n+1}\mu)(1-q^n\mu)^\tau, & (\lambda+q)\mu\le 1, \\ \left[\begin{array}{l} 2(1-q)(1-\lambda\mu)^2\sum_{n=0}^\infty q^{n-1}(1-q^n)[1-q^{n-1}(1-\lambda\mu)]^\tau \\ \quad -(1-q)\mu\sum_{n=0}^\infty q^n(1-\lambda\mu-q^{n+1}\mu)(1-q^n\mu)^\tau \end{array}\right], & (\lambda+q)\mu>1. \end{cases}$$

Lemma 2.6 *Let* $\lambda, \mu \in [0,1]$ *and* $\theta \in [1,\infty)$. *Then we have*

$$\int_0^1 \left|qt-(1-\lambda\mu)\right|^\theta {}_0\mathrm{d}_q t = \begin{cases} (1-q)\sum_{n=0}^\infty q^n(1-\lambda\mu-q^{n+1})^\theta, & 0\le\lambda\mu\le 1-q, \\ \left[\begin{array}{l} (1-q)(1-\lambda\mu)^{\theta+1}\sum_{n=0}^\infty q^{n-1}(1-q^n)^\theta \\ \quad +(1-q)\sum_{n=0}^\infty q^n(q^{n+1}-1+\lambda\mu)^\theta \\ \quad -(1-q)(1-\lambda\mu)^{\theta+1}\sum_{n=0}^\infty q^{n-1}(q^n-1)^\theta \end{array}\right], & 1-q<\lambda\mu\le 1. \end{cases}$$

3 Main results

In 2018, Alp et al. established the q-Hermite–Hadamard inequality in [2]. Here we give a new proof, which is more concise.

Theorem 3.1 *Let $f : I \to \mathbb{R}$ be a convex function on $[a,b]$ with $0 < q < 1$. Then we have*

$$f\left(\frac{qa+b}{1+q}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x \leq \frac{qf(a)+f(b)}{1+q}.$$

Proof It is obvious that $\sum_{n=0}^{\infty}(1-q)q^n = 1$, $0 < q < 1$. Since Jensen's inequality defined on convex sets for infinite sums still remains true, utilizing this fact and Definition 1.2, we have

$$\begin{aligned} f\left(\frac{qa+b}{1+q}\right) &= f\left(\sum_{n=0}^{\infty}(1-q)q^n\bigl(q^n b + (1-q^n)a\bigr)\right) \\ &\leq \sum_{n=0}^{\infty}(1-q)q^n f\bigl(q^n b + (1-q^n)a\bigr) \\ &= \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x. \end{aligned}$$

Using Definition 1.2 and the convexity of f, we get

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x &= \sum_{n=0}^{\infty}(1-q)q^n f\bigl(q^n b + (1-q^n)a\bigr) \\ &\leq \sum_{n=0}^{\infty}(1-q)q^n\bigl[q^n f(b) + (1-q^n)f(a)\bigr] \\ &= \frac{qf(a)+f(b)}{1+q}. \end{aligned}$$

The proof is completed. □

Using Lemma 2.1, we can obtain the following theorem.

Theorem 3.2 *For $0 \leq a < b$ and some fixed $m \in (0,1]$, let $f : [a, \frac{b}{m}] \to \mathbb{R}$ be a continuous and q-differentiable function on $(a, \frac{b}{m})$, and let ${}_aD_q f$ be integrable on $[a, \frac{b}{m}]$ with $0 < q < 1$. Then the inequality*

$$\begin{aligned} &\left|\lambda\bigl[\mu f(b) + (1-\mu)f(a)\bigr] + (1-\lambda)f\bigl(\mu b + (1-\mu)a\bigr) - \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x\right| \\ &\quad\leq \min\bigl\{\mathcal{H}_1(\lambda,\mu,\alpha,m), \mathcal{H}_2(\lambda,\mu,\alpha,m)\bigr\} \end{aligned}$$

holds for all $\lambda,\mu \in [0,1]$ if $|{}_aD_q f|$ is (α,m)-convex on $[a, \frac{b}{m}]$ with $\alpha, m \in (0,1]^2$, where

$$\begin{aligned} \mathcal{H}_1(\lambda,\mu,\alpha,m) = (b-a)\Bigl\{&\bigl[\Phi_1(\lambda,\mu,\alpha) + \Phi_2(\lambda,\mu,\alpha) - \Phi_3(\lambda,\mu,\alpha)\bigr]\bigl|{}_aD_q f(b)\bigr| \\ &+ m\bigl[\Phi_4(\lambda,\mu) + \Phi_5(\lambda,\mu) - \Phi_6(\lambda,\mu) - \Phi_1(\lambda,\mu,\alpha) \\ &- \Phi_2(\lambda,\mu,\alpha) + \Phi_3(\lambda,\mu,\alpha)\bigr]\left|{}_aD_q f\left(\frac{a}{m}\right)\right|\Bigr\}, \end{aligned}$$

$$\begin{aligned}\mathcal{H}_2(\lambda,\mu,\alpha,m) = (b-a)\Big\{&\big[\Phi_7(\lambda,\mu,\alpha)+\Phi_8(\lambda,\mu,\alpha)-\Phi_9(\lambda,\mu,\alpha)\big]\big|{}_aD_qf(a)\big|\\&+m\big[\Phi_4(\lambda,\mu)+\Phi_5(\lambda,\mu)-\Phi_6(\lambda,\mu)-\Phi_7(\lambda,\mu,\alpha)\\&-\Phi_8(\lambda,\mu,\alpha)+\Phi_9(\lambda,\mu,\alpha)\big]\left|{}_aD_qf\left(\frac{b}{m}\right)\right|\Big\},\end{aligned}$$

$$\begin{aligned}\Phi_1(\lambda,\mu,\alpha) &= \int_0^\mu t^\alpha\big|qt-(\lambda-\lambda\mu)\big|\,{}_0d_qt\\&=\begin{cases}\frac{\mu^{\alpha+1}(1-q)(\lambda-\lambda\mu)}{1-q^{\alpha+1}}-\frac{q\mu^{\alpha+2}(1-q)}{1-q^{\alpha+2}}, & (\lambda+q)\mu\le\lambda,\\ \frac{2(1-q)^2(\lambda-\lambda\mu)^{\alpha+2}}{(1-q^{\alpha+1})(1-q^{\alpha+2})}+\frac{q\mu^{\alpha+2}(1-q)}{1-q^{\alpha+2}}-\frac{\mu^{\alpha+1}(1-q)(\lambda-\lambda\mu)}{1-q^{\alpha+1}}, & (\lambda+q)\mu>\lambda,\end{cases}\\ \Phi_2(\lambda,\mu,\alpha) &= \int_0^1 t^\alpha\big|qt-(1-\lambda\mu)\big|\,{}_0d_qt\\&=\begin{cases}\frac{(1-q)(1-\lambda\mu)}{1-q^{\alpha+1}}-\frac{q(1-q)}{1-q^{\alpha+2}}, & \lambda\mu+q\le 1,\\ \frac{2(1-q)^2(1-\lambda\mu)^{\alpha+2}}{(1-q^{\alpha+1})(1-q^{\alpha+2})}+\frac{q(1-q)}{1-q^{\alpha+2}}-\frac{(1-q)(1-\lambda\mu)}{1-q^{\alpha+1}}, & \lambda\mu+q>1,\end{cases}\\ \Phi_3(\lambda,\mu,\alpha) &= \int_0^\mu t^\alpha\big|qt-(1-\lambda\mu)\big|\,{}_0d_qt\\&=\begin{cases}\frac{\mu^{\alpha+1}(1-\lambda\mu)(1-q)}{1-q^{\alpha+1}}-\frac{q\mu^{\alpha+2}(1-q)}{1-q^{\alpha+2}}, & (\lambda+q)\mu\le 1,\\ \frac{2(1-q)^2(1-\lambda\mu)^{\alpha+2}}{(1-q^{\alpha+1})(1-q^{\alpha+2})}+\frac{q\mu^{\alpha+2}(1-q)}{1-q^{\alpha+2}}-\frac{\mu^{\alpha+1}(1-\lambda\mu)(1-q)}{1-q^{\alpha+1}}, & (\lambda+q)\mu>1,\end{cases}\end{aligned} \tag{3.1}$$

