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# Multivariate box spline wavelets in higher-dimensional Sobolev spaces

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**Abstract**

We construct wavelets and derive a density condition of MRA in a higher-dimensional Sobolev space. We give necessary and sufficient conditions for orthonormality of wavelets in $H^s(\mathbb{R}^d)$. We construct nonseparable orthonormal wavelets in a higher-dimensional Sobolev space by using multivariate box spline.

**Keywords:** Wavelets; Box Splines; Multiresolution analysis; Sobolev space

## 1 Introduction

Box splines are refinable functions, and we can easily choose various directions to have a box spline function with a desired order of smoothness. Naturally, they have been used to construct various wavelet functions. Mathematically box splines offer an elegant toolbox for constructing a class of multidimensional elements with flexible shape and support. In multivariate setting, box splines are often considered as a generalization of B-splines [1]. Theoretically, the computational complexity of a box spline is lower than that of an equivalent B-spline, since its support is more compact and its total polynomial degree is lower. To investigate this potential in practice, several attempts were made. Recurrence relation [1, 2] is the most commonly used technique for evaluating box splines at an arbitrary position. There are many papers on multivariate spline wavelet theory, in particular, on orthogonal spline wavelets [3], compactly spline prewavelets [4–6], bivariate and trivariate compactly supported biorthogonal box spline wavelets [7, 8], and multivariate compactly supported tight wavelet frames [9].

Wavelets in a Sobolev space and their properties were instigated by Bastin et al. [10, 11], Dayong and Dengfeng [12], and Pathak [13]. Regular compactly supported wavelets and compactly supported wavelets of integer order in a Sobolev space by B-spline are given in [10, 11]. Further, bivariate box splines in a Sobolev space were introduced in [14].

Inspired by the works mentioned, in this paper, we study nonseparable wavelets in a higher-dimensional Sobolev space by using a multivariate box spline. To the best of our knowledge, no previous studies of multivariate box spline wavelets exist in higher-dimensional Sobolev spaces. This paper is organized as follows. In Sect. 2, we hereby present construction of wavelets and density conditions of MRA in a higher-dimensional Sobolev space. Also, we give necessary and sufficient conditions for the orthonormality of wavelets in $H^s(\mathbb{R}^d)$. In Sect. 3, we construct nonseparable wavelets in a higher-dimensional Sobolev space by using a multivariate box spline.

### 1.1 Sobolev space $H^s(\mathbb{R}^d)$

For any real number $s$, the Sobolev space $H^s(\mathbb{R}^d)$ consists of tempered distributions in $S'(\mathbb{R}^d)$ such that

$$\|f\|_s^2 := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\bigl(1+\|\xi\|^2\bigr)^s\bigl|\hat{f}(\xi)\bigr|^2\,d\xi,$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$, and the corresponding inner product is

$$\langle f,g\rangle_s := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\bigl(1+\|\xi\|^2\bigr)^s\hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi.$$

The Fourier transform $\hat{f}$ of $f\in L^1(\mathbb{R}^d)$ is defined as

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle}f(x)\,dx,$$

where $\langle x,\xi\rangle$ is the inner product of two vectors $x$ and $\xi$ in $\mathbb{R}^d$.

## 2 Multiresolution analysis

To adapt classical theory of MRA over $H^s(\mathbb{R}^d)$, we first derive an orthonormality and density condition. The main problem is that $H^s$-norm is not dilation invariant. We also don't achieve orhtonormality at each level of dilation by a single scaling function. This lead us to a more general construction of MRA, where the scaling function depends on the level of dilation. Throughout this paper, the superscript $j$ of a function $\varphi^{(j)}$ represents level $j$.

