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Approximation by (p,q)-Lupaş–Schurer–Kantorovich operators

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Abstract

In the current paper, we examine the (p,q)-analogue of Kantorovich type Lupaş–Schurer operators with the help of (p,q)-Jackson integral. Then, we estimate the rate of convergence for the constructed operators by using the modulus of continuity in terms of a Lipschitz class function and by means of Peetre's K-functionals based on Korovkin theorem. Moreover, we illustrate the approximation of the (p,q)-Lupaş–Schurer–Kantorovich operators to appointed functions by the help of Matlab algorithm and then show the comparison of the convergence of these operators with Lupaş–Schurer operators based on (p,q)-integers.

Keywords: Lupaş operators; (p,q)-integers; Rate of convergence; Local approximation; Korovkin's approximation theorem

1 Introduction

In 1962, Bernstein–Schurer operators were identified in the paper of Schurer [25]. In 1987, Lupaş [16] initiated the q-generalization of Bernstein operators in rational form. Some other q-Bernstein polynomial was defined by Phillips [22] in 1997. The development q-calculus applications established a precedent in the field of approximation theory. We may refer to some of them as Durrmeyer variant of q-Bernstein–Schurer operators [2], q-Bernstein–Schurer–Kantorovich type operators [3], q-Durrmeyer operators [8], q-Bernstein–Schurer–Durrmeyer type operators [12], q-Bernstein–Schurer operators [19], King's type modified q-Bernstein–Kantorovich operators [20], q-Bernstein–Schurer–Kantorovich operators [23]. Lately, Mursaleen et al. [17] pioneered the research of (p,q)-analogue of Bernstein operators which is a generalization of q-Bernstein operators (Philips). The application of (p,q)-calculus has led to the discovery of various modifications of Bernstein polynomials involving (p,q)-integers. For instance, Mursaleen et al. [18] constructed (p,q)-analogue of Bernstein-Kantorovich operators in 2016, and Khalid et al. [15] generalised q-Bernstein–Lupaş operators. In the (p,q)-calculus, parameter p provides suppleness to the approximation. Some recent articles are [1, 4–6, 9, 10, 13], and [21]. Motivated by the work of Khalid et al. [15], now we define a Kantorovich type Lupaş-Schurer operators based on the (p,q)-calculus.

First of all, we introduce some important notations and definitions for the (p,q)-calculus, which is a generalization of q-oscillator algebras. For $0 < q < p \le 1$ and $m \ge 0$, the (p,q)-

number of m is denoted by $[m]_{p,q}$ and is defined by

$$[m]_{p,q} := p^{m-1} + p^{m-2}q + \cdots + pq^{m-2} + q^{m-1} = \begin{cases} \frac{p^m - q^m}{p-q} & \text{if } p \neq q \neq 1, \\ \frac{1-q^m}{1-q} & \text{if } p = 1, \\ m & \text{if } p = q = 1. \end{cases}$$

The formula for the (p,q)-binomial expansion is defined by

$$(cx + dy)^m_{p,q} := \sum_{l=0}^{m} \begin{bmatrix} m \\ l \end{bmatrix}_{p,q} p^{\frac{(m-l)(m-l-1)}{2}} q^{\frac{l(l-1)}{2}} c^{m-l} d^l x^{m-l} y^l, \tag{1}$$

where

$$\begin{bmatrix} m \\ l \end{bmatrix}_{p,q} = \frac{[m]_{p,q}!}{[l]_{p,q}![m-l]_{p,q}!}$$

are the (p,q)-binomial coefficients. From Eq. (1) we get

$$(x+y)^m_{p,q} = (x+y)(px+qy)\left(p^2x + q^2y\right)\cdots\left(p^{m-1}x + q^{m-1}y\right)$$

and

$$(1-x)^m_{p,q} = (1-x)(p-qx)\left(p^2 - q^2x\right)\cdots\left(p^{m-1} - q^{m-1}x\right).$$

The (p,q)-Jackson integrals are defined by

$$\int_0^a f(x)\,d_{p,q}x = (q-p)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}}a\right), \quad \left|\frac{p}{q}\right| < 1$$

and

$$\int_0^a f(x)\,d_{p,q}x = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} F\left(\frac{q^k}{p^{k+1}}a\right), \quad \left|\frac{q}{p}\right| < 1.$$

For detailed information about the theory of (p,q)-integers, we refer to [11] and [24].

2 Construction of the operator

Definition 1 For any $0 < q < p \leq 1$, we construct a (p,q)-analogue of Kantorovich type Lupaş–Schurer operator by

$$K^{(p,q)}_{m,s}(f;x) = [m]_{p,q} \sum_{l=0}^{m+s} \frac{B^{p,q}_{m,l,s}(x)}{p^{m-l}q^l} \int_{\frac{[l]_{p,q}}{p^{l-m-1}[m]_{p,q}}}^{\frac{[l+1]_{p,q}}{p^{l-m}[m]_{p,q}}} f(t)\,d_{p,q}t, \quad x \in [0,1], \tag{2}$$

where $m \in \mathbb{N}$, $f \in C[0,s+1]$, $s > 0$ is a fixed natural number and

$$B^{p,q}_{m,l,s}(x) = \frac{\begin{bmatrix} m+s \\ l \end{bmatrix}_{p,q} p^{\frac{(m+s-l)(m+s-l-1)}{2}} q^{\frac{l(l-1)}{2}} x^l (1-x)^{m+s-l}}{\prod_{j=1}^{m+s}\{p^{j-1}(1-x) + q^{j-1}x\}}. \tag{3}$$

After some calculations we obtain

$$K_{m,s}^{(p,q)}(f;x) = \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \int_0^1 f\left(\frac{p[l]_{p,q} + q^l t}{p^{l-m}[m]_{p,q}}\right) d_{p,q}t. \tag{4}$$

In the following lemma, we present some equalities for the (p,q)-analogue of Lupaş–Schurer–Kantorovich operators.

Lemma 1 *Let $K_{m,s}^{(p,q)}(\cdot;\cdot)$ be given by Eq. (4). Then we have*

$$K_{m,s}^{(p,q)}(1;x) = 1, \tag{5}$$

$$K_{m,s}^{(p,q)}(t;x) = \left(\frac{[m+s]_{p,q}}{[m]_{p,q}p^{s-1}} - \frac{p^m}{[2]_{p,q}[m]_{p,q}} + \frac{q^{m+s}}{[2]_{p,q}[m]_{p,q}p^s}\right)x + \frac{p^m}{[2]_{p,q}[m]_{p,q}}, \tag{6}$$

$$\begin{aligned} K_{m,s}^{(p,q)}(t^2;x) &= \frac{[m+s]_{p,q}[m+s-1]_{p,q}q^2p^{2-2s}}{[m]_{p,q}^2(p(1-x)+qx)}x^2 + \frac{[m+s]_{p,q}p^{m-s+1}}{[m]_{p,q}^2}x \\ &\quad + \frac{2[m+s]_{p,q}qp^{4m+2s-3}(p^{m+s}(1-x)+q^{m+s}x)}{[2]_{p,q}[m]_{p,q}^2(p(1-x)+qx)}x \\ &\quad + \frac{p^{-2s}(p^{m+s}(1-x)+q^{m+s}x)(p^{m+s+1}(1-x)+q^{m+s+1}x)}{[3]_{p,q}[m]_{p,q}^2(p(1-x)+qx)}, \end{aligned} \tag{7}$$

$$K_{m,s}^{(p,q)}(t-x;x) = \left(\frac{[m+s]_{p,q}}{[m]_{p,q}p^{s-1}} - \frac{p^m}{[2]_{p,q}[m]_{p,q}} + \frac{q^{m+s}}{[2]_{p,q}[m]_{p,q}p^s} - 1\right)x + \frac{p^m}{[2]_{p,q}[m]_{p,q}}, \tag{8}$$