$$\begin{aligned}\Phi_4(\lambda,\mu) &= \int_0^\mu \big|qt-(\lambda-\lambda\mu)\big|\,{}_0d_qt\\&=\begin{cases}\lambda\mu(1-\mu)-\frac{q\mu^2}{1+q}, & (\lambda+q)\mu\le\lambda,\\ \frac{2(\lambda-\lambda\mu)^2}{1+q}+\frac{q\mu^2}{1+q}-\lambda\mu(1-\mu), & (\lambda+q)\mu>\lambda,\end{cases}\\ \Phi_5(\lambda,\mu) &= \int_0^1 \big|qt-(1-\lambda\mu)\big|\,{}_0d_qt\\&=\begin{cases}\frac{1}{1+q}-\lambda\mu, & \lambda\mu+q\le 1,\\ \frac{2(1-\lambda\mu)^2}{1+q}+\lambda\mu-\frac{1}{1+q}, & \lambda\mu+q>1,\end{cases}\\ \Phi_6(\lambda,\mu) &= \int_0^\mu \big|qt-(1-\lambda\mu)\big|\,{}_0d_qt\\&=\begin{cases}\mu(1-\lambda\mu)-\frac{q\mu^2}{1+q}, & (\lambda+q)\mu\le 1,\\ \frac{2(1-\lambda\mu)^2}{1+q}+\frac{q\mu^2}{1+q}-\mu(1-\lambda\mu), & (\lambda+q)\mu>1,\end{cases}\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\Phi_7(\lambda,\mu,\alpha)\\&\quad=\int_0^\mu(1-t)^\alpha\big|qt-(\lambda-\lambda\mu)\big|\,{}_0d_qt\\&\quad=\begin{cases}(1-q)\mu\sum_{n=0}^\infty q^n(\lambda-\lambda\mu-q^{n+1}\mu)(1-q^n\mu)^\alpha, & (\lambda+q)\mu\le\lambda,\\ \left[\begin{array}{c}2(1-q)(\lambda-\lambda\mu)^2\sum_{n=0}^\infty q^{n-1}(1-q^n)[1-q^{n-1}(\lambda-\lambda\mu)]^\alpha\\ -(1-q)\mu\sum_{n=0}^\infty q^n(\lambda-\lambda\mu-q^{n+1}\mu)(1-q^n\mu)^\alpha\end{array}\right], & (\lambda+q)\mu>\lambda,\end{cases}\end{aligned}$$

$$\begin{aligned}
&\Phi_8(\lambda,\mu,\alpha) \\
&\quad = \int_0^1 (1-t)^\alpha \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t \\
&\quad = \begin{cases} (1-q)\sum_{n=0}^{\infty} q^n(1-\lambda\mu-q^{n+1})(1-q^n)^\alpha, & \lambda\mu+q\le 1, \\ \begin{bmatrix} 2(1-q)(1-\lambda\mu)^2\sum_{n=0}^{\infty} q^{n-1}(1-q^n)[1-q^{n-1}(1-\lambda\mu)]^\alpha \\ -(1-q)\sum_{n=0}^{\infty} q^n(1-\lambda\mu-q^{n+1})(1-q^n)^\alpha \end{bmatrix}, & \lambda\mu+q>1, \end{cases}
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\Phi_9(\lambda,\mu,\alpha) \\
&\quad = \int_0^\mu (1-t)^\alpha \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t \\
&\quad = \begin{cases} (1-q)\mu\sum_{n=0}^{\infty} q^n(1-\lambda\mu-q^{n+1}\mu)(1-q^n\mu)^\alpha, & (\lambda+q)\mu\le 1, \\ \begin{bmatrix} 2(1-q)(1-\lambda\mu)^2\sum_{n=0}^{\infty} q^{n-1}(1-q^n)[1-q^{n-1}(1-\lambda\mu)]^\alpha \\ -(1-q)\mu\sum_{n=0}^{\infty} q^n(1-\lambda\mu-q^{n+1}\mu)(1-q^n\mu)^\alpha \end{bmatrix}, & (\lambda+q)\mu>1. \end{cases}
\end{aligned}$$

Proof From Lemma 2.1, utilizing the property of the modulus and the (α,m)-convexity of $|{}_aD_qf|$, we have

$$\begin{aligned}
&\left|\lambda\big[\mu f(b)+(1-\mu)f(a)\big]+(1-\lambda)f\big(\mu b+(1-\mu)a\big)-\frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x\right| \\
&\quad \le (b-a)\biggl\{\int_0^\mu |qt+\lambda\mu-\lambda|\left|{}_aD_qf\big(tb+(1-t)a\big)\right| {}_0\mathrm{d}_q t \\
&\qquad + \int_\mu^1 |qt+\lambda\mu-1|\left|{}_aD_qf\big(tb+(1-t)a\big)\right| {}_0\mathrm{d}_q t\biggr\} \\
&\quad \le (b-a)\biggl\{\int_0^\mu \left|qt-(\lambda-\lambda\mu)\right|\left[t^\alpha\left|{}_aD_qf(b)\right|+m\big(1-t^\alpha\big)\left|{}_aD_qf\left(\frac{a}{m}\right)\right|\right] {}_0\mathrm{d}_q t \\
&\qquad + \int_\mu^1 \left|qt-(1-\lambda\mu)\right|\left[t^\alpha\left|{}_aD_qf(b)\right|+m\big(1-t^\alpha\big)\left|{}_aD_qf\left(\frac{a}{m}\right)\right|\right] {}_0\mathrm{d}_q t\biggr\} \\
&\quad = (b-a)\biggl\{\int_0^\mu \left|qt-(\lambda-\lambda\mu)\right|\left[t^\alpha\left|{}_aD_qf(b)\right|+m\big(1-t^\alpha\big)\left|{}_aD_qf\left(\frac{a}{m}\right)\right|\right] {}_0\mathrm{d}_q t \\
&\qquad + \int_0^1 \left|qt-(1-\lambda\mu)\right|\left[t^\alpha\left|{}_aD_qf(b)\right|+m\big(1-t^\alpha\big)\left|{}_aD_qf\left(\frac{a}{m}\right)\right|\right] {}_0\mathrm{d}_q t \\
&\qquad - \int_0^\mu \left|qt-(1-\lambda\mu)\right|\left[t^\alpha\left|{}_aD_qf(b)\right|+m\big(1-t^\alpha\big)\left|{}_aD_qf\left(\frac{a}{m}\right)\right|\right] {}_0\mathrm{d}_q t\biggr\} \\
&\quad = (b-a)\biggl\{\biggl[\int_0^\mu t^\alpha\left|qt-(\lambda-\lambda\mu)\right| {}_0\mathrm{d}_q t+\int_0^1 t^\alpha\left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t \\
&\qquad - \int_0^\mu t^\alpha\left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t\biggr]\left|{}_aD_qf(b)\right| \\
&\qquad + m\biggl[\int_0^\mu \left|qt-(\lambda-\lambda\mu)\right| {}_0\mathrm{d}_q t+\int_0^1 \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t
\end{aligned}$$

$$- \int_0^{\mu} \left| qt - (1-\lambda\mu) \right| {}_0\mathrm{d}_q t - \int_0^{\mu} t^{\alpha} \left| qt - (\lambda - \lambda\mu) \right| {}_0\mathrm{d}_q t$$
$$- \int_0^{1} t^{\alpha} \left| qt - (1-\lambda\mu) \right| {}_0\mathrm{d}_q t + \int_0^{\mu} t^{\alpha} \left| qt - (1-\lambda\mu) \right| {}_0\mathrm{d}_q t \Bigg] \left| {}_aD_q f\left(\frac{a}{m}\right) \right| \Bigg\}.$$