**Proposition 2.1** *If $s$ is a real number, $\varphi^{(j)}\in H^s(\mathbb{R}^d)$, and $j$ is an integer, then the distributions $\varphi^{(j)}_{j,k}(x)=2^{jd/2}\varphi^{(j)}(2^jx-k), k\in\mathbb{Z}^d$, are orthonormal in $H^s(\mathbb{R}^d)$ iff*

$$\sum_{k\in\mathbb{Z}^d}\bigl(1+2^{2j}\|\xi+2k\pi\|^2\bigr)^s\bigl|\hat{\varphi}^{(j)}(\xi+2k\pi)\bigr|^2=1 \qquad (1)$$

*almost everywhere. It follows that we have the bound*

$$\bigl|\hat{\varphi}^{(j)}\bigl(2^{-j}\xi\bigr)\bigr|\le\bigl(1+\|\xi\|^2\bigr)^{-s/2}.$$

*Proof* Since $\varphi^{(j)}_{j,k}(t)\in H^s(\mathbb{R}^d)$, the series

$$M(\xi)=\sum_{r\in\mathbb{Z}^d}\bigl|\hat{\varphi}^{(j)}(\xi+2\pi r)\bigr|^2\bigl(1+2^{2j}\bigl\|(\xi+2\pi r)\bigr\|^2\bigr)^s$$

converges almost everywhere, belongs to $\mathbb{L}^1_{\text{loc}}(\mathbb{R}^d)$, and is $2\pi\mathbb{Z}^d$-periodic, that is, $M(\xi)\in L^1(\mathbb{T}^d)$, where $\mathbb{T}^d=[0,2\pi]^d$ is the $d$-dimensional torus. Moreover, for every $l\in\mathbb{Z}^d$, we have

$$\begin{aligned}
&\int_{\mathbb{T}^d}M(\xi)e^{-i\langle\xi,(k-l)\rangle}\,d\xi\\
&\quad=\sum_{r\in\mathbb{Z}^d}\int_{\mathbb{T}^d}\bigl|\hat{\varphi}^{(j)}(\xi+2\pi r)\bigr|^2\bigl(1+2^{2j}\bigl\|(\xi+2\pi r)\bigr\|^2\bigr)^s e^{-i\langle\xi,(k-l)\rangle}\,d\xi\\
&\quad=\int_{\mathbb{R}^d}\bigl|\hat{\varphi}^{(j)}(\nu)\bigr|^2\bigl(1+2^{2j}\bigl\|(\nu)\bigr\|^2\bigr)^s e^{-i\langle\nu,(k-l)\rangle}\,d\nu
\end{aligned}$$

$$= 2^{-jd}\int_{\mathbb{R}^d}\left|\hat{\varphi}^{(j)}\left(2^{-j}u\right)\right|^2\left(1+\|u\|^2\right)^s e^{-i2^{-j}\langle u,(k-l)\rangle}\,du$$
$$= \int_{\mathbb{R}^d}\left(1+\|u\|^2\right)^s e^{-i2^{-j}\langle u,k\rangle}2^{-jd/2}\hat{\varphi}^{(j)}\left(2^{-j}u\right)\overline{e^{-i2^{-j}\langle u,l\rangle}2^{-jd/2}\hat{\varphi}^{(j)}\left(2^{-j}u\right)}\,du$$
$$= \int_{\mathbb{R}^d}\left(1+\|u\|^2\right)^s \mathcal{F}\left[\varphi_{j,k}^{(j)}(t)\right](u)\mathcal{F}\left[\varphi_{j,l}^{(j)}(t)\right](u)\,du$$
$$= (2\pi)^d\left\langle\varphi_{j,k}^{(j)}(t),\varphi_{j,l}^{(j)}(t)\right\rangle_s.$$

Since $\{1/(2\pi)^d e^{-i\langle\xi,(k-l)\rangle} : k,l\in\mathbb{Z}^d\}$ is an orthonormal basis for $L^2(\mathbb{T}^d)$, we have

$$\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}M(\xi)e^{-i\langle\xi,(k-l)\rangle}\,d\xi = \left\langle\varphi_{j,k}^{(j)}(t),\varphi_{j,l}^{(j)}(t)\right\rangle_s = \delta_{k,l}$$

if $M(\xi) = 1$.