$$\begin{aligned} K_{m,s}^{(p,q)}\left((t-x)^2;x\right) &= \left(\frac{[m+s]_{p,q}[m+s-1]_{p,q}q^2p^{2-2s}}{[m]_{p,q}^2(p(1-x)+qx)}\right. \\ &\quad \left. + \frac{-2[2]_{p,q}[m+s]_{p,q}p^{1-s} + 2p^m - 2q^{m+s}p^{-s}}{[2]_{p,q}[m]_{p,q}} + 1\right)x^2 \\ &\quad + \left(\frac{[m+s]_{p,q}p^{m-s+1}}{[m]_{p,q}^2} + \frac{2[m+s]_{p,q}qp^{4m+2s-3}(p^{m+s}(1-x)+q^{m+s}x)}{[2]_{p,q}[m]_{p,q}^2(p(1-x)+qx)}\right. \\ &\quad \left. - \frac{2p^m}{[2]_{p,q}[m]_{p,q}}\right)x \\ &\quad + \frac{p^{-2s}(p^{m+s}(1-x)+q^{m+s}x)(p^{m+s+1}(1-x)+q^{m+s+1}x)}{[3]_{p,q}[m]_{p,q}^2(p(1-x)+qx)}. \end{aligned} \tag{9}$$

Proof *(i)* From the definition of the operators in (4), we can easily prove the first claim as follows:

$$\begin{aligned} K_{m,s}^{(p,q)}(1;x) &= \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \int_0^1 d_{p,q}t \\ &= \sum_{l=0}^{m+s} \frac{\left[\begin{smallmatrix} m+s \\ l \end{smallmatrix}\right]_{p,q} p^{\frac{(m+s-l)(m+s-l-1)}{2}} q^{\frac{l(l-1)}{2}} x^l (1-x)^{m+s-l}}{\prod_{j=1}^{m+s}\{p^{j-1}(1-x)+q^{j-1}x\}} \\ &= 1. \end{aligned} \tag{10}$$

(ii) We can calculate the second identity for $K_{m,s}^{(p,q)}(t;x)$ as follows:

$$\begin{aligned} K_{m,s}^{(p,q)}(t;x) &= \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \int_0^1 \frac{p[l]_{p,q}+q^l t}{p^{l-m}[m]_{p,q}}\, d_{p,q}t \\ &= \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \frac{p[l]_{p,q}}{p^{l-m}[m]_{p,q}} \int_0^1 d_{p,q}t + \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \frac{q^l}{p^{l-m}[m]_{p,q}} \int_0^1 t\, d_{p,q}t. \end{aligned}$$

After that, by some simple computations, we have

$$\begin{aligned} K_{m,s}^{(p,q)}(t;x) &= \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \frac{p[l]_{p,q}}{p^{l-m}[m]_{p,q}} + \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \frac{q^l}{p^{l-m}[m]_{p,q}[2]_{p,q}} \\ &= \sum_{l=1}^{m+s} \frac{p^{m-l+1}[m+s]_{p,q}}{[m]_{p,q}} . \frac{\left[{m+s-1 \atop l-1} \right]_{p,q} p^{\frac{(m+s-l)(m+s-l-1)}{2}} q^{\frac{l(l-1)}{2}} x^l (1-x)^{m+s-l}}{\prod_{j=1}^{m+s}\{p^{j-1}(1-x)+q^{j-1}x\}} \\ &\quad + \frac{1}{[m]_{p,q}[2]_{p,q}p^s} \sum_{l=0}^{m+s} \frac{\left[{m+s \atop l} \right]_{p,q} p^{\frac{(m+s-l)(m+s-l-1)}{2}} q^{\frac{l(l-1)}{2}} (\frac{qx}{p(1-x)})^l}{\prod_{j=0}^{m+s-1}\{p^{j-1}+q^{j-1}(\frac{qx}{p(1-x)})\}} \\ &= \frac{[m+s]_{p,q}}{[m]_{p,q}p^s} \sum_{l=0}^{m+s-1} \frac{p^{m+s-l}\left[{m+s-1 \atop l} \right]_{p,q} p^{\frac{(m+s-l-1)(m+s-l-2)}{2}} q^{\frac{l(l+1)}{2}} x^{l+1}(1-x)^{m+s-l-1}}{\prod_{j=1}^{m+s-1}\{p^j(1-x)+q^j x\}} \\ &\quad + \frac{p(1-x)\{p^{m+s-1}+q^{m+s-1}(\frac{qx}{p(1-x)})\}}{[m]_{p,q}[2]_{p,q}p^s} \\ &\quad \times \sum_{l=0}^{m+s} \frac{\left[{m+s \atop l} \right]_{p,q} p^{\frac{(m+s-l)(m+s-l-1)}{2}} q^{\frac{l(l-1)}{2}} (\frac{qx}{p(1-x)})^l}{\prod_{j=1}^{m+s}\{p^{j-1}+q^{j-1}(\frac{qx}{p(1-x)})\}} \\ &= \frac{[m+s]_{p,q}}{[m]_{p,q}p^{s-1}} x + \frac{p(1-x)\{p^{m+s-1}+q^{m+s-1}(\frac{qx}{p(1-x)})\}}{[m]_{p,q}[2]_{p,q}p^s}. \end{aligned}$$

Then, $K_{m,s}^{(p,q)}(t;x)$ is obtained as

$$K_{m,s}^{(p,q)}(t;x) = \left(\frac{[m+s]_{p,q}}{[m]_{p,q}p^{s-1}} - \frac{p^m}{[2]_{p,q}[m]_{p,q}} + \frac{q^{m+s}}{[2]_{p,q}[m]_{p,q}p^s} \right) x + \frac{p^m}{[2]_{p,q}[m]_{p,q}}.$$

Thus, (6) is obtained.

(iii) For the third identity involving $K_{m,s}^{(p,q)}(t^2;x)$, we write

$$\begin{aligned} K_{m,s}^{(p,q)}(t^2;x) &= \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \frac{p^2[l]_{p,q}^2}{p^{2l-2m}[m]_{p,q}^2} \int_0^1 d_{p,q}t + 2 \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \frac{p[l]_{p,q}q^l}{p^{2l-2m}[m]_{p,q}^2} \int_0^1 t\, d_{p,q}t \\ &\quad + \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \frac{q^{2l}}{p^{2l-2m}[m]_{p,q}^2} \int_0^1 t^2\, d_{p,q}t \end{aligned}$$

$$= \underbrace{\sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x)\frac{p^2[l]_{p,q}^2}{p^{2l-2m}[m]_{p,q}^2}}_{B1} + \underbrace{\frac{2}{[2]_{p,q}}\sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x)\frac{p[l]_{p,q}q^l}{p^{2l-2m}[m]_{p,q}^2}}_{B2} + \underbrace{\frac{1}{[3]_{p,q}}\sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x)\frac{q^{2l}}{p^{2l-2m}[m]_{p,q}^2}}_{B3}. \tag{11}$$

Firstly, we calculate $B1$ as

$$B1 = \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x)\frac{p^2[l]_{p,q}^2}{p^{2l-2m}[m]_{p,q}^2} = \sum_{l=0}^{m+s-1} \frac{p^{2m-2l}[l+1]_{p,q}[m+s]_{p,q}}{[m]_{p,q}^2}.\frac{\begin{bmatrix} m+s-1 \\ l \end{bmatrix}_{p,q} p^{\frac{(m+s-l-1)(m+s-l-2)}{2}} q^{\frac{l(l+1)}{2}} x^{l+1}(1-x)^{m+s-l-1}}{\prod_{j=1}^{m+s}\{p^{j-1}(1-x)+q^{j-1}x\}}.$$