Similarly, we get

$$\left| \lambda\big[\mu f(b) + (1-\mu) f(a)\big] + (1-\lambda) f\big(\mu b + (1-\mu)a\big) - \frac{1}{b-a} \int_a^b f(x)\, {}_a\mathrm{d}_q x \right|$$
$$\leq (b-a) \Bigg\{ \Bigg[\int_0^{\mu} (1-t)^{\alpha} \left| qt - (\lambda - \lambda\mu) \right| {}_0\mathrm{d}_q t + \int_0^{1} (1-t)^{\alpha} \left| qt - (1-\lambda\mu) \right| {}_0\mathrm{d}_q t$$
$$- \int_0^{\mu} (1-t)^{\alpha} \left| qt - (1-\lambda\mu) \right| {}_0\mathrm{d}_q t \Bigg] \left| {}_aD_q f(a) \right|$$
$$+ m \Bigg[\int_0^{\mu} \left| qt - (\lambda - \lambda\mu) \right| {}_0\mathrm{d}_q t + \int_0^{1} \left| qt - (1-\lambda\mu) \right| {}_0\mathrm{d}_q t$$
$$- \int_0^{\mu} \left| qt - (1-\lambda\mu) \right| {}_0\mathrm{d}_q t - \int_0^{\mu} (1-t)^{\alpha} \left| qt - (\lambda - \lambda\mu) \right| {}_0\mathrm{d}_q t$$
$$- \int_0^{1} (1-t)^{\alpha} \left| qt - (1-\lambda\mu) \right| {}_0\mathrm{d}_q t + \int_0^{\mu} (1-t)^{\alpha} \left| qt - (1-\lambda\mu) \right| {}_0\mathrm{d}_q t \Bigg] \left| {}_aD_q f\left(\frac{b}{m}\right) \right| \Bigg\}.$$

Using Lemma 2.3, Lemma 2.4 and Lemma 2.5, we get the desired result. This completes the proof. □

Corollary 3.1 *In Theorem 3.2, putting $\mu = \frac{1}{1+q}$, we have*

$$\left| \lambda \frac{qf(a) + f(b)}{1+q} + (1-\lambda) f\left(\frac{qa+b}{1+q}\right) - \frac{1}{b-a} \int_a^b f(x)\, {}_a\mathrm{d}_q x \right|$$
$$\leq \min\left\{ \mathcal{H}_1\left(\lambda, \frac{1}{1+q}, \alpha, m\right), \mathcal{H}_2\left(\lambda, \frac{1}{1+q}, \alpha, m\right) \right\}.$$

Remark 3.1 Consider Corollary 3.1.

(i) Putting $\lambda = 0$, we get the midpoint-like integral inequality

$$\left| f\left(\frac{qa+b}{1+q}\right) - \frac{1}{b-a} \int_a^b f(x)\, {}_a\mathrm{d}_q x \right|$$
$$\leq \min\left\{ \mathcal{H}_1\left(0, \frac{1}{1+q}, \alpha, m\right), \mathcal{H}_2\left(0, \frac{1}{1+q}, \alpha, m\right) \right\},$$

where

$$\mathcal{H}_1\left(0, \frac{1}{1+q}, \alpha, m\right)$$
$$= (b-a) \Bigg\{ \frac{[(1+q)^{\alpha+2} - (1+q^{\alpha+2})](1-q)^2}{(1+q)^{\alpha+2}(1-q^{\alpha+1})(1-q^{\alpha+2})} \left| {}_aD_q f(b) \right|$$
$$+ m \Bigg[\frac{2q}{(1+q)^3} - \frac{[(1+q)^{\alpha+2} - (1+q^{\alpha+2})](1-q)^2}{(1+q)^{\alpha+2}(1-q^{\alpha+1})(1-q^{\alpha+2})} \Bigg] \left| {}_aD_q f\left(\frac{a}{m}\right) \right| \Bigg\}$$

and

$$\begin{aligned}&\mathcal{H}_2\left(0,\frac{1}{1+q},\alpha,m\right)\\&\quad=(b-a)\left\{\left[\Phi_7\left(0,\frac{1}{1+q},\alpha\right)+\Phi_8\left(0,\frac{1}{1+q},\alpha\right)-\Phi_9\left(0,\frac{1}{1+q},\alpha\right)\right]\left|{}_aD_qf(a)\right|\right.\\&\qquad+m\left[\frac{2q}{(1+q)^3}-\Phi_7\left(0,\frac{1}{1+q},\alpha\right)-\Phi_8\left(0,\frac{1}{1+q},\alpha\right)\right.\\&\qquad\left.\left.+\Phi_9\left(0,\frac{1}{1+q},\alpha\right)\right]\left|{}_aD_qf\left(\frac{b}{m}\right)\right|\right\}.\end{aligned}$$

Specially, taking $\alpha=1=m$, we obtain

$$\begin{aligned}&\left|f\left(\frac{qa+b}{1+q}\right)-\frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_qx\right|\\&\quad\le(b-a)\left\{\frac{3q}{(1+q)^3(1+q+q^2)}\left|{}_aD_qf(b)\right|+\frac{-q+2q^2+2q^3}{(1+q)^3(1+q+q^2)}\left|{}_aD_qf(a)\right|\right\},\end{aligned}$$

which is established by Alp et al. in [2, Theorem 13].

(ii) Putting $\lambda=\frac{1}{3}$ and $\alpha=1=m$, we get the Simpson-like integral inequality

$$\begin{aligned}&\left|\frac{1}{3}\left[\frac{qf(a)+f(b)}{1+q}+2f\left(\frac{qa+b}{1+q}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_qx\right|\\&\quad\le\min\left\{\mathcal{H}_1\left(\frac{1}{3},\frac{1}{1+q},1,1\right),\mathcal{H}_2\left(\frac{1}{3},\frac{1}{1+q},1,1\right)\right\}.\end{aligned}$$

Specially, if $q\to1^-$, then we obtain

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{5(b-a)}{72}\left[\left|f'(b)\right|+\left|f'(a)\right|\right],$$

which is established by Alomari et al. in [1, Corollary 1].

(iii) Putting $\lambda=\frac{1}{2}$ and $\alpha=1=m$, we get the averaged midpoint-trapezoid-like integral inequality

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{qf(a)+f(b)}{1+q}+f\left(\frac{qa+b}{1+q}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_qx\right|\\&\quad\le\min\left\{\mathcal{H}_1\left(\frac{1}{2},\frac{1}{1+q},1,1\right),\mathcal{H}_2\left(\frac{1}{2},\frac{1}{1+q},1,1\right)\right\}.\end{aligned}$$

Specially, if $q\to1^-$, then we obtain

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\frac{b-a}{16}\left[\left|f'(b)\right|+\left|f'(a)\right|\right],$$

which is established by Xi and Qi in [30, Corollary 3.4].

