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New generalized systems of nonlinear ordered variational inclusions involving $\oplus$ operator in real ordered Hilbert spaces

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Abstract
This manuscript deals with two general systems of nonlinear ordered variational inclusion problems. We also construct some new iterative algorithms for finding approximation solutions to the general systems of nonlinear ordered variational inclusions and prove the convergence of the sequences obtained by the schemes. The results presented in the manuscript are new and improve some well-known results in the literature.

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1 Introduction

A lot of work has been added into the theory of variational inequalities since its seed was planted by Lions et al. [24]. On account of its wide applications in physics and applied sciences etc., the classical variational inequalities have been extensively studied by many researchers in different ways [1, 4, 5, 7–10].

A useful and important generalization of variational inequality problem is variational inclusion problem which was introduced and studied by Hasounni et al. [16]. Furthermore, they proposed a perturbed iterative algorithm for solving the variational inclusion problem.

Fang et al. [12] introduced and studied H-monotone operators, which was used to design a resolvent operator and to prove its Lipschitz continuity. Furthermore, they also introduced a class of variational inclusions in Hilbert space. Fang et al. [13] additionally presented another class of generalized monotone operators, called (H,η)-monotone operators, which generalize different classes of maximal monotone, maximal η-monotone and H-monotone operators.

Recently, Lan et al. [17] presented another idea of (A,η)-accretive mappings, which generalized the current monotone or accretive operators, and concentrated a few properties of mappings. They examined a class of variational inclusions using the resolvent operator related with (A,η)-accretive mappings.

Amann [6] studied the number of fixed points for a continuous operator $A:[x,y]\to[x,y]$ on a bounded order interval $[x,y]\subset\mathcal{E}$, an ordered Banach space. The nonlinear mapping fixed point theory and applications have been widely studied in ordered Banach spaces [4, 14, 15]. In this manner, it is essential that summed up nonlinear ordered variational inclusions (ordered equation) are contemplated.

Plenty of research concerned with the ordered equations and ordered variational inequalities in ordered Banach spaces has been done by Li et al.; see [21, 23]. Many problems concerning ordered variational inclusions are answered by the resolvent technique linked with RME set-valued mappings [19], (α,λ)-NODM set-valued mapping [20], (γ_G,λ)-weak RRD mapping [2] and (α,λ)-weak ANODD set-valued map with strongly comparison mapping A [21] and many more see; e.g., [3, 22, 25, 26, 29] and the references therein.

In this work, we make use of the resolvent operator approach for the approximation solvability of solutions of implicit system of generalized nonlinear ordered variational inclusions in real ordered Hilbert spaces.

2 Preliminaries

In this part, we present some basic notions and results for the building up the manuscript.

Allow $\mathcal{E}$ to be a real ordered Hilbert space endowed with a norm $\|\cdot\|$, and an inner product $\langle\cdot,\cdot\rangle$, d be a metric induced by the norm $\|\cdot\|$, $CB(\mathcal{E})$ be a collection of all closed and bounded subsets of $\mathcal{E}$ and $D(\cdot,\cdot)$ be a Hausdorff metric on $CB(\mathcal{E})$ defined as

$$D(M,N)=\max\Big\{\sup_{x\in M} d(x,N),\sup_{y\in N} d(M,y)\Big\},$$

where $M,N\in CB(\mathcal{E})$, $d(x,N)=\inf_{y\in N} d(x,y)$ and $d(M,y)=\inf_{x\in M} d(x,y)$.

Definition 2.1 Let $\mathfrak{C}$ be a nonvoid closed, convex subset of $\mathcal{E}$. Then $\mathfrak{C}$ is called a cone if

(a) $x\in\mathfrak{C}$ and $\kappa>0$, $\kappa x\in\mathfrak{C}$;

(b) x and $-x\in\mathfrak{C}$, then $x=\Theta$.

Definition 2.2 ([11]) A cone $\mathfrak{C}$ is said to be normal iff there exists $\lambda_{N_C}>0$ with $0\le x\le y$ implying $\|x\|\le\lambda_{N_C}\|y\|$, where λ_{N_C} is called a normal constant of $\mathfrak{C}$.

Definition 2.3 A relation $\le$ defined as $x\le y$ iff $y-x\in\mathfrak{C}$ for $x,y\in\mathcal{E}$ is known as a partial order relation expounded by $\mathfrak{C}$ in $\mathcal{E}$; then $(\mathcal{E},\le)$ is called a real ordered Hilbert space.

Definition 2.4 ([27]) Members $x,y\in\mathcal{E}$ having the relation $x\le y$ (or $y\le x$) are called comparable with each other.

Definition 2.5 ([27]) For arbitrary elements $x,y\in\mathcal{E}$, $\operatorname{lub}\{x,y\}$ and $\operatorname{glb}\{x,y\}$ mean the least upper bound and the greatest upper bound of the set $\{x,y\}$. Suppose $\operatorname{lub}\{x,y\}$ and $\operatorname{glb}\{x,y\}$ exist; some binary relations are defined as follows:

(a) $x\vee y=\operatorname{lub}\{x,y\}$;

(b) $x\wedge y=\operatorname{glb}\{x,y\}$;

(c) $x\oplus y=(x-y)\vee(y-x)$;

(d) $x\odot y=(x-y)\wedge(y-x)$.

The operations $\vee$, $\wedge$, $\oplus$ and $\odot$ are called **OR, AND, XOR** and **XNOR** operations, respectively.

Proposition 1 ([11]) *For any positive integer n, if $x \propto y_n$ and $y_n \to y^*$ ($n \to \infty$), then $x \propto y^*$.*

Proposition 2 ([11, 20]) *Let **XOR**, **XNOR** be two operations on $\mathcal{E}$. Then the following hold*:

(a) $x \odot x = 0, x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$;
(b) $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$;
(c) $x \odot 0 \le 0$, *if* $x \propto 0$;
(d) $0 \le x \oplus y$, *if* $x \propto y$;
(e) *if* $x \propto y$, *then* $x \oplus y = 0$ *if and only if* $x = y$;
(f) $(x + y) \odot (u + v) \ge (x \odot u) + (y \odot v)$;
(g) $(x + y) \odot (u + v) \ge (x \odot v) + (y \odot u)$;
(h) $(\alpha x \oplus \beta x) = |\alpha - \beta|x$, *if* $x \propto 0$, $\forall x, y, u, v \in \mathcal{E}$ *and* $\alpha, \beta, \lambda \in \mathbb{R}$.

Proposition 3 ([11]) *Let $\mathfrak{C}$ be a normal cone in $\mathcal{E}$ with normal constant λ_{N_C}, then, for each $x, y \in \mathcal{E}$, the following hold*:

(a) $\|0 + 0\| = \|0\| = 0$;
(b) $\|x \vee y\| \le \|x\| \vee \|y\| \le \|x\| + \|y\|$;
(c) $\|x \oplus y\| \le \|x - y\| \le \lambda_{N_C}\|x \oplus y\|$;
(d) *if* $x \propto y$, *then* $\|x \oplus y\| = \|x - y\|$.

Definition 2.6 ([20]) Let $A : \mathcal{E} \to \mathcal{E}$ to be a single-valued map.

(a) A is called a δ-order non-extended map, if there is a positive constant $\delta > 0$ such that

$$\delta(x \oplus y) \le A(x) \oplus A(y) \quad \text{for all } x, y \in \mathcal{E};$$

(b) A is called a strongly comparison map, if it is a comparison map and $A(x) \propto A(y)$ iff $x \propto y$, for all $x, y \in \mathcal{E}$.

Definition 2.7 ([2]) A single-valued map $A : \mathcal{E} \to \mathcal{E}$ is termed a β-ordered compression, if it is comparison map and

$$A(x) \oplus A(y) \le \beta(x \oplus y), \quad \text{for } 0 < \beta < 1.$$

Definition 2.8 ([18]) A map $A : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is called (α_1, α_2)-restricted-accretive map, if it is a comparison and $\exists$ constants $0 \le \alpha_1, \alpha_2 \le 1$ such that

$$\bigl(A(x, \cdot) + I(x)\bigr) \oplus \bigl(A(y, \cdot) + I(y)\bigr) \le \alpha_1\bigl(A(x, \cdot) \oplus A(y, \cdot)\bigr) + \alpha_2(x \oplus y), \quad \text{for all } x, y \in \mathcal{E}$$

where I is the identity map on $\mathcal{E}$.