Now by using the equality

$$[l+1]_{p,q} = p^l + q[l]_{p,q}, \tag{12}$$

we acquire

$$\begin{aligned} B1 &= \frac{[m+s]_{p,q}}{[m]_{p,q}^2}\sum_{l=0}^{m+s-1} p^{2m-l}\frac{\begin{bmatrix} m+s-1 \\ l \end{bmatrix}_{p,q} p^{\frac{(m+s-l-1)(m+s-l-2)}{2}} q^{\frac{l(l+1)}{2}} x^{l+1}(1-x)^{m+s-l-1}}{\prod_{j=1}^{m+s}\{p^{j-1}(1-x)+q^{j-1}x\}} \\ &\quad + \frac{[m+s]_{p,q}}{[m]_{p,q}^2}\sum_{l=0}^{m+s-1} p^{2m-2l}q[l]_{p,q}\frac{\begin{bmatrix} m+s-1 \\ l \end{bmatrix}_{p,q} p^{\frac{(m+s-l-1)(m+s-l-2)}{2}} q^{\frac{l(l+1)}{2}} x^{l+1}(1-x)^{m+s-l-1}}{\prod_{j=1}^{m+s}\{p^{j-1}(1-x)+q^{j-1}x\}} \\ &= \frac{[m+s]_{p,q}p^{2m}x}{[m]_{p,q}^2 p^{m+s-1}}\sum_{l=0}^{m+s-1}\frac{\begin{bmatrix} m+s-1 \\ l \end{bmatrix}_{p,q} p^{\frac{(m+s-l-1)(m+s-l-2)}{2}} q^{\frac{l(l-1)}{2}} (\frac{qx}{p(1-x)})^l(1-x)^{m+s-1}}{\frac{1}{p^{m+s-1}}\prod_{j=1}^{m+s-1}\{p^j(1-x)+q^jx\}} \\ &\quad + \frac{[m+s]_{p,q}[m+s-1]_{p,q}q^2x^2}{[m]_{p,q}^2 p^{2s-2}(p(1-x)+qx)}\sum_{l=0}^{m+s-2}\frac{\begin{bmatrix} m+s-2 \\ l \end{bmatrix}_{p,q} p^{\frac{(m+s-l-2)(m+s-l-3)}{2}} q^{\frac{l(l-1)}{2}} (\frac{q^2x}{p^2(1-x)})^l}{\prod_{j=1}^{m+s-2}\{p^{j-1}+q^{j-1}(\frac{q^2x}{p^2(1-x)})\}} \\ &= \frac{[m+s]_{p,q}p^{m-s+1}}{[m]_{p,q}^2}x + \frac{[m+s]_{p,q}[m+s-1]_{p,q}p^{2-2s}q^2}{[m]_{p,q}^2(p(1-x)+qx)}x^2. \end{aligned} \tag{13}$$

Secondly, we work out $B2$ as follows:

$$\begin{aligned} B2 &= \frac{2}{[2]_{p,q}}\sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x)\frac{p[l]_{p,q}q^l}{p^{2l-2m}[m]_{p,q}^2} \\ &= \frac{2[m+s]_{p,q}}{[2]_{p,q}[m]_{p,q}^2}\sum_{l=1}^{m+s}\frac{q^l}{p^{2l-2m-1}}.\frac{\begin{bmatrix} m+s-1 \\ l-1 \end{bmatrix}_{p,q} p^{\frac{(m+s-l)(m+s-l-1)}{2}} q^{\frac{l(l-1)}{2}} x^l(1-x)^{m+s-l}}{\prod_{j=1}^{m+s}\{p^{j-1}(1-x)+q^{j-1}x\}} \\ &= \frac{2[m+s]_{p,q}x}{[2]_{p,q}[m]_{p,q}^2}\sum_{l=0}^{m+s-1}\frac{q}{p^{-2m+1}}.\frac{\begin{bmatrix} m+s-1 \\ l \end{bmatrix}_{p,q} p^{\frac{(m+s-l-1)(m+s-l-2)}{2}} q^{\frac{l(l-1)}{2}} (\frac{q^2x}{p^2(1-x)})^l(1-x)^{m+s-1}}{\prod_{j=2}^{m+s}\{p^{j-1}(1-x)+q^{j-1}x\}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{2[m+s]_{p,q}qp^{2m-1}x}{[2]_{p,q}[m]_{p,q}^2}\sum_{l=0}^{m+s-1}\frac{\left[{m+s-1 \atop l}\right]_{p,q}p^{\frac{(m+s-l-1)(m+s-l-2)}{2}}q^{\frac{l(l-1)}{2}}(\frac{q^2x}{p^2(1-x)})^l(1-x)^{m+s-1}}{\prod_{j=0}^{m+s-2}\{p^{j+1}(1-x)+q^{j+1}x\}} \\
&= \frac{2[m+s]_{p,q}qp^{4m+2s-3}x}{[2]_{p,q}[m]_{p,q}^2}\sum_{l=0}^{m+s-1}\frac{\left[{m+s-1 \atop l}\right]_{p,q}p^{\frac{(m+s-l-1)(m+s-l-2)}{2}}q^{\frac{l(l-1)}{2}}(\frac{q^2x}{p^2(1-x)})^l}{\prod_{j=0}^{m+s-2}\{p^{j-1}+q^{j-1}(\frac{q^2x}{p^2(1-x)})\}} \\
&= \frac{2[m+s]_{p,q}qp^{4m+2s-3}}{[2]_{p,q}[m]_{p,q}^2}.\frac{(p^{m+s}(1-x)+q^{m+s}x)}{p(1-x+qx)}x. \qquad (14)
\end{aligned}
$$

Thirdly, we deal with $B3$ as

$$
\begin{aligned}
B3 &= \frac{1}{[3]_{p,q}}\sum_{l=0}^{m+s}B_{m,l,s}^{p,q}(x)\frac{q^{2l}}{p^{2l-2m}[m]_{p,q}^2} \\
&= \frac{p^{2m}}{[3]_{p,q}[m]_{p,q}^2}\sum_{l=0}^{m+s}\frac{\left[{m+s \atop l}\right]_{p,q}p^{\frac{(m+s-l)(m+s-l-1)}{2}}q^{\frac{l(l-1)}{2}}(\frac{q^2x}{p^2(1-x)})^l(1-x)^{m+s}}{\prod_{j=0}^{m+s-2}\{p^{j+1}(1-x)+q^{j+1}x\}} \\
&= \frac{p^{-2s}}{[3]_{p,q}[m]_{p,q}^2}.\frac{(p^{m+s}(1-x)+q^{m+s}x)(p^{m+s+1}(1-x)+q^{m+s+1}x)}{p(1-x)+qx}. \qquad (15)
\end{aligned}
$$

As a consequence, $K_{m,s}^{(p,q)}(t^2;x)$ is found as

$$
\begin{aligned}
K_{m,s}^{(p,q)}\big(t^2;x\big) &= \frac{[m+s]_{p,q}p^{m-s+1}}{[m]_{p,q}^2}x+\frac{[m+s]_{p,q}[m+s-1]_{p,q}p^{2-2s}q^2}{[m]_{p,q}^2(p(1-x)+qx)}x^2 \\
&\quad+\frac{2[m+s]_{p,q}qp^{4m+2s-3}}{[m]_{p,q}^2[2]_{p,q}}.\frac{(p^{m+s}(1-x)+q^{m+s}x)}{p(1-x+qx)}x \\
&\quad+\frac{p^{-2s}}{[3]_{p,q}[m]_{p,q}^2}.\frac{(p^{m+s}(1-x)+q^{m+s}x)(p^{m+s+1}(1-x)+q^{m+s+1}x)}{p(1-x)+qx}.
\end{aligned}
$$

If we reorganize, we obtain

$$
\begin{aligned}
K_{m,s}^{(p,q)}\big(t^2;x\big) &= \frac{[m+s]_{p,q}[m+s-1]_{p,q}q^2p^{2-2s}}{[m]_{p,q}^2(p(1-x)+qx)}x^2+\frac{[m+s]_{p,q}p^{m-s+1}}{[m]_{p,q}^2}x \\
&\quad+\frac{2[m+s]_{p,q}qp^{4m+2s-3}(p^{m+s}(1-x)+q^{m+s}x)}{[2]_{p,q}(p(1-x)+qx)[m]_{p,q}^2}x \\
&\quad+\frac{p^{-2s}(p^{m+s}(1-x)+q^{m+s}x)(p^{m+s+1}(1-x)+q^{m+s+1}x)}{[3]_{p,q}[m]_{p,q}^2(p(1-x)+qx)}, \qquad (16)
\end{aligned}
$$

as desired.