(iv) Putting $\lambda = 1$, we get the trapezoid-like integral inequality

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_qx\right|\\&\quad\le\min\left\{\mathcal{H}_1\left(1,\frac{1}{1+q},\alpha,m\right),\mathcal{H}_2\left(1,\frac{1}{1+q},\alpha,m\right)\right\},\end{aligned}$$

where

$$\begin{aligned}&\mathcal{H}_1\left(1,\frac{1}{1+q},\alpha,m\right)\\&\quad=(b-a)\left\{\frac{2q^{\alpha+2}(1-q)^2+q^2(1+q)^{\alpha+1}(1-q)(1-q^\alpha)}{(1+q)^{\alpha+2}(1-q^{\alpha+1})(1-q^{\alpha+2})}\left|{}_aD_qf(b)\right|\right.\\&\qquad\left.+m\left[\frac{2q^2}{(1+q)^3}-\frac{2q^{\alpha+2}(1-q)^2+q^2(1+q)^{\alpha+1}(1-q)(1-q^\alpha)}{(1+q)^{\alpha+2}(1-q^{\alpha+1})(1-q^{\alpha+2})}\right]\left|{}_aD_qf\left(\frac{a}{m}\right)\right|\right\}\end{aligned}$$

and

$$\begin{aligned}&\mathcal{H}_2\left(1,\frac{1}{1+q},\alpha,m\right)\\&\quad=(b-a)\left\{\left[\Phi_7\left(1,\frac{1}{1+q},\alpha\right)+\Phi_8\left(1,\frac{1}{1+q},\alpha\right)-\Phi_9\left(1,\frac{1}{1+q},\alpha\right)\right]\left|{}_aD_qf(a)\right|\right.\\&\qquad+m\left[\frac{2q^2}{(1+q)^3}-\Phi_7\left(1,\frac{1}{1+q},\alpha\right)-\Phi_8\left(1,\frac{1}{1+q},\alpha\right)\right.\\&\qquad\left.\left.+\Phi_9\left(1,\frac{1}{1+q},\alpha\right)\right]\left|{}_aD_qf\left(\frac{b}{m}\right)\right|\right\}.\end{aligned}$$

Specially, taking $\alpha = 1 = m$, we obtain

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_qx\right|\\&\quad\le(b-a)\left\{\frac{q^2(1+4q+q^2)}{(1+q)^4(1+q+q^2)}\left|{}_aD_qf(b)\right|+\frac{q^2(1+3q^2+2q^3)}{(1+q)^4(1+q+q^2)}\left|{}_aD_qf(a)\right|\right\},\end{aligned}$$

which is established by Sudsutad et al. in [26, Theorem 4.1].

If $|{}_aD_qf|^r$ for $r \ge 1$ is (α, m)-convex, then the following theorem can be obtained.

Theorem 3.3 *For $0 \le a < b$ and some fixed $m \in (0,1]$, let $f : [a, \frac{b}{m}] \to \mathbb{R}$ be a continuous and q-differentiable function on $(a, \frac{b}{m})$, and let ${}_aD_qf$ be integrable on $[a, \frac{b}{m}]$ with $0 < q < 1$. Then the inequality*

$$\begin{aligned}&\left|\lambda\bigl[\mu f(b)+(1-\mu)f(a)\bigr]+(1-\lambda)f\bigl(\mu b+(1-\mu)a\bigr)-\frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_qx\right|\\&\quad\le(b-a)\min\bigl\{\mathcal{J}_1(\lambda,\mu,\alpha,m,r),\mathcal{J}_2(\lambda,\mu,\alpha,m,r)\bigr\}\end{aligned}$$

holds for all $\lambda, \mu \in [0,1]$ *if* $|{}_aD_qf|^r$ *for* $r \geq 1$ *is* (α, m)*-convex on* $[a, \frac{b}{m}]$ *with* $\alpha, m \in (0,1]^2$, *where*

$$\begin{aligned}
&\mathcal{J}_1(\lambda,\mu,\alpha,m,r) \\
&\quad = \Phi_5^{1-\frac{1}{r}}(\lambda,\mu)\biggl[\Phi_2(\lambda,\mu,\alpha)\big|{}_aD_qf(b)\big|^r + m\bigl(\Phi_5(\lambda,\mu)-\Phi_2(\lambda,\mu,\alpha)\bigr)\biggl|{}_aD_qf\biggl(\frac{a}{m}\biggr)\biggr|^r\biggr]^{\frac{1}{r}} \\
&\qquad + (1-\lambda)\mu^{1-\frac{1}{r}}\biggl[\Upsilon_1(\mu,\alpha)\big|{}_aD_qf(b)\big|^r + m\Upsilon_2(\mu,\alpha)\biggl|{}_aD_qf\biggl(\frac{a}{m}\biggr)\biggr|^r\biggr]^{\frac{1}{r}}, \\
&\mathcal{J}_2(\lambda,\mu,\alpha,m,r) \\
&\quad = \Phi_5^{1-\frac{1}{r}}(\lambda,\mu)\biggl[\Phi_8(\lambda,\mu,\alpha)\big|{}_aD_qf(a)\big|^r + m\bigl(\Phi_5(\lambda,\mu)-\Phi_8(\lambda,\mu,\alpha)\bigr)\biggl|{}_aD_qf\biggl(\frac{b}{m}\biggr)\biggr|^r\biggr]^{\frac{1}{r}} \\
&\qquad + (1-\lambda)\mu^{1-\frac{1}{r}}\biggl[\Upsilon_3(\mu,\alpha)\big|{}_aD_qf(a)\big|^r + m\Upsilon_4(\mu,\alpha)\biggl|{}_aD_qf\biggl(\frac{b}{m}\biggr)\biggr|^r\biggr]^{\frac{1}{r}},
\end{aligned}$$

$$\Upsilon_1(\mu,\alpha) = \int_0^\mu t^\alpha \, {}_0d_qt = \frac{\mu^{\alpha+1}(1-q)}{1-q^{\alpha+1}}, \tag{3.4}$$

$$\Upsilon_2(\mu,\alpha) = \int_0^\mu \bigl(1-t^\alpha\bigr) \, {}_0d_qt = \mu - \frac{\mu^{\alpha+1}(1-q)}{1-q^{\alpha+1}}, \tag{3.5}$$

$$\Upsilon_3(\mu,\alpha) = \int_0^\mu (1-t)^\alpha \, {}_0d_qt = (1-q)\mu\sum_{n=0}^{\infty} q^n\bigl(1-q^n\mu\bigr)^\alpha, \tag{3.6}$$

$$\Upsilon_4(\mu,\alpha) = \int_0^\mu \bigl(1-(1-t)^\alpha\bigr) \, {}_0d_qt = \mu - (1-q)\mu\sum_{n=0}^{\infty} q^n\bigl(1-q^n\mu\bigr)^\alpha, \tag{3.7}$$

and $\Phi_2(\lambda,\mu,\alpha)$, $\Phi_5(\lambda,\mu)$, $\Phi_8(\lambda,\mu,\alpha)$ *are defined by* (3.1), (3.2) *and* (3.3), *respectively.*