Lemma 2.1 ([28]) *Let $\theta \in (0, 1)$ be a constant. Then the function $f(\lambda) = 1 - \lambda + \lambda\theta$ for $\lambda \in [0, 1]$ is nonnegative and strictly decreases and $f(\lambda) \in [0, 1]$. Furthermore, if $\lambda \neq 0$, then $f(\lambda) \in (0, 1)$.*

Lemma 2.2 ([30]) *Assume that $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that*

$$a_{n+1} \le \theta a_n + b_n,$$

where $\theta \in (0, 1)$ and $\lim_{n\to\infty} b_n = 0$. Then $\lim_{n\to\infty} a_n = 0$.

3 Ordered weak-ARD mapping in ordered Hilbert spaces

Definition 3.1 Let $A : \mathcal{E} \to \mathcal{E}$ be a strong comparison and β-ordered compression mapping and $M : \mathcal{E} \to CB(\mathcal{E})$ be a set-valued mapping. Then

(a) M is said to be a comparison mapping, if for any $v_x \in M(x)$, $x \propto v_x$ and if $x \propto y$, then, for any $v_x \in M(x)$ and any $v_y \in M(y)$, $v_x \propto v_y$, for all $x, y \in \mathcal{E}$;

(b) a comparison mapping M is said to be ordered rectangular, if for each $x, y \in \mathcal{E}$, $v_x \in M(x)$ and $v_y \in M(y)$ such that

$$\langle v_x \odot v_y, -(x \oplus y)\rangle = 0;$$

(c) a comparison mapping M is said to be a γ-ordered rectangular with respect to A, if there exists a constant $\gamma_A > 0$ for any $x, y \in \mathcal{E}$, there exist $v_x \in M(A(x))$ and $v_y \in M(A(y))$ such that

$$\langle v_x \odot v_y, -(A(x) \oplus A(y))\rangle \geq \gamma_A \|A(x) \oplus A(y)\|^2,$$

holds, where v_x and v_y are said to be γ_A-elements, respectively;

(d) M is said to be a weak comparison mapping with respect to A, if, for any $x, y \in \mathcal{E}$, $x \propto y$, there exist $v_x \in M(A(x))$ and $v_y \in M(A(y))$ such that $x \propto v_x$, $y \propto v_y$, where v_x and v_y are said to be weak comparison elements, respectively;

(e) M is said to be a λ-weak ordered different comparison mapping with respect to A, if there exists a constant $\lambda > 0$ such that, for any $x, y \in \mathcal{E}$, there exist $v_x \in M(A(x))$ and $v_y \in M(A(y))$, $\lambda(v_x - v_y) \propto (x - y)$ holds, where v_x and v_y are said to be λ-elements, respectively;

(f) a weak comparison mapping M is said to be a (γ_A, λ)-weak ARD mapping with respect to A, if M is a γ_A-ordered rectangular and λ-weak ordered different comparison mapping with respect to A and $(A + \lambda M)(\mathcal{E}) = \mathcal{E}$, for $\lambda > 0$ and there exist $v_x \in M(A(x))$ and $v_y \in M(A(y))$ such that v_x and v_y are (γ_A, λ)-elements, respectively.

Definition 3.2 A set-valued mapping $A : \mathcal{E} \to CB(\mathcal{E})$ is said to be δ_A-Lipschitz continuous, if for each $x, y \in \mathcal{E}, x \propto y$, there exists a constant δ_A such that

$$D(A(x), A(y)) \leq \delta_A \|x \oplus y\|, \quad \forall x, y \in \mathcal{E}.$$

Definition 3.3 Let $M : \mathcal{E} \to CB(\mathcal{E})$ be a set-valued mapping, $A : \mathcal{E} \to \mathcal{E}$ be a single-valued mapping and $I : \mathcal{E} \to \mathcal{E}$ be an identity mapping. Then a weak comparison mapping M is said to be a (γ', λ)-weak-ARD mapping with respect to $(I - A)$, if M is a γ'-ordered rectangular and λ-weak ordered different comparison mapping with respect to $(I - A)$ and $[(I - A) + \lambda M](\mathcal{E}) = \mathcal{E}$, for $\lambda > 0$ and there exist $v_x \in M((I - A)(x))$ and $v_y \in M((I - A)(y))$ such that v_x and v_y are called (γ', λ)-elements, respectively.

Definition 3.4 Let $\mathfrak{C}$ be a normal cone with normal constant λ_{N_C} and $M : \mathcal{E} \to CB(\mathcal{E})$ be a weak-ARD set-valued mapping. Let $I : \mathcal{E} \to \mathcal{E}$ be the identity mapping and $A : \mathcal{E} \to \mathcal{E}$ be a set-valued mapping and $A : \mathcal{E} \to \mathcal{E}$ be a single-valued mapping. The relaxed resolvent operator $R^{(I-A)}_{M,\lambda} : \mathcal{E} \to \mathcal{E}$ associated with I, A and M is defined by

$$R^{(I-A)}_{M,\lambda}(x) = [(I - A) + \lambda M]^{-1}(x), \quad \forall x \in \mathcal{E} \text{ and } \lambda > 0. \tag{1}$$

The relaxed resolvent operator defined by (1) is single-valued, a comparison mapping and Lipschitz continuous.

Proposition 4 ([2]) *Let $A : \mathcal{E} \to \mathcal{E}$ be a β-ordered compression mapping and $M : \mathcal{E} \to CB(\mathcal{E})$ be the set-valued ordered rectangular mapping. Then the resolvent $R_{M,\lambda}^{(I-A)} : \mathcal{E} \to \mathcal{E}$ is single-valued, for all $\lambda > 0$.*

Proposition 5 ([2]) *Let $M : \mathcal{E} \to CB(\mathcal{E})$ be a (γ_A, λ)-weak-ARD set-valued mapping with respect to $R_{M,\lambda}^{(I-A)}$. Let $A : \mathcal{E} \to \mathcal{E}$ be a strongly comparison mapping with respect to $R_{M,\lambda}^{(I-A)}$ and $I : \mathcal{E} \to \mathcal{E}$ be the identity mapping. Then the resolvent operator $R_{M,\lambda}^{(I-A)} : \mathcal{E} \to \mathcal{E}$ is a comparison mapping.*

Proposition 6 ([2]) *Let $M : \mathcal{E} \to CB(\mathcal{E})$ be a (γ_A, λ)-weak-ARD set-valued mapping with respect to $R_{M,\lambda}^{(I-A)}$. Let $A : \mathcal{E} \to \mathcal{E}$ be a strongly comparison and β-ordered compression mapping with respect to $R_{M,\lambda}^{(I-A)}$ with condition $\lambda\gamma_A > \beta + 1$. Then the following condition survives:*

$$\left\| R_{M,\lambda}^{(I-A)}(x) \oplus R_{M,\lambda}^{(I-A)}(y) \right\| \le \left(\frac{1}{\lambda\gamma_A - \beta - 1} \right) \|x \oplus y\|.$$

4 Formulation of the problems

Let $F_i : \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m \to \mathcal{E}_i$, $A_i : \mathcal{E}_i \to \mathcal{E}_i$ and $g_i : \mathcal{E}_i \to \mathcal{E}_i$ to be single-valued mappings, for $i, j = 1, 2, 3, \ldots, m$. Let $U_{ij} : \mathcal{E}_j \to CB(\mathcal{E}_j)$ be a set-valued map and $M_i : \mathcal{E}_i \to CB(\mathcal{E}_i)$ be set-valued weak-ARD mapping. Then we have the problem:

Find $(x_1^*, x_2^*, \ldots, x_m^*) \in \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m$ and $u_{ij}^* \in U_{ij}(x_j^*)$, for $i, j = 1, 2, 3, \ldots m$, such that

$$0 \in \rho_i F_i\left(u_{i1}^*, u_{i2}^*, \ldots, u_{im}^*\right) \oplus \lambda_i M_i\left(g_i\left(x_i^*\right)\right), \tag{2}$$

where ρ_i and λ_i are given positive constants. Problem (2) is called a generalized set-valued system of nonlinear ordered variational inclusions problem for weak-ARD mappings.

If $U_{ij} = T_{ij}$ is a single-valued mapping, then problem (2) becomes:

Find $x_j \in \mathcal{E}_j$, such that

$$0 \in \rho_i F_i\left(T_{i1}x_1^*, T_{i2}x_2^*, \ldots, T_{im}x_m^*\right) \oplus \lambda_i M_i\left(g_i\left(x_i^*\right)\right). \tag{3}$$

This problem is known as a generalized system of nonlinear ordered variational inclusions problem involving weak-ARD mappings.