(iv) By using the linearity of the operators $K_{m,s}^{(p,q)}$, we acquire the first central moment $K_{m,s}^{(p,q)}(t-x;x)$ as

$$
\begin{aligned}
K_{m,s}^{(p,q)}(t-x;x) &= K_{m,s}^{(p,q)}(t;x)-xK_{m,s}^{(p,q)}(1;x) \\
&= \left(\frac{[m+s]_{p,q}}{[m]_{p,q}p^{s-1}}-\frac{p^m}{[2]_{p,q}[m]_{p,q}}+\frac{q^{m+s}}{[2]_{p,q}[m]_{p,q}p^s}-1\right)x \\
&\quad+\frac{p^m}{[2]_{p,q}[m]_{p,q}}. \qquad (17)
\end{aligned}
$$

(v) Similarly, we write the second central moment $K_{m,s}^{(p,q)}((t-x)^2;x)$ as

$$K_{m,s}^{(p,q)}\big((t-x)^2;x\big) = K_{m,s}^{(p,q)}\big(t^2;x\big) - 2xK_{m,s}^{(p,q)}(t;x) + x^2K_{m,s}^{(p,q)}(1;x). \tag{18}$$

We now plug-in into equation (18) expressions (5), (6) and (7). Then we get

$$\begin{aligned} K_{m,s}^{(p,q)}\big((t-x)^2;x\big) = {} & \bigg(\frac{[m+s]_{p,q}[m+s-1]_{p,q}q^2p^{2-2s}}{[m]_{p,q}^2(p(1-x)+qx)} \\ & + \frac{-2[2]_{p,q}[m+s]_{p,q}p^{1-s}+2p^m-2q^{m+s}p^{-s}}{[2]_{p,q}[m]_{p,q}}+1\bigg)x^2 \\ & + \bigg(\frac{[m+s]_{p,q}p^{m-s+1}}{[m]_{p,q}^2} + \frac{2[m+s]_{p,q}qp^{4m+2s-3}(p^{m+s}(1-x)+q^{m+s}x)}{[2]_{p,q}[m]_{p,q}^2(p(1-x)+qx)} \\ & - \frac{2p^m}{[2]_{p,q}[m]_{p,q}}\bigg)x \\ & + \frac{p^{-2s}(p^{m+s}(1-x)+q^{m+s}x)(p^{m+s+1}(1-x)+q^{m+s+1}x)}{[3]_{p,q}(p(1-x)+qx)[m]_{p,q}^2}. \end{aligned} \tag{19}$$

□

We can easily see that $K_{m,s}^{(p,q)}(f;x)$ are linear positive operators.

Remark 1 [15] Let p, q satisfy $0 < q < p \leq 1$ and $\lim_{m\to\infty}[m]_{p,q} = \frac{1}{p-q}$. To obtain the convergence results for operators $K_{m,s}^{(p,q)}(f;x)$, we take sequences $q_m \in (0,1)$, $p_m \in (q_m,1]$ such that $\lim_{m\to\infty} p_m = 1$, $\lim_{m\to\infty} q_m = 1$, $\lim_{m\to\infty} p_m^m = 1$ and $\lim_{m\to\infty} q_m^m = 1$. Such sequences can be constructed by taking $p_m = 1-1/m^2$ and $q_m = 1-1/2m^2$.

Now we will present the next theorem, which ensures the approximation process according to Korovkin's approximation theorem.

Theorem 1 *Let $K_{m,s}^{(p,q)}(f;x)$ satisfy the conditions $p_m \to 1$, $q_m \to 1$, $p_m^m \to 1$ and $q_m^m \to 1$ as $m \to \infty$ for $q_m \in (0,1)$, $p_m \in (q_m,1]$. Then for every monotone increasing function $f \in C[0,s+1]$, operators $K_{m,s}^{(p,q)}(f;x)$ converge uniformly to f.*

Proof By the Korovkin theorem, it is sufficient to prove that

$$\lim_{m\longrightarrow\infty}\big\|K_{m,s}^{(p,q)}e_k - e_k\big\| = 0, \qquad k = 0,1,2,$$

where $e_k(x) = x^k$, $k = 0,1,2$.

(i) By using Eq. (5), it can be clearly seen that

$$\lim_{m\longrightarrow\infty}\big\|K_{m,s}^{(p,q)}e_0 - e_0\big\| = \lim_{m\longrightarrow\infty}\sup_{x\in[0,1]}\big|K_{m,s}^{(p,q)}(1;x) - 1\big| = 0.$$

(ii) By Eq. (6), we write

$$\begin{aligned} & \lim_{m\longrightarrow\infty}\big\|K_{m,s}^{(p,q)}e_1 - e_1\big\| \\ & \quad = \lim_{m\longrightarrow\infty}\sup_{x\in[0,1]}\big|K_{m,s}^{(p,q)}(t;x) - x\big| \end{aligned}$$

$$
\begin{aligned}
&= \lim_{m\longrightarrow\infty}\sup_{x\in[0,1]}\left|\left(\frac{[m+s]_{p,q}}{p^{s-1}[m]_{p,q}}-\frac{p^m}{[2]_{p,q}[m]_{p,q}}+\frac{q^{m+s}}{[2]_{p,q}[m]_{p,q}p^s}-1\right)x+\frac{p^m}{[2]_{p,q}[m]_{p,q}}\right| \\
&\le \lim_{m\longrightarrow\infty}\left(\frac{[m+s]_{p,q}}{p^{s-1}[m]_{p,q}}-1+\frac{q^{m+s}}{[2]_{p,q}[m]_{p,q}p^s}\right) \\
&= 0.
\end{aligned}
$$