From (1) we get

$$\left(1+2^{2j}\|\xi\|^2\right)^s\left|\hat{\varphi}^{(j)}(\xi)\right|^2 \le 1,$$

which implies

$$\left|\hat{\varphi}^{(j)}(\xi)\right| \le \left(1+2^{2j}\|\xi\|^2\right)^{-s/2}.$$ □

**Proposition 2.2** *Let $\varphi^{(j)}, j\in\mathbb{Z}$, be a sequence of elements of $H^s(\mathbb{R}^d)$ such that, for every $j$, the distributions $\varphi_{j,k}^{(j)}(x) = 2^{jd/2}\varphi^{(j)}(2^jx-k), k\in\mathbb{Z}^d$, are orthonormal in $H^s(\mathbb{R}^d)$. If $P_j$ is the orthogonal projection from $H^s(\mathbb{R}^d)$ onto $V_j := \overline{\operatorname{span}}\{\varphi_{j,k}^{(j)} : k\in\mathbb{Z}^d\}$, then, for every $h\in H^s(\mathbb{R}^d)$, we have*

$$\lim_{j\to+\infty}\left(\|P_jh\|_s^2 - \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(1+\|\xi\|^2\right)^{2s}\left|\hat{h}(\xi)\right|^2\left|\hat{\varphi}^{(j)}\left(2^{-j}\xi\right)\right|^2 d\xi\right) = 0.$$

*Moreover, if there are $A,\alpha>0$ such that*

$$\int_{\mathbb{R}^d}\left(1+\|\xi\|\right)^\alpha\left|\hat{\varphi}^{(j)}(\xi)\right|^2 d\xi \le A$$

*for every $j\le 0$, then $\bigcap_{j=-\infty}^{j=\infty}V_j = \{0\}^d$.*

*Proof* Let us prove the first part with $h\in C_0^\infty(\mathbb{R}^d)$. By the definition of $P_j$ we get

$$\|P_jh\|_s^2 = \sum_{k\in\mathbb{Z}^d}\left|\left\langle h,\varphi_{j,k}^{(j)}\right\rangle_s\right|^2 = \frac{2^{-jd}}{(2\pi)^{2d}}\sum_{k\in\mathbb{Z}^d}\left|\int_{\mathbb{R}^d}\left(1+\|\xi\|^2\right)^s\hat{h}(\xi)\overline{\hat{\varphi}^{(j)}\left(2^{-j}\xi\right)}e^{i2^{-j}\langle k,\xi\rangle}\,d\xi\right|^2.$$

Moreover, since $h$ and $\varphi^{(j)}$ belong to $H^s(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\left(1+\|\xi\|^2\right)^s\hat{h}(\xi)\overline{\hat{\varphi}^{(j)}\left(2^{-j}\xi\right)}e^{i2^{-j}\langle k,\xi\rangle}\,d\xi$$
$$= \int_{]0,2^j2\pi[^d}e^{i2^{-j}\langle k,\xi\rangle}\sum_{p\in\mathbb{Z}^d}\left(1+\left\|\xi+2^j2\pi p\right\|^2\right)^s\hat{h}\left(\xi+2^j2\pi p\right)\overline{\hat{\varphi}^{(j)}\left(2^{-j}\xi+2\pi p\right)}\,d\xi.$$

Hence, using the Parseval formula in $L^2(]0,2^j2\pi[^d)$, we get

$$\begin{aligned}
&\|P_jh\|_s^2 \\
&= \frac{1}{(2\pi)^d}\int_{]0,2^j2\pi[^d}\left|\sum_{p\in\mathbb{Z}^d}\left(1+\left\|\xi+2^j2\pi p\right\|^2\right)^s\hat{h}\left(\xi+2^j2\pi p\right)\overline{\hat{\varphi}^{(j)}\left(2^{-j}\xi+2\pi p\right)}\right|^2 d\xi \\
&= \frac{1}{(2\pi)^d}\sum_{q\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\left(1+\|\xi\|^2\right)^s\left(1+\left\|\xi+2^j2\pi q\right\|^2\right)^s\hat{h}(\xi)\overline{\hat{\varphi}^{(j)}\left(2^{-j}\xi\right)} \\
&\quad\times\overline{\hat{h}\left(\xi+2^j2\pi q\right)}\hat{\varphi}^{(j)}\left(2^{-j}\xi+2\pi q\right)d\xi \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(1+\|\xi\|^2\right)^{2s}\left|\hat{h}(\xi)\right|^2\left|\hat{\varphi}^{(j)}\left(2^{-j}\xi\right)\right|^2 \\
&\quad+\frac{1}{(2\pi)^d}\sum_{q\in\mathbb{Z}^d\setminus\{0\}^d}\int_{\mathbb{R}^d}\left(1+\|\xi\|^2\right)^s\left(1+\left\|\xi+2^j2\pi q\right\|^2\right)^s\hat{h}(\xi)\overline{\hat{\varphi}^{(j)}\left(2^{-j}\xi\right)} \\
&\quad\times\overline{\hat{h}\left(\xi+2^j2\pi q\right)}\hat{\varphi}^{(j)}\left(2^{-j}\xi+2\pi q\right)d\xi.
\end{aligned}$$