Remark Here, we discuss special cases for our problem (2), which was encountered by Li et al.

Case 1. For $i, j = 1$, $\rho_i = 1$, $\lambda_i = 1$ and $U_{ij} = g_i = I$, then problem (2) is reduced to finding $x \in \mathcal{E}_1$ such that

$$0 \in F_1(x) \oplus M_1(x). \tag{4}$$

This problem was considered by Li et al. [23] and coined a general nonlinear mixed-order quasi-variational inclusion (GNMOQVI) involving the $\oplus$ operator in an ordered Banach space.

Case 2. If $F = 0$ (zero mapping), then problem (4) is reduced to finding $x \in \mathcal{E}$ such that

$$0 \in M(x). \tag{5}$$

This problem were considered by Li for ordered RME set-valued mappings [19] and (α,λ)-NODM set-valued mappings [20].

Lemma 4.1 *Let* $(x_1^*, x_2^*, \dots, x_m^*) \in \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m$ *and* $u_{ij}^* \in U_{ij}(x_j^*)$ *for* $i,j = 1,2,3,\dots,m$. *Then* $(x_1^*, x_2^*, \dots, x_m^*, u_{11}^*, u_{12}^*, \dots, u_{1m}^*, \dots, u_{m1}^*, u_{m2}^*, \dots, u_{mm}^*)$ *is a solution of problem* (2) *if and only if it satisfies*

$$g_i(x_i^*) = J_{\lambda_i,M_i}^{I_i-A_i}\big[(I_i - A_i)\big(g_i(x_i^*)\big) + \rho_i F_i\big(u_{i1}^*, u_{i2}^*, \dots, u_{im}^*\big)\big], \tag{6}$$

where $J_{\lambda_i,M_i}^{I_i-A_i}(x) = [(I_i - A_i) + \lambda_i M_i]^{-1}(x)$ *and* $\rho_i, \lambda_i > 0$ *for* $i = 1,2,\dots,m$.

Proof The proof follows from the definition of the relaxed resolvent operator. □

5 Design of the algorithms

Remark If we choose $\lambda = 1$ and $U_{ij} = T_{ij}$ for $i,j = 1,2,\dots,m$, is single-valued operator, then Algorithm 1 reduces to Algorithm 2 for problem (3).

Algorithm 1 for the problem (2):

For $i,j = 1,2,\dots,m$, choose $(x_1^0, x_2^0, \dots, x_m^0) \in \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m$ and $u_{ij}^0 \in U_{ij}(x_j^0)$. For $n = 0,1,2,3,\dots$, set:

$$\begin{aligned} x_i^{n+1} = {} & (1-\lambda)x_i^n + \lambda\big[x_i^n - g_i(x_i^n) + J_{\lambda_i,M_i}^{I_i-A_i}\big[(I_i - A_i)\big(g_i(x_i^n)\big) \\ & + \rho_i F_i\big(u_{i1}^n, u_{i2}^n, \dots, u_{im}^n\big)\big]\big]. \end{aligned} \tag{7}$$

From Nadler's result, choose $u_{ij}^{n+1} \in U_{ij}(x_j^{n+1})$ such that

$$\|u_{ij}^{n+1} \oplus u_{ij}^n\| \le \left(1 + \frac{1}{(n+1)}\right) D_j\big(U_{ij}(x_j^{n+1}), U_{ij}(x_j^n)\big).$$

Algorithm 2 for the problem (3):

For $n = 0,1,2,\dots$, $i = 1,2,\dots,m$, choose $(x_1^0, x_2^0, \dots, x_m^0) \in \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m$, x_i^n is computed as follows:

$$\begin{aligned} x_i^{n+1} = {} & x_i^n - g_i(x_i^n) + J_{\lambda_i,M_i}^{I_i-A_i}\big[(I_i - A_i)\big(g_i(x_i^n)\big) \\ & + \rho_i F_i\big(T_{i1}x_1^n, T_{i2}x_2^n, \dots, T_{im}x_m^n\big)\big] + w_i^n, \end{aligned} \tag{8}$$

where $w_i^n \in \mathcal{E}_i$ is the error to take into account a possible inexact computation of the resolvent operator point satisfying condition $\lim_{n\to\infty} \|w_i^n\| = 0$.

6 Main results

Theorem 6.1 *Let $A_i : \mathcal{E}_i \to \mathcal{E}_i$, $g_i : \mathcal{E}_i \to \mathcal{E}_i$ and $F_i : \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m \to \mathcal{E}_i$ be the single-valued mappings such that A_i be λ_{A_i}-ordered compression mapping, g_i be λ_{g_i}-ordered compression, (α_1^i, α_2^i)-ordered restricted-accretive mapping and F_i be λ_{ij}-ordered compression mapping with respect to the jth argument. Let $U_{ij} : \mathcal{E}_j \to CB(\mathcal{E}_j)$ be a D_i-δ_{ij}-ordered Lipschitz continuous set-valued mapping. Let $M_i : \mathcal{E}_i \to CB(\mathcal{E}_i)$ be a $(\gamma_{A_i}, \lambda_i)$-weak rectangular different compression mapping with respect to A_i and if $x_i \propto y_i$, $J_{\lambda_i,M_i}^{I_i-A_i}(x_i) \propto J_{\lambda_i,M_i}^{I_i-A_i}(y_i)$ and for all λ_i, $\rho_i > 0$, then the following condition holds*:

$$\theta_j = \left\{\alpha_1^j + \alpha_2^j\lambda_{g_j} + L_j(\lambda_{g_j} + \lambda_{A_j}\lambda_{g_j}) + \sum_{i\neq j,i=1}^{m} L_i\rho_i\lambda_{F_{ij}}\delta_{D_{ij}}\right\} < 1, \tag{9}$$

for all $j = 1, 2, 3, \ldots, m$, which in turn, implies that problem (2) admits a solution $(x_1^*, x_2^*, \ldots, x_m^*, u_{11}^*, u_{12}^*, \ldots, u_{1m}^*, \ldots, u_{m1}^*, \ldots, u_{mm}^*)$, where $(x_1^*, x_2^*, \ldots, x_m^*) \in \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m$ and $u_{ij}^* \in U_{ij}(x_j^*)$. Moreover, iterative sequences $\{x_j^n\}$ and $\{u_{ij}^n\}$ generated by Algorithm 1, converge strongly to x_j^* and u_{ij}^*, for $i, j = 1, 2, \ldots, m$, respectively.

Proof Using Algorithm 1 and Proposition 2, for $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} x_i^{n+1} \oplus x_i^n &= \big((1-\lambda)x_i^n + \lambda\big[x_i^n - g_i(x_i^n) + J_{\lambda_i,M_i}^{I_i-A_i}\big[(I_i - A_i)(g_i(x_i^n)) \\ &\quad + \rho_i F_i(u_{i1}^n, u_{i2}^n, \ldots, u_{im}^n)\big]\big]\big) \oplus \big((1-\lambda)x_i^{n-1} \\ &\quad + \lambda\big[x_i^{n-1} - g_i(x_i^{n-1}) + J_{\lambda_i,M_i}^{I_i-A_i}\big[(I_i - A_i)(g_i(x_i^{n-1})) \\ &\quad + \rho_i F_i(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{im}^{n-1})\big]\big]\big) \\ &\leq (1-\lambda)(x_i^n \oplus x_i^{n-1}) + \lambda\big[(x_i^n - g_i(x_i^n)) \oplus (x_i^{n-1} - g_i(x_i^{n-1}))\big] \\ &\quad + \lambda\big\{J_{\lambda_i,M_i}^{I_i-A_i}\big[(I_i - A_i)(g_i(x_i^n)) + \rho_i F_i(u_{i1}^n, u_{i2}^n, \ldots, u_{im}^n)\big] \\ &\quad \oplus J_{\lambda_i,M_i}^{I_i-A_i}\big[(I_i - A_i)(g_i(x_i^{n-1})) + \rho_i F_i(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{im}^{n-1})\big]\big\} \\ &\leq (1-\lambda)(x_i^n \oplus x_i^{n-1}) + \lambda(\alpha_1^i(x_i^n \oplus x_i^{n-1}) + \alpha_2^i(g_i(x_i^n) \oplus g_i(x_i^{n-1})) \\ &\quad + \lambda\big\{J_{\lambda_i,M_i}^{I_i-A_i}\big[(I_i - A_i)(g_i(x_i^n)) + \rho_i F_i(u_{i1}^n, u_{i2}^n, \ldots, u_{im}^n)\big] \\ &\quad \oplus J_{\lambda_i,M_i}^{I_i-A_i}\big[(I_i - A_i)(g_i(x_i^{n-1})) + \rho_i F_i(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{im}^{n-1})\big]\big\} \\ &\leq (1-\lambda)(x_i^n \oplus x_i^{n-1}) + \lambda(\alpha_1^i + \alpha_2^i\lambda_{g_i})(x_i^n \oplus x_i^{n-1}) \\ &\quad + \lambda\big\{J_{\lambda_i,M_i}^{I_i-A_i}\big[(I_i - A_i)(g_i(x_i^n)) + \rho_i F_i(u_{i1}^n, u_{i2}^n, \ldots, u_{im}^n)\big] \\ &\quad \oplus J_{\lambda_i,M_i}^{I_i-A_i}\big[(I_i - A_i)(g_i(x_i^{n-1})) + \rho_i F_i(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{im}^{n-1})\big]\big\}. \end{aligned} \tag{10}$$