(iii) From Eq. (7), we have

$$
\begin{aligned}
&\lim_{m\longrightarrow\infty}\left\|K_{m,s}^{(p,q)}e_2-e_2\right\| \\
&= \lim_{m\longrightarrow\infty}\sup_{x\in[0,1]}\left|K_{m,s}^{(p,q)}(t^2;x)-x^2\right| \\
&= \lim_{m\longrightarrow\infty}\sup_{x\in[0,1]}\left|\left(\frac{[m+s]_{p,q}[m+s-1]_{p,q}q^2p^{2-2s}}{[m]_{p,q}^2(p(1-x)+qx)}-1\right)x^2\right. \\
&\quad+\frac{[m+s]_{p,q}p^{m-s+1}}{[m]_{p,q}^2}x+\frac{2[m+s]_{p,q}qp^{4m+2s-3}(p^{m+s}(1-x)+q^{m+s}x)}{[2]_{p,q}[m]_{p,q}^2(p(1-x)+qx)}x \\
&\quad\left.+\frac{p^{-2s}(p^{m+s}(1-x)+q^{m+s}x)(p^{m+s+1}(1-x)+q^{m+s+1}x)}{[3]_{p,q}[m]_{p,q}^2(p(1-x)+qx)}\right| \\
&\le \lim_{m\longrightarrow\infty}\left(\left(\frac{[m+s]_{p,q}[m+s-1]_{p,q}q^2p^{2-2s}}{[m]_{p,q}^2(p(1-x)+qx)}-1\right)+\frac{[m+s]_{p,q}p^{m-s+1}}{[m]_{p,q}^2}\right. \\
&\quad+\frac{2[m+s]_{p,q}qp^{4m+2s-3}(p^{m+s}(1-x)+q^{m+s}x)}{[2]_{p,q}[m]_{p,q}^2(p(1-x)+qx)} \\
&\quad\left.+\frac{p^{-2s}(p^{m+s}(1-x)+q^{m+s}x)(p^{m+s+1}(1-x)+q^{m+s+1}x)}{[3]_{p,q}[m]_{p,q}^2(p(1-x)+qx)}\right) \\
&= 0.
\end{aligned}
$$

Consequently, the proof is finished. □

Before mentioning local approximation properties, we will give two lemmas as follows.

Lemma 2 *If f is a monotone increasing function, then the constructed operators $K_{m,s}^{(p,q)}(f;x)$ are linear and positive.*

Lemma 3 *Let $0<q<p\le 1$, $0<u<v$, and $\frac{1}{u}+\frac{1}{v}=1$. Then the operators $K_{m,s}^{(p,q)}(f;x)$ satisfy the following Hölder inequality:*

$$K_{m,s}^{(p,q)}\left(|fg|;x\right)\le\left(K_{m,s}^{(p,q)}\left(|f|^u;x\right)\right)^{\frac{1}{u}}\left(K_{m,s}^{(p,q)}\left(|g|^v;x\right)\right)^{\frac{1}{v}}.$$

3 Local approximation properties

Let f be a continuous function on $C[0,s+1]$. The modulus of continuity of f is denoted by $w(f,\sigma)$ and given as

$$w(f,\sigma)=\sup_{\substack{|y-x|\le\sigma\\ x,y\in[0,1]}}\left|f(y)-f(x)\right|. \tag{20}$$

Then we know from the properties of modulus of continuity that for each $\sigma > 0$, we have

$$\left|f(y) - f(x)\right| \le w(f,\sigma)\left(\frac{|y-x|}{\sigma} + 1\right), \quad x, y \in [0,1]. \tag{21}$$

And also, for $f \in C[0,s+1]$ we have $\lim_{\sigma\to 0^+} w(f,\sigma) = 0$. First of all, we begin by giving the rate of convergence of the operators $K_{m,s}^{(p,q)}(f;x)$ by using the modulus of continuity.

Theorem 2 *Let the sequences $p := (p_m)$ and $q := (q_m)$, $0 < q_m < p_m \le 1$, satisfy the conditions $p_m \to 1$, $q_m \to 1$, $p_m^m \to 1$ and $q_m^m \to 1$ as $m \to \infty$. Then for each $f \in C[0,s+1]$,*

$$\left\|K_{m,s}^{(p,q)} f - f\right\|_{C[0,s+1]} \le 2\omega\bigl(f;\sigma_m(x)\bigr),$$

where

$$\sigma_m(x) = \sqrt{K_{m,s}^{(p,q)}\bigl((t-x)^2;x\bigr)} \tag{22}$$

and $K_{m,s}^{(p,q)}((t-x)^2;x)$ is as given by (19).

Proof By the positivity and linearity of the operators $K_{m,s}^{(p,q)}(f;x)$, we get

$$\begin{aligned}\left|K_{m,s}^{(p,q)}(f;x) - f(x)\right| &= \left|K_{m,s}^{(p,q)}\bigl(f(t)-f(x);x\bigr)\right| \\ &\le K_{m,s}^{(p,q)}\bigl(\left|f(t)-f(x)\right|;q;x\bigr).\end{aligned}$$

After that we apply (21) and obtain

$$\begin{aligned}\left|K_{m,s}^{(p,q)}(f;x) - f(x)\right| &\le K_{m,s}^{(p,q)}\left(w(f,\sigma_m)\left(\frac{|t-x|}{\sigma_m}+1\right);x\right) \\ &= \frac{w(f,\sigma_m)}{\sigma_m}\sqrt{K_{m,s}^{(p,q)}\bigl((t-x)^2;x\bigr)} + w(f,\sigma_m) \\ &= w(f,\sigma_m)\left(1 + \frac{1}{\sigma_m}\sqrt{K_{m,s}^{(p,q)}\bigl((t-x)^2;x\bigr)}\right).\end{aligned} \tag{23}$$

Then, taking supremum of the last equation, we have

$$\begin{aligned}\left\|K_{m,s}^{(p,q)} f - f\right\| &= \sup_{x\in[0,1]}\left|K_{m,s}^{(p,q)}(f;x) - f(x)\right| \\ &\le w(f,\sigma_m)\left(1 + \frac{1}{\sigma_m}\sqrt{K_{m,s}^{(p,q)}\bigl((t-x)^2;x\bigr)}\right).\end{aligned}$$

Choose

$$\begin{aligned}\sigma_m(x) = \Biggl\{&\left(\frac{q^2[m+l]_{p,q}[m+l-1]_{p,q}}{([m]_{p,q}+\beta)^2(p(1-x)+qx)} - \frac{2[m+l]_{p,q}}{[m]_{p,q}+\beta} + 1\right)x^2 \\ &+ \left(-\frac{2\alpha}{[m]_{p,q}+\beta} + \frac{[m+l]_{p,q}(p^{m+l-1}+2\alpha)}{([m]_{p,q}+\beta)^2}\right)x + \left(\frac{\alpha}{[m]_{p,q}+\beta}\right)^2\Biggr\}^{1/2}.\end{aligned}$$

Thus, we achieve

$$\left\| K_{m,s}^{(p,q)} f - f \right\|_{C[0,s+1]} \leq 2\omega\big(f; \sigma_m(x)\big).$$

This result completes the proof of the theorem. □

In what follows, by using Lipschitz functions, we will give the rate of convergence of the operators $K_{m,s}^{(p,q)}(f;x)$. We remember that if the inequality

$$\big|f(y) - f(x)\big| \leq M|y-x|^{\alpha}; \quad \forall x, y \in [0,1] \tag{24}$$

is satisfied, then f belongs to the class $\mathrm{Lip}_M(\alpha)$.

Theorem 3 *Denote $p := (p_m)$ and $q := (q_m)$ satisfying $0 < q_m < p_m \leq 1$. Then, for every $f \in \mathrm{Lip}_M(\alpha)$, we have*

$$\left\| K_{m,s}^{(p,q)} f - f \right\| \leq M\sigma_m^{\alpha}(x),$$

where $\sigma_m(x)$ is the same as in (22).