Proof Using Lemma 2.1 and the power mean inequality, we have

$$\begin{aligned}
&\biggl|\lambda\bigl[\mu f(b)+(1-\mu)f(a)\bigr] + (1-\lambda)f\bigl(\mu b+(1-\mu)a\bigr) - \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\biggr| \\
&\quad \leq (b-a)\biggl\{\biggl(\int_0^1 \big|qt-(1-\lambda\mu)\big|\,{}_0d_qt\biggr)^{1-\frac{1}{r}} \\
&\qquad \times \biggl(\int_0^1 \big|qt-(1-\lambda\mu)\big|\big|{}_aD_qf\bigl(tb+(1-t)a\bigr)\big|^r\,{}_0d_qt\biggr)^{\frac{1}{r}} \\
&\qquad + (1-\lambda)\biggl(\int_0^\mu 1\,{}_0d_qt\biggr)^{1-\frac{1}{r}}\biggl(\int_0^\mu \big|{}_aD_qf\bigl(tb+(1-t)a\bigr)\big|^r\,{}_0d_qt\biggr)^{\frac{1}{r}}\biggr\}.
\end{aligned} \tag{3.8}$$

Utilizing the (α, m)-convexity of $|{}_aD_qf|^r$, we get

$$\begin{aligned}
&\int_0^1 \big|qt-(1-\lambda\mu)\big|\big|{}_aD_qf\bigl(tb+(1-t)a\bigr)\big|^r\,{}_0d_qt \\
&\quad \leq \int_0^1 \big|qt-(1-\lambda\mu)\big|\biggl[t^\alpha\big|{}_aD_qf(b)\big|^r + m\bigl(1-t^\alpha\bigr)\biggl|{}_aD_qf\biggl(\frac{a}{m}\biggr)\biggr|^r\biggr]\,{}_0d_qt
\end{aligned}$$

$$= \left(\int_0^1 t^\alpha \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t\right) \left|{}_aD_q f(b)\right|^r + m\left(\int_0^1 \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t - \int_0^1 t^\alpha \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t\right) \left|{}_aD_q f\left(\frac{a}{m}\right)\right|^r \tag{3.9}$$

and

$$\begin{aligned} &\int_0^\mu \left|{}_aD_q f\big(tb+(1-t)a\big)\right|^r {}_0\mathrm{d}_q t \\ &\quad\leq \int_0^\mu \left[t^\alpha \left|{}_aD_q f(b)\right|^r + m\big(1-t^\alpha\big)\left|{}_aD_q f\left(\frac{a}{m}\right)\right|^r\right] {}_0\mathrm{d}_q t \\ &\quad= \left(\int_0^\mu t^\alpha {}_0\mathrm{d}_q t\right)\left|{}_aD_q f(b)\right|^r + m\left(\int_0^\mu \big(1-t^\alpha\big) {}_0\mathrm{d}_q t\right)\left|{}_aD_q f\left(\frac{a}{m}\right)\right|^r. \end{aligned} \tag{3.10}$$

Using (3.9) and (3.10) in (3.8), we get

$$\begin{aligned} &\left|\lambda\big[\mu f(b)+(1-\mu)f(a)\big]+(1-\lambda)f\big(\mu b+(1-\mu)a\big)-\frac{1}{b-a}\int_a^b f(x)\, {}_a\mathrm{d}_q x\right| \\ &\quad\leq (b-a)\left\{\left[\int_0^1 \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t\right]^{1-\frac{1}{r}}\right. \\ &\qquad\times\left[\left(\int_0^1 t^\alpha \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t\right)\left|{}_aD_q f(b)\right|^r\right. \\ &\qquad\left.+ m\left(\int_0^1 \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t - \int_0^1 t^\alpha \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t\right)\left|{}_aD_q f\left(\frac{a}{m}\right)\right|^r\right]^{\frac{1}{r}} \\ &\qquad+ (1-\lambda)\mu^{1-\frac{1}{r}}\left[\left(\int_0^\mu t^\alpha {}_0\mathrm{d}_q t\right)\left|{}_aD_q f(b)\right|^r\right. \\ &\qquad\left.\left.+ m\left(\int_0^\mu \big(1-t^\alpha\big) {}_0\mathrm{d}_q t\right)\left|{}_aD_q f\left(\frac{a}{m}\right)\right|^r\right]^{\frac{1}{r}}\right\}. \end{aligned} \tag{3.11}$$

Similarly, we get

$$\begin{aligned} &\int_0^1 \left|qt-(1-\lambda\mu)\right|\left|{}_aD_q f\big(tb+(1-t)a\big)\right|^r {}_0\mathrm{d}_q t \\ &\quad\leq \int_0^1 \left|qt-(1-\lambda\mu)\right|\left[(1-t)^\alpha\left|{}_aD_q f(a)\right|^r + m\big(1-(1-t)^\alpha\big)\left|{}_aD_q f\left(\frac{b}{m}\right)\right|^r\right] {}_0\mathrm{d}_q t \\ &\quad= \left(\int_0^1 (1-t)^\alpha \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t\right)\left|{}_aD_q f(a)\right|^r \\ &\qquad+ m\left(\int_0^1 \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t - \int_0^1 (1-t)^\alpha \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t\right) \\ &\qquad\times\left|{}_aD_q f\left(\frac{b}{m}\right)\right|^r \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &\int_0^\mu \left|{}_aD_q f\big(tb+(1-t)a\big)\right|^r {}_0\mathrm{d}_q t \\ &\quad\leq \int_0^\mu \left[(1-t)^\alpha\left|{}_aD_q f(a)\right|^r + m\big(1-(1-t)^\alpha\big)\left|{}_aD_q f\left(\frac{b}{m}\right)\right|^r\right] {}_0\mathrm{d}_q t \end{aligned}$$

$$= \left(\int_0^{\mu} (1-t)^{\alpha} \, {}_0\mathrm{d}_q t\right) \left|{}_aD_q f(a)\right|^r + m\left(\int_0^{\mu} \left(1-(1-t)^{\alpha}\right) {}_0\mathrm{d}_q t\right) \left|{}_aD_q f\left(\frac{b}{m}\right)\right|^r. \tag{3.13}$$

Using (3.12) and (3.13) in (3.8), we get

$$\begin{aligned}
&\left|\lambda\left[\mu f(b) + (1-\mu) f(a)\right] + (1-\lambda) f\left(\mu b + (1-\mu)a\right) - \frac{1}{b-a}\int_a^b f(x) \, {}_a\mathrm{d}_q x\right| \\
&\quad \le (b-a)\left\{\left[\int_0^1 \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t\right]^{1-\frac{1}{r}} \right. \\
&\qquad \times \left[\left(\int_0^1 (1-t)^{\alpha}\left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t\right)\left|{}_aD_q f(a)\right|^r \right. \\
&\qquad \left. + m\left(\int_0^1 \left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t - \int_0^1 (1-t)^{\alpha}\left|qt-(1-\lambda\mu)\right| {}_0\mathrm{d}_q t\right)\left|{}_aD_q f\left(\frac{b}{m}\right)\right|^r\right]^{\frac{1}{r}} \\
&\qquad + (1-\lambda)\mu^{1-\frac{1}{r}}\left[\left(\int_0^{\mu} (1-t)^{\alpha} \, {}_0\mathrm{d}_q t\right)\left|{}_aD_q f(a)\right|^r \right. \\
&\qquad \left.\left. + m\left(\int_0^{\mu}\left(1-(1-t)^{\alpha}\right) {}_0\mathrm{d}_q t\right)\left|{}_aD_q f\left(\frac{b}{m}\right)\right|^r\right]^{\frac{1}{r}}\right\}.
\end{aligned} \tag{3.14}$$

From (3.11) and (3.14), utilizing (3.1), (3.2), (3.3) and Lemma 2.2, we can deduce the desired result. The proof is complete. □

If $|{}_aD_q f|^r$ for $r > 1$ is (α, m)-convex, then the following theorem can be obtained.