Using Definition 2.2, Proposition 6 and Eq. (10), we get

$$\begin{aligned} \|x_i^{n+1} \oplus x_i^n\| &\leq \lambda_{N_C}\big[(1-\lambda) + \lambda(\alpha_1^i + \lambda_{g_i}\alpha_2^i)\big]\|x_i^n \oplus x_i^{n-1}\| \\ &\quad + \lambda\lambda_{N_C}L_i\big\|\big[(I_i - A_i)(g_i(x_i^n)) + \rho_i F_i(u_{i1}^n, u_{i2}^n, \ldots, u_{im}^n)\big] \\ &\quad \oplus \big[(I_i - A_i)(g_i(x_i^{n-1})) + \rho_i F_i(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{im}^{n-1})\big]\big\| \\ &\leq \lambda_{N_C}\big[1 - \lambda(1 - (\alpha_1^i + \lambda_{g_i}\alpha_2^i))\big]\|x_i^n \oplus x_i^{n-1}\| \\ &\quad + \lambda\lambda_{N_C}L_i\big[\big\|(I_i - A_i)(g_i(x_i^n)) \oplus (I_i - A_i)(g_i(x_i^{n-1}))\big\|\big] \end{aligned}$$

$$
\begin{aligned}
&+ \lambda\lambda_{N_C} L_i \rho_i \big\| F_i\big(u_{i1}^n, u_{i2}^n, \dots, u_{im}^n\big) \oplus F_i\big(u_{i1}^{n-1}, u_{i2}^{n-1}, \dots, u_{im}^{n-1}\big) \big\| \\
&\le \lambda_{N_C} \big[1 - \lambda\big(1 - \big(\alpha_1^i + \lambda_{g_i}\alpha_2^i\big)\big)\big] \big\| x_i^n \oplus x_i^{n-1} \big\| \\
&\quad + \lambda\lambda_{N_C} L_i \big[\big\| \big(g_i(x_i^n) \oplus g_i(x_i^n)\big) \big\| + \big\| A_i\big(g_i(x_i^n)\big) \oplus A_i\big(g_i(x_i^{n-1})\big) \big\|\big] \\
&\quad + \lambda\lambda_{N_C} L_i \rho_i \big[\big\| F_i\big(u_{i1}^n, u_{i2}^n, \dots, u_{im}^n\big) \oplus F_i\big(u_{i1}^{n-1}, u_{i2}^{n-1}, \dots, u_{im}^{n-1}\big) \big\|\big] \\
&\le \lambda_{N_C} \big[1 - \lambda\big(1 - \big(\alpha_1^i + \lambda_{g_i}\alpha_2^i\big)\big)\big] \big\| x_i^n \oplus x_i^{n-1} \big\| \\
&\quad + \lambda\lambda_{N_C} L_i \big[(\lambda_{g_i} + \lambda_{A_i}\lambda_{g_i}) \big\| x_i^n \oplus x_i^{n-1} \big\|\big] \\
&\quad + \lambda\lambda_{N_C} L_i \rho_i \big[\big\| F_i\big(u_{i1}^n, u_{i2}^n, \dots, u_{im}^n\big) \oplus F_i\big(u_{i1}^{n-1}, u_{i2}^{n-1}, \dots, u_{im}^{n-1}\big) \big\|\big]. \qquad (11)
\end{aligned}
$$

Now, from Eq. (11), we compute

$$
\begin{aligned}
&\big\| F_i\big(u_{i1}^n, u_{i2}^n, \dots, u_{ii-1}^n, u_{ii}^n, u_{ii+1}^n, \dots, u_{im}^n\big) \oplus F_i\big(u_{i1}^{n-1}, u_{i2}^{n-1}, \dots, u_{ii-1}^{n-1}, u_{ii}^{n-1}, u_{ii+1}^{n-1}, \dots, u_{im}^{n-1}\big) \big\| \\
&\le \big\| F_i\big(u_{i1}^n, u_{i2}^n, \dots, u_{ii-1}^n, u_{ii}^n, u_{ii+1}^n, \dots, u_{im}^n\big) \\
&\quad \oplus F_i\big(u_{i1}^{n-1}, u_{i2}^n, \dots, u_{ii-1}^n, u_{ii}^n, u_{ii+1}^n, \dots, u_{im}^n\big) \big\| \\
&\quad + \big\| F_i\big(u_{i1}^{n-1}, u_{i2}^n, \dots, u_{ii-1}^n, u_{ii}^n, u_{ii+1}^n, \dots, u_{im}^n\big) \\
&\quad \oplus F_i\big(u_{i1}^{n-1}, u_{i2}^{n-1}, \dots, u_{ii-1}^{n-1}, u_{ii}^{n-1}, u_{ii+1}^{n-1}, \dots, u_{im}^{n-1}\big) \big\| + \cdots \\
&\quad + \big\| F_i\big(u_{i1}^{n-1}, u_{i2}^{n-1}, \dots, u_{ii-1}^{n-1}, u_{ii}^{n-1}, u_{im-1}^{n-1}, u_{im}^n\big) \\
&\quad \oplus F_i\big(u_{i1}^{n-1}, u_{i2}^{n-1}, \dots, u_{ii-1}^{n-1}, u_{ii}^{n-1}, u_{ii+1}^{n-1}, \dots, u_{im}^{n-1}\big) \big\|. \qquad (12)
\end{aligned}
$$

By the definition of F_i as a $\lambda_{F_{ij}}$-ordered compression map with respect to the jth argument, we have

$$
\begin{aligned}
&\big\| F_i\big(u_{i1}^n, u_{i2}^n, \dots, u_{ii-1}^n, u_{ii}^n, u_{ii+1}^n, \dots, u_{im}^n\big) \oplus F_i\big(u_{i1}^{n-1}, u_{i2}^{n-1}, \dots, u_{ii-1}^{n-1}, u_{ii}^{n-1}, u_{ii+1}^{n-1}, \dots, u_{im}^{n-1}\big) \big\| \\
&\le \lambda_{F_{i1}} \big\| u_{i1}^n \oplus u_{i1}^{n-1} \big\| + \lambda_{F_{i2}} \big\| u_{i2}^n \oplus u_{i2}^{n-1} \big\| + \cdots + \lambda_{F_{im}} \big\| u_{im}^n \oplus u_{im}^{n-1} \big\| \\
&= \sum_{i \neq j, j=1}^{m} \lambda_{F_{ij}} \big\| u_{ij}^n \oplus u_{ij}^{n-1} \big\| \\
&\le \sum_{i \neq j, j=1}^{m} \lambda_{F_{ij}} \left(1 + \frac{1}{(n+1)}\right) D_j\big(U_{ij}(x_j^n), U_{ij}(x_j^{n-1})\big) \\
&\le \left(1 + \frac{1}{(n+1)}\right) \sum_{i \neq j, j=1}^{m} \lambda_{F_{ij}} \delta_{D_{ij}} \big\| x_j^n \oplus x_j^{n-1} \big\|. \qquad (13)
\end{aligned}
$$