Proof Let f belong to the class $\mathrm{Lip}_M(\alpha)$ for some $0 < \alpha \leq 1$. Using the monotonicity of the operators $K_{m,s}^{(p,q)}(f;x)$ and (24), we obtain

$$\begin{aligned} \left|K_{m,s}^{(p,q)}(f;x) - f(x)\right| &\leq K_{m,s}^{(p,q)}\big(\big|f(t) - f(x)\big|; x\big) \\ &\leq MK_{m,s}^{(p,q)}\big(|t-x|^{\alpha}; x\big). \end{aligned}$$

Taking $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$ and applying Hölder inequality yields

$$\begin{aligned} \left|K_{m,s}^{(p,q)}(f;x) - f(x)\right| &\leq M\big\{K_{m,s}^{(p,q)}\big((t-x)^2; x\big)\big\}^{\frac{\alpha}{2}} \\ &\leq M\sigma_m^{\alpha}(x). \end{aligned}$$

By choosing $\sigma_m(x)$ as in Theorem 2, we complete the proof as desired. □

Finally, in the light of Peetre-K functionals, we obtain the rate of convergence of the constructed operators $K_{m,s}^{(p,q)}(f;x)$. We recall the properties of Peetre-K functionals, which are defined as

$$K(f,\delta) := \inf_{g \in C^2[0,s+1]} \big\{ \|f-g\|_{C[0,s+1]} + \delta\|g\|_{C^2[0,s+1]} \big\}.$$

Here $C^2[0,s+1]$ defines the space of the functions f such that $f, f', f'' \in C[0,s+1]$. The norm in this space is given by

$$\|f\|_{C^2[0,s+1]} = \left\|f''\right\|_{C[0,s+1]} + \left\|f'\right\|_{C[0,s+1]} + \|f\|_{C[0,s+1]}.$$

Also we consider the second modulus of smoothness of $f \in C[0,s+1]$, namely

$$\omega_2(f,\delta) := \sup_{0<h<\delta} \sup_{x,x+h \in [0,s+1]} \big|f(x+2h) - 2f(x+h) + f(x)\big|, \quad \delta > 0.$$

We know from [7] that for $M > 0$

$$K(f,\delta) \le M\omega_2(f,\sqrt{\sigma}).$$

Before giving the main theorem, we present an auxiliary lemma, which will be used in the proof of the theorem.

Lemma 4 *For any $f \in C[0,s+1]$, we have*

$$\left|K_{m,s}^{(p,q)}(f;x)\right| \le \|f\|. \tag{25}$$

Proof

$$\begin{aligned} \left|K_{m,s}^{(p,q)}(f;x)\right| &= \left|\sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \int_0^1 f\left(\frac{p[l]_{p,q}+q^l t}{p^{l-m}[m]_{p,q}}\right) d_{p,q}t\right| \\ &\le \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \left|\int_0^1 f\left(\frac{p[l]_{p,q}+q^l t}{p^{l-m}[m]_{p,q}}\right) d_{p,q}t\right| \\ &\le \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \int_0^1 \left|f\left(\frac{p[l]_{p,q}+q^l t}{p^{l-m}[m]_{p,q}}\right)\right| d_{p,q}t \\ &\le \|f\| K_{m,s}^{(p,q)}(1;x) \\ &= \|f\|. \end{aligned}$$

□

Theorem 4 *Let $0 < q_m < p_m \le 1$, $m \in \mathbb{N}$ and $f \in C[0,s+1]$. There exists a constant $M > 0$ such that*

$$\left|K_{m,s}^{(p,q)}(f;x) - f(x)\right| \le M\omega_2\big(f,\alpha_m(x)\big) + \omega\big(f,\beta_m(x)\big),$$

where

$$\alpha_m(x) = \sqrt{K_{m,s}^{(p,q)}\big((t-x)^2;x\big) + \frac{1}{2}\left(\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s} - p^m + p^{-s}q^{m+s})x + p^m}{[2]_{p,q}[m]_{p,q}} - x\right)^2} \tag{26}$$

and

$$\beta_m(x) = \frac{([2]_{p,q}[m+s]_{p,q}p^{1-s} - p^m + p^{-s}q^{m+s})x + p^m}{[2]_{p,q}[m]_{p,q}} - x. \tag{27}$$

Proof Define an auxiliary operator $K_{m,s}^*$ as follows:

$$K_{m,s}^*(f;x) = K_{m,s}^{(p,q)}(f;x) - f\left(\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s} - p^m + p^{-s}q^{m+s})x + p^m}{[2]_{p,q}[m]_{p,q}}\right) + f(x). \tag{28}$$

From Lemma 1, we have

$$K_{m,s}^*(1;x) = 1,$$

$$\begin{aligned}
K_{m,s}^{*}(t-x;x) &= K_{m,s}^{(p,q)}\big((t-x);x\big) - \left(\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s} - p^{m} + p^{-s}q^{m+s})x + p^{m}}{[2]_{p,q}[m]_{p,q}} - x\right) \\
&= \left(\frac{[m+s]_{p,q}}{p^{s-1}[m]_{p,q}} - \frac{p^{m}}{[2]_{p,q}[m]_{p,q}} + \frac{q^{m+s}}{[2]_{p,q}[m]_{p,q}p^{s}} - 1\right)x + x \\
&\quad + \frac{p^{m}}{[2]_{p,q}[m]_{p,q}} - \frac{([2]_{p,q}[m+s]_{p,q}p^{1-s} - p^{m} + p^{-s}q^{m+s})x + p^{m}}{[2]_{p,q}[m]_{p,q}} \\
&= 0. \qquad (29)
\end{aligned}$$

Taylor's expansion for a function $g \in C^{2}[0,s+1]$ can be written as follows:

$$g(t) = g(x) + (t-x)g'(x) + \int_{x}^{t}(t-u)g''(u)\,du, \quad t \in [0,1]. \qquad (30)$$

Then applying operator $K_{m,s}^{*}$ to both sides of (30), we get

$$\begin{aligned}
K_{m,s}^{*}(g;x) &= K_{m,s}^{*}\left(g(x) + (t-x)g'(x) + \int_{x}^{t}(t-u)g''(u)\,du\right) \\
&= g(x) + K_{m,s}^{*}\big((t-x)g'(x);x\big) + K_{m,s}^{*}\left(\int_{x}^{t}(t-u)g''(u)\,du\right).
\end{aligned}$$

So,

$$K_{m,s}^{*}(g;x) - g(x) = g'(x)K_{m,s}^{*}\big((t-x);x\big) + K_{m,s}^{*}\left(\int_{x}^{t}(t-u)g''(u)\,du\right).$$

Using (29) and (28), we obtain

$$\begin{aligned}
&K_{m,s}^{*}(g;x) - g(x) \\
&\quad = K_{m,s}^{*}\left(\int_{x}^{t}(t-u)g''(u)\,du\right) \\
&\quad = K_{m,s}^{(p,q)}\left(\int_{x}^{t}(t-u)g''(u)\,du\right) \\
&\qquad - \int_{x}^{\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s} - p^{m} + p^{-s}q^{m+s})x + p^{m}}{[2]_{p,q}[m]_{p,q}}}\left(\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s} - p^{m} + p^{-s}q^{m+s})x + p^{m}}{[2]_{p,q}[m]_{p,q}}\right. \\
&\qquad \left. - u\right)g''(u)\,du \\
&\qquad + \int_{x}^{x}\left(\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s} - p^{m} + p^{-s}q^{m+s})x + p^{m}}{[2]_{p,q}[m]_{p,q}} - u\right)g''(u)\,du. \qquad (31)
\end{aligned}$$

Moreover,

$$\left|\int_{x}^{t}(t-u)g''(u)\,du\right| \leq \int_{x}^{t}|t-u|\big|g''(u)\big|\,du \leq \big\|g''\big\|\int_{x}^{t}|t-u|\,du \leq (t-x)^{2}\big\|g''\big\| \qquad (32)$$

and

$$\begin{aligned}&\left|\int_x^{\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s}-p^m+p^{-s}q^{m+s})x+p^m}{[2]_{p,q}[m]_{p,q}}}\left(\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s}-p^m+p^{-s}q^{m+s})x+p^m}{[2]_{p,q}[m]_{p,q}}\right.\right.\\&\qquad\left.\left.-u\right)g''(u)\,du\right|\\&\quad\le\|g''\|\int_x^{\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s}-p^m+p^{-s}q^{m+s})x+p^m}{[2]_{p,q}[m]_{p,q}}}\left(\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s}-p^m+p^{-s}q^{m+s})x+p^m}{[2]_{p,q}[m]_{p,q}}\right.\\&\qquad\left.-u\right)du\\&\quad=\frac{\|g''\|}{2}\left(\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s}-p^m+p^{-s}q^{m+s})x+p^m}{[2]_{p,q}[m]_{p,q}}-x\right)^2.\end{aligned}\tag{33}$$