Theorem 3.4 *For* $0 \le a < b$ *and some fixed* $m \in (0,1]$, *let* $f : [a, \frac{b}{m}] \to \mathbb{R}$ *be a continuous and q-differentiable function on* $(a, \frac{b}{m})$, *and let* ${}_aD_q f$ *be integrable on* $[a, \frac{b}{m}]$ *with* $0 < q < 1$. *Then the inequality*

$$\begin{aligned}
&\left|\lambda\left[\mu f(b) + (1-\mu) f(a)\right] + (1-\lambda) f\left(\mu b + (1-\mu)a\right) - \frac{1}{b-a}\int_a^b f(x) \, {}_a\mathrm{d}_q x\right| \\
&\quad \le (b-a)\min\left\{\mathcal{K}_1(\lambda,\mu,\alpha,m), \mathcal{K}_2(\lambda,\mu,\alpha,m)\right\}
\end{aligned}$$

holds for all $\lambda, \mu \in [0,1]$ *if* $|{}_aD_q f|^r$ *for* $r > 1$ *with* $r^{-1} + s^{-1} = 1$ *is* (α, m)*-convex on* $[a, \frac{b}{m}]$ *with* $\alpha, m \in (0,1]^2$, *where*

$$\begin{aligned}
&\mathcal{K}_1(\lambda,\mu,\alpha,m) \\
&\quad = \Psi_1^{\frac{1}{s}}(\lambda,\mu)\left[\Psi_2(\alpha)\left|{}_aD_q f(b)\right|^r + m\left(1-\Psi_2(\alpha)\right)\left|{}_aD_q f\left(\frac{a}{m}\right)\right|^r\right]^{\frac{1}{r}} \\
&\qquad + (1-\lambda)\mu^{\frac{1}{s}}\left[\Upsilon_1(\mu,\alpha)\left|{}_aD_q f(b)\right|^r + m\Upsilon_2(\mu,\alpha)\left|{}_aD_q f\left(\frac{a}{m}\right)\right|^r\right]^{\frac{1}{r}},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{K}_2(\lambda,\mu,\alpha,m)\\
&\quad= \Psi_1^{\frac{1}{s}}(\lambda,\mu)\left[\Psi_3(\alpha)\left|{}_aD_qf(a)\right|^r + m\bigl(1-\Psi_3(\alpha)\bigr)\left|{}_aD_qf\left(\frac{b}{m}\right)\right|^r\right]^{\frac{1}{r}}\\
&\qquad+(1-\lambda)\mu^{\frac{1}{s}}\left[\Upsilon_3(\mu,\alpha)\left|{}_aD_qf(a)\right|^r + m\Upsilon_4(\mu,\alpha)\left|{}_aD_qf\left(\frac{b}{m}\right)\right|^r\right]^{\frac{1}{r}},\\
&\Psi_1(\lambda,\mu) = \int_0^1 \left|qt-(1-\lambda\mu)\right|^s {}_0d_qt\\
&\quad= \begin{cases} (1-q)\sum_{n=0}^{\infty} q^n(1-\lambda\mu-q^{n+1})^s, & 0\le\lambda\mu\le 1-q,\\ \begin{bmatrix} (1-q)(1-\lambda\mu)^{s+1}\sum_{n=0}^{\infty} q^{n-1}(1-q^n)^s\\ +(1-q)\sum_{n=0}^{\infty} q^n(q^{n+1}-1+\lambda\mu)^s\\ -(1-q)(1-\lambda\mu)^{s+1}\sum_{n=0}^{\infty} q^{n-1}(q^n-1)^s \end{bmatrix}, & 1-q<\lambda\mu\le 1,\end{cases}\\
&\Psi_2(\alpha) = \int_0^1 t^\alpha\, {}_0d_qt = \frac{1-q}{1-q^{\alpha+1}},\\
&\Psi_3(\alpha) = \int_0^1 (1-t)^\alpha\, {}_0d_qt = (1-q)\sum_{n=0}^{\infty} q^n\bigl(1-q^n\bigr)^\alpha,
\end{aligned}$$

and $\Upsilon_1(\mu,\alpha)$, $\Upsilon_2(\mu,\alpha)$, $\Upsilon_3(\mu,\alpha)$, $\Upsilon_4(\mu,\alpha)$ *are defined by* (3.4), (3.5), (3.6) *and* (3.7), *respectively.*

Proof Using Lemma 2.1 and the Hölder inequality, we have

$$\begin{aligned}
&\left|\lambda\bigl[\mu f(b)+(1-\mu)f(a)\bigr]+(1-\lambda)f\bigl(\mu b+(1-\mu)a\bigr)-\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right|\\
&\quad\le(b-a)\left\{\left(\int_0^1\left|qt-(1-\lambda\mu)\right|^s{}_0d_qt\right)^{\frac{1}{s}}\right.\\
&\qquad\times\left(\int_0^1\left|{}_aD_qf\bigl(tb+(1-t)a\bigr)\right|^r{}_0d_qt\right)^{\frac{1}{r}}\\
&\qquad\left.+(1-\lambda)\left(\int_0^\mu 1^s\,{}_0d_qt\right)^{\frac{1}{s}}\left(\int_0^\mu\left|{}_aD_qf\bigl(tb+(1-t)a\bigr)\right|^r{}_0d_qt\right)^{\frac{1}{r}}\right\}.
\end{aligned}\tag{3.15}$$

Utilizing the (α,m)-convexity of $|{}_aD_qf|^r$, we get

$$\begin{aligned}
&\int_0^1\left|{}_aD_qf\bigl(tb+(1-t)a\bigr)\right|^r{}_0d_qt\\
&\quad\le\int_0^1\left[t^\alpha\left|{}_aD_qf(b)\right|^r+m\bigl(1-t^\alpha\bigr)\left|{}_aD_qf\left(\frac{a}{m}\right)\right|^r\right]{}_0d_qt\\
&\quad=\left(\int_0^1 t^\alpha\,{}_0d_qt\right)\left|{}_aD_qf(b)\right|^r+m\left(\int_0^1\bigl(1-t^\alpha\bigr)\,{}_0d_qt\right)\left|{}_aD_qf\left(\frac{a}{m}\right)\right|^r
\end{aligned}\tag{3.16}$$

and

$$\int_0^\mu\left|{}_aD_qf\bigl(tb+(1-t)a\bigr)\right|^r{}_0d_qt$$

$$\begin{aligned}
&\leq \int_0^{\mu}\left[t^{\alpha}\left|{}_aD_qf(b)\right|^r + m\left(1-t^{\alpha}\right)\left|{}_aD_qf\left(\frac{a}{m}\right)\right|^r\right]{}_0d_qt \\
&= \left(\int_0^{\mu} t^{\alpha}\,{}_0d_qt\right)\left|{}_aD_qf(b)\right|^r + m\left(\int_0^{\mu}\left(1-t^{\alpha}\right){}_0d_qt\right)\left|{}_aD_qf\left(\frac{a}{m}\right)\right|^r.
\end{aligned} \tag{3.17}$$