Using Proposition 6 and Eq. (13) in Eq. (11), we obtain

$$
\begin{aligned}
\big\| x_i^{n+1} \oplus x_i^n \big\| = \big\| x_i^{n+1} - x_i^n \big\|
&\le \lambda_{N_C} \big[1 - \lambda\big(1 - \big(\alpha_1^i + \lambda_{g_i}\alpha_2^i\big)\big)\big] \big\| x_i^n - x_i^{n-1} \big\| \\
&\quad + \lambda\lambda_{N_C} L_i \big[(\lambda_{g_i} + \lambda_{A_i}\lambda_{g_i}) \big\| x_i^n - x_i^{n-1} \big\|\big] \\
&\quad + \lambda\lambda_{N_C} L_i \rho_i \left(1 + \frac{1}{(n+1)}\right) \sum_{i \neq j, j=1}^{m} \lambda_{F_{ij}} \delta_{D_{ij}} \big\| x_j^n - x_j^{n-1} \big\|
\end{aligned}
$$

$$\le \{\lambda_{N_C}[1-\lambda(1-(\alpha_1^i+\alpha_2^i\lambda_{g_i}))]+\lambda\lambda_{N_C}L_i(\lambda_{g_i}+\lambda_{A_i}\lambda_{g_i})\}\|x_i^n-x_i^{n-1}\|$$
$$+\lambda\lambda_{N_C}L_i\rho_i\left(1+\frac{1}{(n+1)}\right)\sum_{i\neq j,j=1}^{m}\lambda_{F_{ij}}\delta_{D_{ij}}\|x_j^n-x_j^{n-1}\|,$$

which implies that

$$\begin{aligned}
&\sum_{j=1}^{m}\|x_j^{n+1}-x_j^n\| \\
&\quad=\sum_{i=1}^{m}\|x_i^{n+1}-x_i^n\| \\
&\quad\le\sum_{i=1}^{m}\Bigg\{[\lambda_{N_C}[1-\lambda(1-(\alpha_1^i+\alpha_2^i\lambda_{g_i}))]+\lambda\lambda_{N_C}L_i(\lambda_{g_i}+\lambda_{A_i}\lambda_{g_i})]\|x_i^n-x_i^{n-1}\| \\
&\quad\quad+\lambda\lambda_{N_C}\left(1+\frac{1}{(n+1)}\right)\sum_{i\neq j,j=1}^{m}L_j\rho_j\lambda_{F_{ij}}\delta_{D_{ij}}\|x_j^n-x_j^{n-1}\|\Bigg\} \\
&\quad=\sum_{i=1}^{m}\lambda_{N_C}[1-\lambda\{1-(\alpha_1^i+\lambda_{g_i}\alpha_2^i)+L_i(\lambda_{g_i}+\lambda_{A_i}\lambda_{g_i})\}]\|x_i^n-x_i^{n-1}\| \\
&\quad\quad+\lambda\lambda_{N_C}\left(1+\frac{1}{(n+1)}\right)\sum_{i=1}^{m}\sum_{i\neq j,j=1}^{m}L_j\rho_j\lambda_{F_{ij}}\delta_{D_{ij}}\|x_j^n-x_j^{n-1}\| \\
&\quad=\sum_{j=1}^{m}\lambda_{N_C}[1-\lambda\{1-(\alpha_1^j+\alpha_2^j\lambda_{g_j})+L_j(\lambda_{g_j}+\lambda_{A_j}\lambda_{g_j})\}]\|x_j^n-x_j^{n-1}\| \\
&\quad\quad+\lambda_{N_C}\lambda\left(1+\frac{1}{(n+1)}\right)\sum_{j=1}^{m}\sum_{i\neq j,i=1}^{m}L_j\rho_j\lambda_{F_{ij}}\delta_{D_{ij}}\|x_j^n-x_j^{n-1}\| \\
&\quad=\sum_{j=1}^{m}\lambda_{N_C}\Bigg[1-\lambda+\lambda\Bigg\{\alpha_1^j+\alpha_2^j\lambda_{g_j}+L_j(\lambda_{g_j}+\lambda_{A_j}\lambda_{g_j}) \\
&\quad\quad+\left(1+\frac{1}{(n+1)}\right)\sum_{i\neq j,i=1}^{m}L_i\rho_i\lambda_{F_{ij}}\delta_{D_{ij}}\Bigg\}\Bigg]\|x_j^n-x_j^{n-1}\| \\
&\quad=\sum_{j=1}^{m}\lambda_{N_C}(1-\lambda+\lambda\theta_j^n)\|x_j^n-x_j^{n-1}\| \\
&\quad\le\lambda_{N_C}f_n(\lambda)\sum_{j=1}^{m}\|x_j^n-x_j^{n-1}\|, \qquad (14)
\end{aligned}$$

where

$$\theta_j^n=\left\{\alpha_1^j+\alpha_2^j\lambda_{g_j}+L_j(\lambda_{g_j}+\lambda_{A_j}\lambda_{g_j})+\left(1+\frac{1}{(n+1)}\right)\sum_{i\neq j,i=1}^{m}L_i\rho_i\lambda_{F_{ij}}\delta_{D_{ij}}\right\}<1$$

and

$$f_n(\lambda)=\max_{1\le j\le m}\{1-\lambda+\lambda\theta_j^n\}.$$

From Eq. (14), we know that the sequence $\{\theta_j^n\}$ is monotonic decreasing and $\theta_j^n \to \theta_j$ as $n \to \infty$. Thus, $f(\lambda) = \lim_{n\to\infty} f_n(\lambda) = \max_{1\le j\le m}\{1-\lambda+\lambda\theta_j\}$. Since $0 < \theta_j < 1$ for $j = 1,2,\dots,m$. We get $\theta = \max_{1\le j\le m}\{\theta_j\} \in (0,1)$. By Lemma 2.1, we have $f(\lambda) = 1-\lambda+\lambda\theta \in (0,1)$, from Eq. (14), it follows that $\{x_j^n\}$ is a Cauchy sequence and there exists $x_j^* \in \mathcal{E}_j$ such that $x_j^n \to x_j^*$ as $n \to \infty$ for $j = 1,2,\dots,m$. Next, we show that $u_{ij}^n \to u_{ij}^* \in U_{ij}(x_j^*)$ as $n \to \infty$ for $i,j = 1,2,\dots,m$. It follows from Eq. (13) that the $\{u_{ij}^n\}$ are also Cauchy sequences. Hence, there exists $u_{ij}^* \in \mathcal{E}_j$ such that $u_{ij}^n \to u_{ij}^*$ as $n \to \infty$ for $i,j = 1,2,\dots,m$. Furthermore,

$$\begin{aligned} d(u_{ij}^*, U_{ij}(x_j^*)) &= \inf\{\|u_{ij}^* \oplus t\| : t \in U_{ij}(x_j^*)\} \\ &\le \|u_{ij}^* \oplus u_{ij}^n\| + d(u_{ij}^n, U_{ij}(x_j^*)) \\ &\le \|u_{ij}^* \oplus u_{ij}^n\| + d(U_{ij}(x_j^n), U_{ij}(x_j^*)) \\ &\le \|u_{ij}^* \oplus u_{ij}^n\| + \delta_{D_{ij}}\|x_j^* \oplus x_j^n\| \\ &\le \|u_{ij}^* - u_{ij}^n\| + \delta_{D_{ij}}\|x_j^* - x_j^n\| \to 0 \quad (n \to \infty). \end{aligned}$$

Since $U_{ij}(x_j^*)$ is closed for $i,j = 1,2,\dots,m$, we have $u_{ij}^* \in U_{ij}(x_j^*)$ for $i,j = 1,2,\dots,m$. By using continuity $(x_1^*, x_2^*, \dots, x_m^*) \in \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m$ and $u_{ij}^* \in U_{ij}(x_j^*)$ for $i,j = 1,2,\dots,m$ satisfy Eq. (6) and so by Lemma 4.1, problem (2) has a solution $(x_1^*, x_2^*, \dots, x_m^*, u_{11}^*, u_{12}^*, \dots, u_{1m}^*, \dots, u_{m1}^*, u_{m2}^*, \dots, u_{mm}^*)$, where $u_{ij}^* \in U_{ij}(x_j^*)$ for $i,j = 1,2,\dots,m$ and $(x_1^*, x_2^*, \dots, x_m^*) \in \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m$. This completes the proof. □

Theorem 6.2 *Suppose that A_i, g_i and M_i are the same as in Theorem 6.1 for $i = 1,2,\dots,m$. Let $T_{ij} : \mathcal{E}_j \to \mathcal{E}_j$ be γ_{ij}-Lipschitz continuous and $F_i : \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m \to \mathcal{E}_i$ be $\lambda_{F_{ij}}$-ordered compression mapping with respect to the jth argument. Let there be constants $\lambda_j > 0$, for $j = 1,2,\dots,m$ such that*

$$\theta_j = \left[\lambda_{N_C}\{(\alpha_1^j + \alpha_2^j\lambda_{g_i}) + L_j(\lambda_{g_j} + \lambda_{A_j}\lambda_{g_j})\} + \lambda_{N_C}\sum_{i\ne j, i=1}^{m} L_i\rho_i\lambda_{F_{ij}}\gamma_{ij}\right] < 1.$$

Then problem (3) has a unique solution $(x_1^*, x_2^*, \dots, x_m^*) \in \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m$. Moreover, the iterative sequence $\{x_j^n\}$ generated by Algorithm 2 converges strongly to x_j^* for $j = 1,2,\dots,m$.