Let us employ (32) and (33) when taking the absolute value of (31). We obtain

$$\begin{aligned}\left|K^*_{m,s}(g;x)-g(x)\right|&\le\|g''\|K^{(p,q)}_{m,s}\left((t-x)^2;x\right)\\&\quad+\frac{\|g''\|}{2}\left(\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s}-p^m+p^{-s}q^{m+s})x+p^m}{[2]_{p,q}[m]_{p,q}}-x\right)^2\\&=\|g''\|\alpha_m^2(x),\end{aligned}$$

where

$$\begin{aligned}&\alpha_m(x)\\&\quad=\sqrt{K^{(p,q)}_{m,s}\left((t-x)^2;x\right)+\frac{1}{2}\left(\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s}-p^m+p^{-s}q^{m+s})x+p^m}{[2]_{p,q}[m]_{p,q}}-x\right)^2}.\end{aligned}\tag{34}$$

We now give an upper bound for the auxiliary operator $K^*_{m,l,p,q}(f;x)$. From Lemma 4 we get

$$\begin{aligned}\left|K^*_{m,s}(f;x)\right|&=\left|K^{(p,q)}_{m,s}(f;x)-f\left(\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s}-p^m+p^{-s}q^{m+s})x+p^m}{[2]_{p,q}[m]_{p,q}}\right)+f(x)\right|\\&\le\left|K^{(p,q)}_{m,s}(f;x)\right|+\left|f\left(\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s}-p^m+p^{-s}q^{m+s})x+p^m}{[2]_{p,q}[m]_{p,q}}\right)\right|+\left|f(x)\right|\\&\le 3\|f\|.\end{aligned}$$

Accordingly,

$$\begin{aligned}&\left|K^{(p,q)}_{m,s}(f;x)-f(x)\right|\\&\quad=\left|K^*_{m,s}(f;x)-f(x)+f\left(\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s}-p^m+p^{-s}q^{m+s})x+p^m}{[2]_{p,q}[m]_{p,q}}\right)-f(x)\right.\\&\qquad\left.\mp g(x)\mp K^*_{m,s}(g;x)\right|,\end{aligned}$$

$$
\begin{aligned}
&\left|K_{m,s}^{(p,q)}(f;x)-f(x)\right| \\
&\quad\le \left|K_{m,s}^{*}(f-g;x)-(f-g)(x)\right| + \left|K_{m,s}^{*}(g;x)-g(x)\right| \\
&\qquad + \left|f\left(\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s}-p^{m}+p^{-s}q^{m+s})x+p^{m}}{[2]_{p,q}[m]_{p,q}}\right)-f(x)\right| \\
&\quad\le 4\|f-g\| + \left\|g''\right\|\alpha_m^2(x) + \omega\bigl(f,\beta_m(x)\bigr)\left(\frac{\bigl(\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s}-p^{m}+p^{-s}q^{m+s})x+p^{m}}{[2]_{p,q}[m]_{p,q}}-x\bigr)}{\beta_m(x)}+1\right) \\
&\quad= 4\|f-g\| + \left\|g''\right\|\alpha_m^2(x) \\
&\qquad + 2\omega\left(f,\left(\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s}-p^{m}+p^{-s}q^{m+s})x+p^{m}}{[2]_{p,q}[m]_{p,q}}-x\right)\right),
\end{aligned}
\tag{35}
$$

where

$$
\beta_m(x)=\frac{([2]_{p,q}[m+s]_{p,q}p^{1-s}-p^{m}+p^{-s}q^{m+s})x+p^{m}}{[2]_{p,q}[m]_{p,q}}-x. \tag{36}
$$

Finally, for all $g\in C^2[0,s+1]$, taking the infimum of (35), we get

$$
\left|K_{m,s}^{(p,q)}(f;x)-f(x)\right| \le 4K\bigl(f,\alpha_m^2(x)\bigr)+\omega\bigl(f,\beta_m(x)\bigr). \tag{37}
$$

Consequently, using the property of Peetre-K functional, we obtain

$$
\left|K_{m,s}^{(p,q)}(f;x)-f(x)\right| \le M\omega_2\bigl(f,\alpha_m(x)\bigr)+\omega\bigl(f,\beta_m(x)\bigr). \tag{38}
$$

This completes the proof. □

4 Graphical illustrations

In this section, we illustrate an approximation of the operators $K_{m,s}^{(p,q)}$ for a function $f(x)$ by employing Matlab codes. Let us specially choose

$$
f(x)=\frac{1}{96}\tan\left(\frac{x}{16}\right)\left(\frac{x}{8}\right)^2\left(1-\frac{x}{4}\right)^3,
$$

and take $p=0.8$, $q=0.7$ and $s=5$.

Algorithm 1

```
function y=pqinteger(n,p,q)
    y=(p^(n)-q^(n))/(p-q);
end

clear all close all clc format short
syms x
n=[5, 10, 15];
s=5;
p=0.8;
q=0.7;
for i=1:3
   f=1/96*tan(x/16)*(x/8)^2*(1-x/4)^3;
    m=n(i);
 f2=abs(diff(f,2))
 error=inline(char([((q^2)*(p^(2-2*s))*pqinteger(m+s,p,q)*pqinteger(m+s-1,p,q))/(((pqinteger(m,p,q))^2)*(p*(1-x)+q*x))+...
     (-2*pqinteger(2,p,q)*pqinteger(m+s,p,q)*p^(1-s)+2*p^m-2*q^(m+s)*p^(-s))/(pqinteger(2,p,q)*pqinteger(m,p,q))+1]*x^2+...
     [(pqinteger(m+s,p,q)*p^(m-s+1))/(pqinteger(m,p,q))^2+(2*pqinteger(m+s,p,q)*q*(p^(4*m+2*s-3))*((p^(m+s))*(1-x)+...
     (q^(m+s))*x))/(pqinteger(2,p,q)*(pqinteger(m,p,q))^2*(p*(1-x)+q*x))-(2*p^m)/(pqinteger(2,p,q)*pqinteger(m,p,q))]*x+...
     (((p^(m+s))*(1-x)+(q^(m+s))*x)*((p^(m+s+1))*(1-x)+(q^(m+s+1))*x)*p^(-2*s))/(pqinteger(3,p,q)*(pqinteger(m,p,q))^2*(p*(1-x)+q*x))+...
     (1/2)*(((pqinteger(2,p,q)*pqinteger(m+s,p,q)*p^(1-s)-p^m+p^(-s)*q^(m+s))*x+p^m)/(pqinteger(2,p,q)*pqinteger(m,p,q))-x)^2)*f2)
dat(i,1)=n(i);
dat(i,2)=error(0.1);
dat(i,3)=error(0.5);
dat(i,4)=error(0.9);
end
dat
er=abs(dat(:,2:4))
```