Using (3.16) and (3.17) in (3.15), we get

$$\begin{aligned}
&\left|\lambda\left[\mu f(b)+(1-\mu)f(a)\right]+(1-\lambda)f\left(\mu b+(1-\mu)a\right)-\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right| \\
&\quad\leq (b-a)\left\{\left(\int_0^1\left|qt-(1-\lambda\mu)\right|^s{}_0d_qt\right)^{\frac{1}{s}}\right. \\
&\qquad\times\left[\left(\int_0^1 t^{\alpha}\,{}_0d_qt\right)\left|{}_aD_qf(b)\right|^r + m\left(\int_0^1\left(1-t^{\alpha}\right){}_0d_qt\right)\left|{}_aD_qf\left(\frac{a}{m}\right)\right|^r\right]^{\frac{1}{r}} \\
&\qquad+(1-\lambda)\mu^{\frac{1}{s}}\left[\left(\int_0^{\mu} t^{\alpha}\,{}_0d_qt\right)\left|{}_aD_qf(b)\right|^r\right. \\
&\qquad\left.\left.+m\left(\int_0^{\mu}\left(1-t^{\alpha}\right){}_0d_qt\right)\left|{}_aD_qf\left(\frac{a}{m}\right)\right|^r\right]^{\frac{1}{r}}\right\}.
\end{aligned} \tag{3.18}$$

Similarly, we get

$$\begin{aligned}
&\int_0^1\left|{}_aD_qf\left(tb+(1-t)a\right)\right|^r{}_0d_qt \\
&\quad\leq \int_0^1\left[(1-t)^{\alpha}\left|{}_aD_qf(a)\right|^r + m\left(1-(1-t)^{\alpha}\right)\left|{}_aD_qf\left(\frac{b}{m}\right)\right|^r\right]{}_0d_qt \\
&\quad= \left(\int_0^1 (1-t)^{\alpha}\,{}_0d_qt\right)\left|{}_aD_qf(a)\right|^r \\
&\qquad+ m\left(\int_0^1\left(1-(1-t)^{\alpha}\right){}_0d_qt\right)\left|{}_aD_qf\left(\frac{b}{m}\right)\right|^r
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
&\int_0^{\mu}\left|{}_aD_qf\left(tb+(1-t)a\right)\right|^r{}_0d_qt \\
&\quad\leq \int_0^{\mu}\left[(1-t)^{\alpha}\left|{}_aD_qf(a)\right|^r + m\left(1-(1-t)^{\alpha}\right)\left|{}_aD_qf\left(\frac{b}{m}\right)\right|^r\right]{}_0d_qt \\
&\quad= \left(\int_0^{\mu} (1-t)^{\alpha}\,{}_0d_qt\right)\left|{}_aD_qf(a)\right|^r \\
&\qquad+ m\left(\int_0^{\mu}\left(1-(1-t)^{\alpha}\right){}_0d_qt\right)\left|{}_aD_qf\left(\frac{b}{m}\right)\right|^r.
\end{aligned} \tag{3.20}$$

Using (3.19) and (3.20) in (3.15), we get

$$\begin{aligned}
&\left|\lambda\left[\mu f(b)+(1-\mu)f(a)\right]+(1-\lambda)f\left(\mu b+(1-\mu)a\right)-\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right| \\
&\quad\leq (b-a)\left\{\left(\int_0^1\left|qt-(1-\lambda\mu)\right|^s{}_0d_qt\right)^{\frac{1}{s}}\right.
\end{aligned}$$

$$\times \left[\left(\int_0^1 (1-t)^\alpha \, {}_0\mathrm{d}_q t \right) \left| {}_aD_q f(a) \right|^r + m \left(\int_0^1 \left(1-(1-t)^\alpha\right) {}_0\mathrm{d}_q t \right) \left| {}_aD_q f\left(\frac{b}{m}\right) \right|^r \right]^{\frac{1}{r}} + (1-\lambda)\mu^{\frac{1}{s}} \left[\left(\int_0^\mu (1-t)^\alpha \, {}_0\mathrm{d}_q t \right) \left| {}_aD_q f(a) \right|^r + m \left(\int_0^\mu \left(1-(1-t)^\alpha\right) {}_0\mathrm{d}_q t \right) \left| {}_aD_q f\left(\frac{b}{m}\right) \right|^r \right]^{\frac{1}{r}} \Bigg\}. \tag{3.21}$$

From (3.18) and (3.21), utilizing (3.4), (3.5), (3.6), (3.7), Lemma 2.2 and Lemma 2.6, we can deduce the desired result. The proof is completed. □

Remark 3.2 For $\mu = \frac{1}{1+q}$, if we put $\lambda = 0$, $\lambda = \frac{1}{3}$, $\lambda = \frac{1}{2}$ and $\lambda = 1$ in Theorem 3.3 and Theorem 3.4, then we can get the midpoint-like integral inequality, the Simpson-like integral inequality, the averaged midpoint-trapezoid-like integral inequality and the trapezoid-like integral inequality, respectively.

Next we establish the q-integral inequalities involving the product of two (α, m)-convex functions.

Theorem 3.5 *For $0 \le a < b$ and some fixed $m \in (0,1]$, let $f, g : [a, \frac{b}{m}] \to \mathbb{R}$ be continuous and nonnegative functions. Then the inequality*

$$\frac{1}{b-a} \int_a^b f(x) g(x) \, {}_a\mathrm{d}_q x \le \min\left\{ \mathcal{L}_1(\alpha_1, \alpha_2, m), \mathcal{L}_2(\alpha_1, \alpha_2, m) \right\}$$

holds if f and g are (α_1, m)-convex and (α_2, m)-convex on $[a, \frac{b}{m}]$ with $\alpha_1, \alpha_2 \in (0,1]^2$, respectively, where

$$\begin{aligned}
&\mathcal{L}_1(\alpha_1, \alpha_2, m) \\
&\quad = \left[\frac{1-q}{1-q^{\alpha_1+\alpha_2+1}} - \frac{1-q}{1-q^{\alpha_1+1}} - \frac{1-q}{1-q^{\alpha_2+1}} + 1 \right] m^2 f\left(\frac{a}{m}\right) g\left(\frac{a}{m}\right) \\
&\qquad + \frac{1-q}{1-q^{\alpha_1+\alpha_2+1}} f(b) g(b) + \left[\frac{1-q}{1-q^{\alpha_2+1}} - \frac{1-q}{1-q^{\alpha_1+\alpha_2+1}} \right] m f\left(\frac{a}{m}\right) g(b) \\
&\qquad + \left[\frac{1-q}{1-q^{\alpha_1+1}} - \frac{1-q}{1-q^{\alpha_1+\alpha_2+1}} \right] m f(b) g\left(\frac{a}{m}\right), \\
&\mathcal{L}_2(\alpha_1, \alpha_2, m) \\
&\quad = \left[\Theta(\alpha_1, \alpha_2) - \Theta(\alpha_1) - \Theta(\alpha_2) + 1 \right] m^2 f\left(\frac{b}{m}\right) g\left(\frac{b}{m}\right) \\
&\qquad + \Theta(\alpha_1, \alpha_2) f(a) g(a) + \left[\Theta(\alpha_1) - \Theta(\alpha_1, \alpha_2) \right] m f(a) g\left(\frac{b}{m}\right) \\
&\qquad + \left[\Theta(\alpha_2) - \Theta(\alpha_1, \alpha_2) \right] m f\left(\frac{b}{m}\right) g(a), \\
&\Theta(\alpha_1, \alpha_2) = \int_0^1 (1-t)^{\alpha_1+\alpha_2} \, {}_0\mathrm{d}_q t = (1-q) \sum_{n=0}^{\infty} q^n \left(1-q^n\right)^{\alpha_1+\alpha_2},
\end{aligned}$$

and

$$\Theta(\alpha_i) = \int_0^1 (1-t)^{\alpha_i}\,{}_0\mathrm{d}_q t = (1-q)\sum_{n=0}^{\infty} q^n\left(1-q^n\right)^{\alpha_i}, \quad i = 1,2.$$