Proof Let us define a norm $\|\cdot\|_*$ on the product space $\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m$ by

$$\|(x_1, x_2, \dots, x_m)\|_* = \sum_{i=1}^{m}\|x_j\|, \quad \forall (x_1, x_2, \dots, x_m) \in \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m. \tag{15}$$

Then it can easily be seen that $(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m, \|\cdot\|_*)$ is a Banach space.

Setting

$$\begin{aligned} y_i &= x_i - g_i(x_i) + J_{\lambda_i, M_i}^{I_i - A_i}\big[(I_i - A_i)(g_i(x_i)) \\ &\quad + \rho_i F_i(T_{i1}x_1, \dots, T_{ii-1}x_{i-1}, T_{ii}x_i, T_{ii+1}x_{i+1}, \dots, T_{im}x_m)\big]. \end{aligned}$$

Define a mapping $Q : \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m \to \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m$ as

$$Q(x_1, x_2, \dots, x_m) = (y_1, y_2, \dots, y_m), \quad \forall (x_1, x_2, \dots, x_m) \in \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m.$$

For any $(x_1^1, x_2^1, \ldots, x_m^1)$, $(x_1^2, x_2^2, \ldots, x_m^2) \in \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m$ we have

$$\begin{aligned}
&\big\| Q(x_1^1, x_2^1, \ldots, x_m^1) \oplus Q(x_1^2, x_2^2, \ldots, x_m^2) \big\|_* \\
&\quad \leq \big\| Q(x_1^1, x_2^1, \ldots, x_m^1) - Q(x_1^2, x_2^2, \ldots, x_m^2) \big\|_* \\
&\quad \leq \big\| (y_1^1, y_2^1, \ldots, y_m^1) - (y_1^2, y_2^2, \ldots, y_m^2) \big\|_* \\
&\quad \leq \sum_{i=1}^{m} \big\| y_i^1 - y_i^2 \big\|.
\end{aligned} \tag{16}$$

First of all, we have to calculate $(y_i^1 \oplus y_i^2)$ as follows:

$$\begin{aligned}
(y_i^1 \oplus y_i^2) &= \big(x_i^1 - g_i(x_i^1) + J_{\lambda_i, M_i}^{I_i - A_i}\big[(I_i - A_i)\big(g_i(x_i^1)\big) \\
&\qquad + \rho_i F_i\big(T_{i1}x_1^1, \ldots, T_{ii-1}x_{i-1}^1, T_{ii}x_i^1, T_{ii+1}x_{i+1}^1, \ldots, T_{im}x_m^1\big)\big]\big) \\
&\qquad \oplus \big(x_i^2 - g_i(x_i^2) + J_{\lambda_i, M_i}^{I_i - A_i}\big[(I_i - A_i)\big(g_i(x_i^2)\big) \\
&\qquad + \rho_i F_i\big(T_{i1}x_1^2, \ldots, T_{ii-1}x_{i-1}^2, T_{ii}x_i^2, T_{ii+1}x_{i+1}^2, \ldots, T_{im}x_m^2\big)\big]\big) \\
&= \big(\big(x_i^1 - g_i(x_i^1)\big) \oplus \big(x_i^2 - g_i(x_i^2)\big)\big) + \big(J_{\lambda_i, M_i}^{I_i - A_i}\big[(I_i - A_i)\big(g_i(x_i^1)\big) \\
&\qquad + \rho_i F_i\big(T_{i1}x_1^1, \ldots, T_{ii-1}x_{i-1}^1, T_{ii}x_i^1, T_{ii+1}x_{i+1}^1, \ldots, T_{im}x_m^1\big)\big] \\
&\qquad \oplus J_{\lambda_i, M_i}^{I_i - A_i}\big[(I_i - A_i)\big(g_i(x_i^2)\big) \\
&\qquad + \rho_i F_i\big(T_{i1}x_1^2, \ldots, T_{ii-1}x_{i-1}^2, T_{ii}x_i^2, T_{ii+1}x_{i+1}^2, \ldots, T_{im}x_m^2\big)\big]\big).
\end{aligned}$$

From Definition 2.2 and Proposition 3, we have

$$\begin{aligned}
\big\| y_i^1 \oplus y_i^2 \big\| &\leq \big\| y_i^1 - y_i^2 \big\| \\
&\leq \lambda_{N_C} \big\| \big\{\big(\big(x_i^1 - g_i(x_i^1)\big) \\
&\qquad \oplus \big(x_i^2 - g_i(x_i^2)\big)\big) + \big(J_{\lambda_i, M_i}^{I_i - A_i}\big[(I_i - A_i)\big(g_i(x_i^1)\big) \\
&\qquad + \rho_i F_i\big(T_{i1}x_1^1, \ldots, T_{ii-1}x_{i-1}^1, T_{ii}x_i^1, T_{ii+1}x_{i+1}^1, \ldots, T_{im}x_m^1\big)\big] \\
&\qquad \oplus J_{\lambda_i, M_i}^{I_i - A_i}\big[(I_i - A_i)\big(g_i(x_i^2)\big) \\
&\qquad + \rho_i F_i\big(T_{i1}x_1^2, \ldots, T_{ii-1}x_{i-1}^2, T_{ii}x_i^2, T_{ii+1}x_{i+1}^2, \ldots, T_{im}x_m^2\big)\big]\big)\big\}\big\| \\
&\leq \lambda_{N_C} \big\{\alpha_1^j \big\| x_i^1 - x_i^2 \big\| + \alpha_2^j \lambda_{g_i} \big\| x_i^1 - x_i^2 \big\|\big\} \\
&\qquad + \lambda_{N_C} \big\| \big(J_{\lambda_i, M_i}^{I_i - A_i}\big[(I_i - A_i)\big(g_i(x_i^1)\big) + \rho_i F_i\big(T_{i1}x_1^1, \ldots, T_{ii-1}x_{i-1}^1, \\
&\qquad T_{ii}x_i^1, T_{ii+1}x_{i+1}^1, \ldots, T_{im}x_m^1\big)\big] \oplus J_{\lambda_i, M_i}^{I_i - A_i}\big[(I_i - A_i)\big(g_i(x_i^2)\big) \\
&\qquad + \rho_i F_i\big(T_{i1}x_1^2, \ldots, T_{ii-1}x_{i-1}^2, T_{ii}x_i^2, T_{ii+1}x_{i+1}^2, \ldots, T_{im}x_m^2\big)\big]\big)\big\| \\
&\leq \lambda_{N_C} \big(\alpha_1^j + \alpha_2^j \lambda_{g_i}\big) \big\| x_i^1 - x_i^2 \big\| + \lambda_{N_C} \big\| \big(J_{\lambda_i, M_i}^{I_i - A_i}\big[(I_i - A_i)\big(g_i(x_i^1)\big) \\
&\qquad + \rho_i F_i\big(T_{i1}x_1^1, \ldots, T_{ii-1}x_{i-1}^1, T_{ii}x_i^1, T_{ii+1}x_{i+1}^1, \ldots, T_{im}x_m^1\big)\big] \\
&\qquad \oplus J_{\lambda_i, M_i}^{I_i - A_i}\big[(I_i - A_i)\big(g_i(x_i^2)\big) \\
&\qquad + \rho_i F_i\big(T_{i1}x_1^2, \ldots, T_{ii-1}x_{i-1}^2, T_{ii}x_i^2, T_{ii+1}x_{i+1}^2, \ldots, T_{im}x_m^2\big)\big]\big)\big\|.
\end{aligned} \tag{17}$$