Algorithm 2

```
clear all
close all
clc
syms t
n=[5,10,15];
for j=1:3
    m=n(j);
s=5;
p1=[0.7, 0.8, 0.9]
q1=[0.6, 0.7, 0.8]
 subplot(1,3,j)
for i=1:3
    p=p1(i)
    q=q1(i)
    a=[1:100];
    u=1;
    for x=0:0.01:1
        ts=0;
        for l=0:m+s
            z=1;
            for j=1:m+s
                z=z*((p^(j-1)).*(1-x)+(q^(j-1)).*x);
            end
  h1=1;
  for a1=0:m+s-1
  h1=h1*pqinteger(m+s-a1,p,q);
  end
  if (l==0)
      h2=1;
  end
  if (l~=0)
       h2=1;
       for a2=0:l-1
       h2=h2*pqinteger(l-a2,p,q);
       end
  end
  h3=1;
  for a3=0:m+s-l-1
  h3=h3*pqinteger(m+s-l-a3,p,q);
  end
  fact=h1/(h2*h3);
     f12=0;
     for k=0:100
     x1=(p*pqinteger(l,p,q)+(q^k)/(p^(k+1))*q^l)/(pqinteger(m,p,q)*p^(l-m));
    f11=(q^k)/(p^(k+1))*1/96*tan(x1/16)*(1*x1/8)^2*(1-x1*1/4)^3;
     f12=f12+f11;
     end
     f1=(p-q)*f12;
     z;
     B=fact*(p^((m+s-l)*(m+s-l-1)/2))*(q^(l*(l-1)/2))*(x^l)*(1-x)^(m+s-l)*f1;
     ts=ts+B/z;
     end
     a(u)=ts;
     u=u+1;
    end
     x=0:0.01:1;
    if (i==1)
        c=plot(x,a,'g');
         hold on
   elseif (i==2)
        c=plot(x,a,'r');
         hold on
    else (i==3)
        c=plot(x,a,'m');
    end
end
 x=0:0.01:1;
    y=1/96*tan(x/16).*(1*x/8).^2.*(1-x*1/4).^3;
    plot(x,y,'--p')
     legend('p=0.7, q=0.6','p=0.8, q=0.7','p=0.9, q=0.8','function')
end
a;
```

Initially, we discuss the error estimates of the Kantorovich type Lupaş–Schurer operators based on (p,q)-integers for different values of x and m in Table 1 by using Algorithm 1.

And then, we illustrate the convergence of the (p,q)-Lupaş–Schurer–Kantorovich operators $K_{m,s}^{(p,q)}(f;x)$ for the selected function $f(x) = \frac{1}{96}\tan(\frac{x}{16})(\frac{x}{8})^2(1-\frac{x}{4})^3$ in Fig. 1 for several values of m by using Algorithm 2. Furthermore, we give the error estimates in Table 2 in order to indicate that the (p,q)-analogue Lupaş–Schurer operators [14] converge and

Table 1 Error estimates for different values of x when $s=5$, $p=0.8$ and $q=0.7$

m	Error at $x=0.1$	Error at $x=0.5$	Error at $x=0.9$
5	$0.1494 \cdot 10^{-6}$	$0.0583 \cdot 10^{-6}$	$0.0441 \cdot 10^{-6}$
10	$0.0326 \cdot 10^{-6}$	$0.3298 \cdot 10^{-6}$	$0.1599 \cdot 10^{-6}$
15	$0.0135 \cdot 10^{-6}$	$0.2398 \cdot 10^{-6}$	$0.0078 \cdot 10^{-6}$

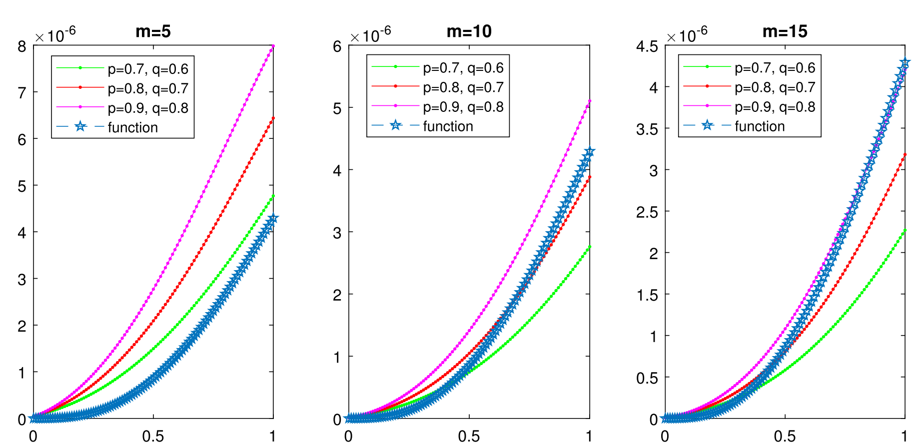

Figure 1 Convergence of (p,q)-analogue Lupaş–Schurer–Kantorovich operators $K_{m,s}^{(p,q)}(f;x)$ for various values of p, 1 and m with fixed $s=5$

Table 2 Error estimates of (p,q)-Lupaş–Schurer operators for various values of x

m	Error at $x = 0.1$	Error at $x = 0.5$	Error at $x = 0.9$
5	$0.0067 \cdot 10^{-5}$	$0.3011 \cdot 10^{-5}$	$0.4464 \cdot 10^{-5}$
10	$0.0073 \cdot 10^{-5}$	$0.3821 \cdot 10^{-5}$	$0.5743 \cdot 10^{-5}$
15	$0.0075 \cdot 10^{-5}$	$0.4077 \cdot 10^{-5}$	$0.6141 \cdot 10^{-5}$

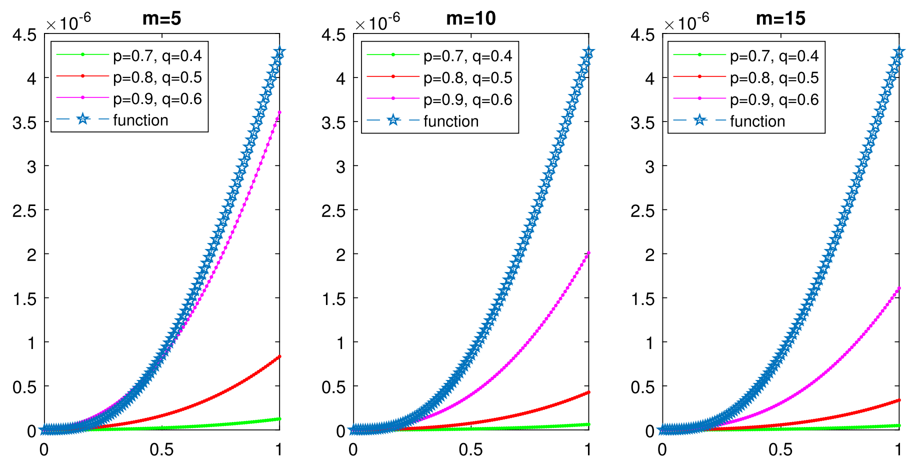

Figure 2 Convergence of the (p,q)-analogue Lupaş–Schurer operators $L^{p,q}_{m,l}(f;x)$ with fixed $l = 5$ for various values of p and q

then plot Fig. 2. It can be clearly seen that the (p,q)-Lupaş–Schurer–Kantorovich operators converge faster than the (p,q)-analogue Lupaş–Schurer operators.

5 Conclusion

In this paper, we constructed a new kind of Lupaş operators based on (p,q)-integers to provide a better error estimation. Firstly, we investigated some local approximation results by the help of the well-known Korovkin theorem. Also, we calculated the rate of convergence of the constructed operators employing the modulus of continuity, by using Lipschitz functions and then with the help of Peetre's K-functional. Additionally, we presented a table of error estimates of the (p,q)-Lupaş–Schurer–Kantorovich operators for a certain function. Finally, we compared the convergence of the new operator to that of the (p,q)-analogue of Lupaş–Schurer operator.

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Authors' contributions
The authors declare that they have studied in collaboration and share the same responsibility for this paper. All authors read and approved the final manuscript.

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