Proof Using the (α_1, m)-convexity of f and the (α_2, m)-convexity of g, respectively, for all $t \in [0,1]$, we have

$$f\big(tb+(1-t)a\big) \le t^{\alpha_1} f(b) + m\big(1-t^{\alpha_1}\big) f\left(\frac{a}{m}\right) \tag{3.22}$$

and

$$g\big(tb+(1-t)a\big) \le t^{\alpha_2} g(b) + m\big(1-t^{\alpha_2}\big) g\left(\frac{a}{m}\right). \tag{3.23}$$

Multiplying (3.22) with (3.23), we get

$$\begin{aligned} &f\big(tb+(1-t)a\big)g\big(tb+(1-t)a\big) \\ &\quad\le t^{\alpha_1+\alpha_2} f(b)g(b) + \big(1-t^{\alpha_1}\big)\big(1-t^{\alpha_2}\big)m^2 f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right) \\ &\qquad + t^{\alpha_2}\big(1-t^{\alpha_1}\big)mf\left(\frac{a}{m}\right)g(b) + t^{\alpha_1}\big(1-t^{\alpha_2}\big)mf(b)g\left(\frac{a}{m}\right). \end{aligned} \tag{3.24}$$

Taking the q-integral for (3.24) with respect to t on $(0,1)$ and using Lemma 2.2, we obtain

$$\begin{aligned} &\int_0^1 f\big(tb+(1-t)a\big)g\big(tb+(1-t)a\big)\,{}_0\mathrm{d}_q t \\ &\quad\le \left[\frac{1-q}{1-q^{\alpha_1+\alpha_2+1}} - \frac{1-q}{1-q^{\alpha_1+1}} - \frac{1-q}{1-q^{\alpha_2+1}} + 1\right] m^2 f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right) \\ &\qquad + \frac{1-q}{1-q^{\alpha_1+\alpha_2+1}} f(b)g(b) + \left[\frac{1-q}{1-q^{\alpha_2+1}} - \frac{1-q}{1-q^{\alpha_1+\alpha_2+1}}\right] mf\left(\frac{a}{m}\right)g(b) \\ &\qquad + \left[\frac{1-q}{1-q^{\alpha_1+1}} - \frac{1-q}{1-q^{\alpha_1+\alpha_2+1}}\right] mf(b)g\left(\frac{a}{m}\right). \end{aligned} \tag{3.25}$$

Similarly, we get

$$\begin{aligned} &\int_0^1 f\big(tb+(1-t)a\big)g\big(tb+(1-t)a\big)\,{}_0\mathrm{d}_q t \\ &\quad\le \left(\int_0^1 (1-t)^{\alpha_1+\alpha_2}\,{}_0\mathrm{d}_q t - \int_0^1 (1-t)^{\alpha_1}\,{}_0\mathrm{d}_q t \right. \\ &\qquad \left. - \int_0^1 (1-t)^{\alpha_2}\,{}_0\mathrm{d}_q t + 1\right) m^2 f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right) \\ &\qquad + \left(\int_0^1 (1-t)^{\alpha_1+\alpha_2}\,{}_0\mathrm{d}_q t\right) f(a)g(a) \end{aligned}$$

$$+\left(\int_0^1 (1-t)^{\alpha_1}\,{}_0\mathrm{d}_qt-\int_0^1 (1-t)^{\alpha_1+\alpha_2}\,{}_0\mathrm{d}_qt\right)mf(a)g\left(\frac{b}{m}\right)$$
$$+\left(\int_0^1 (1-t)^{\alpha_2}\,{}_0\mathrm{d}_qt-\int_0^1 (1-t)^{\alpha_1+\alpha_2}\,{}_0\mathrm{d}_qt\right)mf\left(\frac{b}{m}\right)g(a). \tag{3.26}$$

A simple calculation shows that

$$\int_0^1 f\bigl(tb+(1-t)a\bigr)g\bigl(tb+(1-t)a\bigr)\,{}_0\mathrm{d}_qt=\frac{1}{b-a}\int_a^b f(x)g(x)\,{}_a\mathrm{d}_qx. \tag{3.27}$$

From (3.25), (3.26) and (3.27), we obtain the desired result. This ends the proof. □

Corollary 3.2 *In Theorem 3.5, choosing $\alpha_1=\alpha_2=\alpha$, we obtain*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,{}_a\mathrm{d}_qx\le\min\bigl\{\mathcal{T}_1(\alpha,m),\mathcal{T}_2(\alpha,m)\bigr\},$$

where

$$\mathcal{T}_1(\alpha,m)=\frac{1-q}{1-q^{2\alpha+1}}f(b)g(b)+\left[\frac{1-q}{1-q^{2\alpha+1}}-\frac{2(1-q)}{1-q^{\alpha+1}}+1\right]m^2f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)$$
$$+\frac{q^{\alpha+1}(1-q)(1-q^{\alpha})}{(1-q^{\alpha+1})(1-q^{2\alpha+1})}m\left[f\left(\frac{a}{m}\right)g(b)+f(b)g\left(\frac{a}{m}\right)\right]$$

and

$$\mathcal{T}_2(\alpha,m)=\left[(1-q)\sum_{n=0}^{\infty}q^n\bigl(1-q^n\bigr)^{2\alpha}-2(1-q)\sum_{n=0}^{\infty}q^n\bigl(1-q^n\bigr)^{\alpha}+1\right]m^2f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)$$
$$+(1-q)\sum_{n=0}^{\infty}q^n\bigl(1-q^n\bigr)^{2\alpha}f(a)g(a)$$
$$+\left[(1-q)\sum_{n=0}^{\infty}q^n\bigl(1-q^n\bigr)^{\alpha}-(1-q)\sum_{n=0}^{\infty}q^n\bigl(1-q^n\bigr)^{2\alpha}\right]$$
$$\times\left[mf(a)g\left(\frac{b}{m}\right)+mf\left(\frac{b}{m}\right)g(a)\right].$$

Further, taking $\alpha=1=m$, we get

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,{}_a\mathrm{d}_qx$$
$$\le\frac{1}{1+q+q^2}f(b)g(b)+\frac{q(1+q^2)}{(1+q)(1+q+q^2)}f(a)g(a)$$
$$+\frac{q^2}{(1+q)(1+q+q^2)}\bigl[f(a)g(b)+f(b)g(a)\bigr],$$

which is established by Sudsutad et al. in [26, Theorem 4.3].

4 Conclusions

In the present research, based on a new quantum integral identity with multiple parameters, we have developed some quantum error estimations of different type inequalities through (α, m)-convexity, such as the midpoint-like inequalities, the Simpson-like inequalities, the averaged midpoint-trapezoid-like inequalities and the trapezoid-like inequalities. The inequalities derived in this work are very helpful in error estimations involved in various approximation processes. We expect that the ideas of this article will facilitate further study concerning quantum integral inequalities.

Funding
This work was partially supported by the National Natural Science Foundation of China (No. 61374028) and sponsored by Research Fund for Excellent Dissertation of China Three Gorges University (No. 2018SSPY132, No. 2018SSPY134).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Publisher's Note

Received: 19 May 2018 Accepted: 20 September 2018 Published online: 27 September 2018

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