Further, we calculate

$$\begin{aligned}
&\big\| J^{I_i-A_i}_{\lambda_i,M_i}\big[(I_i-A_i)\big(g_i\big(x_i^1\big)\big)+\rho_i F_i\big(T_{i1}x_1^1,\dots,T_{ii-1}x_{i-1}^1,T_{ii}x_i^1,T_{ii+1}x_{i+1}^1,\dots,T_{im}x_m^1\big)\big]\\
&\quad\oplus J^{I_i-A_i}_{\lambda_i,M_i}\big[(I_i-A_i)\big(g_i\big(x_i^2\big)\big)+\rho_i F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^2,T_{ii}x_i^2,T_{ii+1}x_{i+1}^2,\dots,T_{im}x_m^2\big)\big]\big\|\\
&\quad\le L_i\big[\big\|\big\{(I_i-A_i)\big(g_i\big(x_i^1\big)\big)\oplus(I_i-A_i)\big(g_i\big(x_i^2\big)\big)\big\}\big\|\\
&\qquad+\rho_i\big\|F_i\big(T_{i1}x_1^1,\dots,T_{ii-1}x_{i-1}^1,T_{ii}x_i^1,T_{ii+1}x_{i+1}^1,\dots,T_{im}x_m^1\big)\\
&\qquad\oplus F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^2,T_{ii}x_i^2,T_{ii+1}x_{i+1}^2,\dots,T_{im}x_m^2\big)\big\|\big]\\
&\quad\le L_i\big[\big\|\big(g_i\big(x_i^1\big)\oplus g_i\big(x_i^2\big)\big)+\big(A_i\big(g_i\big(x_i^1\big)\big)\oplus A_i\big(g_i\big(x_i^2\big)\big)\big)\big\|\\
&\qquad+\rho_i\big\|F_i\big(T_{i1}x_1^1,\dots,T_{ii-1}x_{i-1}^1,T_{ii}x_i^1,T_{ii+1}x_{i+1}^1,\dots,T_{im}x_m^1\big)\\
&\qquad\oplus F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^2,T_{ii}x_i^2,T_{ii+1}x_{i+1}^2,\dots,T_{im}x_m^2\big)\big\|\big]\\
&\quad\le L_i\big[\big\{\lambda_{g_i}\big\|x_i^1\oplus x_i^2\big\|+\lambda_{A_i}\lambda_{g_i}\big\|x_i^1\oplus x_i^2\big\|\big\}\\
&\qquad+\rho_i\big\|F_i\big(T_{i1}x_1^1,\dots,T_{ii-1}x_{i-1}^1,T_{ii}x_i^1,T_{ii+1}x_{i+1}^1,\dots,T_{im}x_m^1\big)\\
&\qquad\oplus F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^2,T_{ii}x_i^2,T_{ii+1}x_{i+1}^2,\dots,T_{im}x_m^2\big)\big\|\big]\\
&\quad\le L_i\big[(\lambda_{g_i}+\lambda_{A_i}\lambda_{g_i})\big\|x_i^1\oplus x_i^2\big\|\\
&\qquad+\rho_i\big\|F_i\big(T_{i1}x_1^1,\dots,T_{ii-1}x_{i-1}^1,T_{ii}x_i^1,T_{ii+1}x_{i+1}^1,\dots,T_{im}x_m^1\big)\\
&\qquad F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^2,T_{ii}x_i^2,T_{ii+1}x_{i+1}^2,\dots,T_{im}x_m^2\big)\big\|\big].
\end{aligned}\tag{18}$$

Now we calculate the inner part estimate of the above expression with the help of the properties of the F_i-operator for $i=1,2,\dots,m$. We have

$$\begin{aligned}
&\big\|F_i\big(T_{i1}x_1^1,\dots,T_{ii-1}x_{i-1}^1,T_{ii}x_i^1,T_{ii+1}x_{i+1}^1,\dots,T_{im}x_m^1\big)\\
&\quad\oplus F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^2,T_{ii}x_i^2,T_{ii+1}x_{i+1}^2,\dots,T_{im}x_m^2\big)\big\|\\
&\quad=\big\|F_i\big(T_{i1}x_1^1,\dots,T_{ii-1}x_{i-1}^1,T_{ii}x_i^1,T_{ii+1}x_{i+1}^1,\dots,T_{im}x_m^1\big)\oplus F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^1,\\
&\qquad T_{ii}x_i^1,T_{ii+1}x_{i+1}^1,\dots,T_{im}x_m^1\big)\oplus F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^1,T_{ii}x_i^1,\\
&\qquad T_{ii+1}x_{i+1}^1,\dots,T_{im}x_m^1\big)\oplus F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^2,T_{ii}x_i^2,T_{ii+1}x_{i+1}^2,\dots,T_{im}x_m^2\big)\big\|\\
&\quad\le\big\|F_i\big(T_{i1}x_1^1,\dots,T_{ii-1}x_{i-1}^1,T_{ii}x_i^1,T_{ii+1}x_{i+1}^1,\dots,T_{im}x_m^1\big)\\
&\qquad\oplus F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^1,T_{ii}x_i^1,T_{ii+1}x_{i+1}^1,\dots,T_{im}x_m^1\big)\big\|+\cdots\\
&\qquad+\big\|F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^1,T_{ii}x_i^1,T_{ii+1}x_{i+1}^1,\dots,T_{im}x_m^1\big)\\
&\qquad\oplus F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^2,T_{ii}x_i^1,T_{ii+1}x_{i+1}^1,\dots,T_{im}x_m^1\big)\big\|\\
&\qquad+\big\|F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^2,T_{ii}x_i^1,T_{ii+1}x_{i+1}^1,\dots,T_{im}x_m^1\big)\\
&\qquad\oplus F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^2,T_{ii}x_i^1,T_{ii+1}x_{i+1}^1,\dots,T_{im}x_m^1\big)\big\|\\
&\qquad+\big\|F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^2,T_{ii}x_i^2,T_{ii+1}x_{i+1}^1,\dots,T_{im}x_m^1\big)\\
&\qquad\oplus F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^2,T_{ii}x_i^2,T_{ii+1}x_{i+1}^2,\dots,T_{im}x_m^1\big)\big\|+\cdots\\
&\qquad+\big\|F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^2,T_{ii}x_i^2,T_{ii+1}x_{i+1}^2,\dots,T_{im}x_m^1\big)\\
&\qquad\oplus F_i\big(T_{i1}x_1^2,\dots,T_{ii-1}x_{i-1}^2,T_{ii}x_i^2,T_{ii+1}x_{i+1}^2,\dots,T_{im}x_m^2\big)\big\|
\end{aligned}$$

$$\begin{aligned}&\le \lambda_{F_{i1}}\left\|T_{i1}x_1^1 \oplus T_{i1}x_1^2\right\| + \lambda_{F_{i2}}\left\|T_{i2}x_2^1 \oplus T_{i2}x_2^2\right\| + \cdots \\ &\quad + \lambda_{F_{ii-1}}\left\|T_{ii-1}x_{i-1}^1 \oplus T_{ii-1}x_{i-1}^2\right\| + \lambda_{F_{ii}}\left\|T_{ii}x_i^1 \oplus T_{ii}x_i^2\right\| \\ &\quad + \lambda_{F_{ii+1}}\left\|T_{ii+1}x_{i+1}^1 \oplus T_{ii+1}x_{i+1}^2\right\| + \cdots + \lambda_{F_{im}}\left\|T_{im}x_m^1 \oplus T_{im}x_m^2\right\|. \end{aligned} \tag{19}$$

By using the Lipschitz continuity of T_{ij}-operator in Eq. (19), we have

$$\begin{aligned}&\left\|F_i\left(T_{i1}x_1^1,\ldots,T_{ii-1}x_{i-1}^1,T_{ii}x_i^1,T_{ii+1}x_{i+1}^1,\ldots,T_{im}x_m^1\right)\right. \\ &\quad \left.\oplus F_i\left(T_{i1}x_1^2,\ldots,T_{ii-1}x_{i-1}^2,T_{ii}x_i^2,T_{ii+1}x_{i+1}^2,\ldots,T_{im}x_m^2\right)\right\| \\ &\le \sum_{i\neq j,j=1}^{m} \lambda_{F_{ij}}\gamma_{ij}\left\|x_j^1 \oplus x_j^2\right\|. \end{aligned} \tag{20}$$

Using Eq. (20) in Eq. (18) and then use it in Eq. (17), we have

$$\begin{aligned}\left\|y_i^1 \oplus y_i^2\right\| &\le \left\|y_i^1 - y_i^2\right\| \\ &\le \lambda_{N_C}\left(\alpha_1^i + \lambda_{g_i}\alpha_2^i\right)\left\|x_i^1 - x_i^2\right\| \\ &\quad + \lambda_{N_C}\left[L_i(\lambda_{g_i} + \lambda_{A_i}\lambda_{g_i})\left\|x_i^1 - x_i^2\right\| + L_i\rho_i \sum_{i\neq j,j=1}^{m} \lambda_{F_{ij}}\gamma_{ij}\left\|x_j^1 - x_j^2\right\|\right]. \end{aligned} \tag{21}$$

Now, Eq. (16) can be rewritten as

$$\begin{aligned}&\left\|Q\left(x_1^1,x_2^1,\ldots,x_m^1\right) - Q\left(x_1^2,x_2^2,\ldots,x_m^2\right)\right\|_* \\ &\le \sum_{i=1}^{m}\left\|y_i^1 - y_i^2\right\| \\ &\le \sum_{i=1}^{m}\left\{\lambda_{N_C}\left[\left(\alpha_1^i + \lambda_{g_i}\alpha_2^i\right) + L_i(\lambda_{g_i} + \lambda_{A_i}\lambda_{g_i})\right]\left\|x_i^1 - x_i^2\right\|\right. \\ &\quad \left. + \lambda_{N_C}L_i\rho_i \sum_{i\neq j,j=1}^{m} \lambda_{F_{ij}}\gamma_{ij}\left\|x_j^1 - x_j^2\right\|\right\} \\ &\le \sum_{i=1}^{m}\left\{\lambda_{N_C}\left[\left(\alpha_1^i + \lambda_{g_i}\alpha_2^i\right) + L_i(\lambda_{g_i} + \lambda_{A_i}\lambda_{g_i})\right]\right\}\left\|x_i^1 - x_i^2\right\| \\ &\quad + \sum_{i=1}^{m}\lambda_{N_C}L_i\rho_i \sum_{i\neq j,j=1}^{m} \lambda_{F_{ij}}\gamma_{ij}\left\|x_j^1 - x_j^2\right\| \\ &\le \sum_{j=1}^{m}\left[\lambda_{N_C}\left\{\left(\alpha_1^j + \alpha_2^j\lambda_{g_i}\right) + L_j(\lambda_{g_j} + \lambda_{A_j}\lambda_{g_j})\right\}\right. \\ &\quad \left. + \lambda_{N_C}\sum_{i\neq j,i=1}^{m} L_i\rho_i\lambda_{F_{ij}}\gamma_{ij}\right]\left\|x_j^1 - x_j^2\right\| \\ &= \sum_{j=1}^{m}\theta_j\left\|x_j^1 - x_j^2\right\| \end{aligned}$$

$$\begin{aligned} &\le \theta \sum_{j=1}^{m} \|x_j^1 - x_j^2\| \\ &\le \theta \left\| \left(x_1^1, x_2^1, \dots, x_m^1\right) - \left(x_1^2, x_2^2, \dots, x_m^2\right) \right\|_*, \end{aligned} \tag{22}$$

where $\theta = \max_{1\le j\le m} \theta_j$. Finally, from Eq. (22), Eq. (16) can be written as

$$\begin{aligned} \left\| Q\left(x_1^1, x_2^1, \dots, x_m^1\right) - Q\left(x_1^2, x_2^2, \dots, x_m^2\right) \right\|_* &\le \theta \sum_{j=1}^{m} \|x_j^1 - x_j^2\| \\ &= \theta \left\| \left(x_1^1, x_2^1, \dots, x_m^1\right) - \left(x_1^2, x_2^2, \dots, x_m^2\right) \right\|_*. \end{aligned} \tag{23}$$

It follows from the condition (9) that $0 < \theta < 1$. This implies that $Q : \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m \to \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m$ is a contraction which in turn implies that there exists a unique $(x_1^*, x_2^*, \dots, x_m^*) \in \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_m$ such that $Q(x_1^*, x_2^*, \dots, x_m^*) = (x_1^*, x_2^*, \dots, x_m^*)$. Thus, $(x_1^*, x_2^*, \dots, x_m^*)$ is the unique solution of problem (3). Now, we prove that $x_i^n \to x_i^*$ as $n \to \infty$ for $i = 1, 2, \dots, m$. In fact, it follows from Eq. (8) and the Lipschitz continuity of the relaxed resolvent operator that

$$\begin{aligned} \|x_i^{n+1} \oplus x_i^*\| = {}& \Big\| \Big[x_i^n - g_i\left(x_i^n\right) + J_{\lambda_i, M_i}^{I_i - A_i}\big[(I_i - A_i)\left(g_i\left(x_i^n\right)\right) \\ & + \rho_i F_i\left(T_{i1}x_1^n, T_{i2}x_2^n, \dots, T_{im}x_m^n\right)\big] + w_i^n \oplus \big[x_i^* - g_i\left(x_i^*\right) \\ & + J_{\lambda_i, M_i}^{I_i - A_i}\big[(I_i - A_i)\left(g_i\left(x_i^*\right)\right) \\ & + \rho_i F_i\left(T_{i1}x_1^*, T_{i2}x_2^*, \dots, T_{im}x_m^*\right)\big]\big]\Big]\Big\| \\ \le {}& \big\| \left(x_i^n - g_i\left(x_i^n\right)\right) \oplus \left(x_i^* - g_i\left(x_i^*\right)\right) \big\| \\ & + \big\| J_{\lambda_i, M_i}^{I_i - A_i}\big[(I_i - A_i)\left(g_i\left(x_i^n\right)\right) \\ & + \rho_i F_i\left(T_{i1}x_1^n, T_{i2}x_2^n, \dots, T_{im}x_m^n\right)\big] \\ & \oplus J_{\lambda_i, M_i}^{I_i - A_i}\big[(I_i - A_i)\left(g_i\left(x_i^*\right)\right) \\ & + \rho_i F_i\left(T_{i1}x_1^*, T_{i2}x_2^*, \dots, T_{im}x_m^*\right)\big]\big\| + \big\| w_i^n \oplus 0 \big\|. \end{aligned} \tag{24}$$

From the previous calculations, we have

$$\begin{aligned} \sum_{i=1}^{m} \|x_i^{n+1} \oplus x_i^*\| &= \sum_{i=1}^{m} \|x_i^{n+1} - x_i^*\| \\ &\le \Bigg[\lambda_{N_C} \left\{ \left(\alpha_1^j + \alpha_2^j \lambda_{g_i}\right) + L_j(\lambda_{g_j} + \lambda_{A_j}\lambda_{g_j}) \right\} \\ &\quad + \lambda_{N_C} \sum_{i\ne j, i=1}^{m} L_i \rho_i \lambda_{F_{ij}} \gamma_{ij} \Bigg] \sum_{j=1}^{m} \|x_j^n - x_j^*\| + \sum_{j=1}^{m} \|w_j^n\| \\ &= \sum_{j=1}^{m} \theta_j \|x_j^n - x_j^*\| + \sum_{j=1}^{m} \|w_j^n\|, \end{aligned} \tag{25}$$

where $a_n = \sum_{j=1}^{m} \|x_j^n - x_j^*\|$, $b_n = \sum_{j=1}^{m} \|w_j^n\|$. Algorithm 2 yields $\lim_{n\to\infty} b_n = 0$. Now, Lemma 2.2 implies that $\lim_{n\to\infty} a_n = 0$, and so $x_j^n \to x_j^*$ as $n \to \infty$ for $j = 1, 2, \dots, m$. This completes the proof. □

7 Conclusion

Two of the most troublesome and imperative issues identified with inclusions are the foundation of generalized inclusions and the improvement of an iterative calculation. In this article, two systems of variational inclusions were presented and contemplated, which is a broader aim than the numerous current systems of generalized ordered variational inclusions in the literature. An iterative calculation is performed with a weak ARD mapping to an inexact solution of our systems, and the convergence criterion is likewise addressed.

We comment that our outcomes are new and valuable for additionally investigations. Considerably more work is required in every one of these regions to address utilizations of the system of general ordered variational inclusions in engineering and physical sciences.

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