The term associated with $q=\{0\}^d$, $\{0\}^d=(0,0,\dots,0)\in\mathbb{Z}^d$ is used as an approximation for $\|P_jh\|_s^2$. Using Proposition 2.1, the inequality $|\varphi^{(j)}(2^{-j}\xi)|\le(1+\|\xi\|^2)^{-s/2}$, and the fact that $\hat{h}$ belongs to the Schwartz space $S(\mathbb{R}^d)$ (i.e., $|\hat{h}(\xi)|\le C(1+\|\xi\|^2)^{-\alpha}$ for any $\alpha>0$), we obtain that the sum of the other ones is bounded by

$$\begin{aligned}
&\sum_{q\in\mathbb{Z}^d\setminus\{0\}^d}\int_{\mathbb{R}^d}\left(1+\|\xi\|^2\right)^{s/2}\left(1+\left\|\xi+2^j2\pi q\right\|^2\right)^{s/2}\left|\hat{h}(\xi)\overline{\hat{h}\left(\xi+2^j2\pi q\right)}\right|d\xi \\
&\le C\sum_{q\in\mathbb{Z}^d\setminus\{0\}^d}\frac{1}{(1+\|2^j2\pi q\|^2)^2}\int_{\mathbb{R}^d}\frac{1}{(1+\|\xi\|^2)^2}d\xi \\
&\le C\sum_{q\in\mathbb{Z}^d\setminus\{0\}^d}\frac{1}{(\|2^j2\pi q\|^2)^2}\int_{\mathbb{R}^d}\frac{1}{(1+\|\xi\|^2)^2}d\xi \\
&\le C2^{-4(j+1)}\left(\sum_{q\in\mathbb{Z}^d\setminus\{0\}^d}\frac{1}{|q|^2}\right)\int_{\mathbb{R}^d}\frac{1}{(1+\|\xi\|^2)^2}d\xi,
\end{aligned}$$

where $|q|=(\sum_{r=1}^d|q_r|^2)^{1/2}$, $q=(q_1,q_2,\dots,q_d)\in\mathbb{Z}^d$. This expression converges to 0 as $j\to+\infty$.

Now let $h\in H^s(\mathbb{R}^d)$. Recall the inequality

$$\|f+g\|^2\le(1+\varepsilon)\|f\|^2+\left(1+\frac{1}{\varepsilon}\right)\|g\|^2,$$

which is valid for every $\varepsilon>0$ and any seminorm. For any $\chi\in C_0^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\|P_jh\|_s^2-\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(1+\|\xi\|^2\right)^{2s}\left|\hat{h}(\xi)\right|^2\left|\hat{\varphi}^{(j)}\left(2^{-j}\xi\right)\right|^2d\xi \\
&\le(1+\varepsilon)\|P_j\chi\|_s^2+\left(1+\frac{1}{\varepsilon}\right)\left\|P_j(h-\chi)\right\|^2 \\
&\quad-\frac{1}{(2\pi)^d(1+\varepsilon)}\int_{\mathbb{R}^d}\left(1+\|\xi\|^2\right)^{2s}\left|\hat{\chi}(\xi)\right|^2\left|\hat{\varphi}^{(j)}\left(2^{-j}\xi\right)\right|^2d\xi
\end{aligned}$$

$$+\frac{1}{(2\pi)^d(\varepsilon)}\int_{\mathbb{R}^d}\bigl(1+\|\xi\|^2\bigr)^{2s}\bigl|\hat{h}(\xi)-\hat{\chi}(\xi)\bigr|^2\bigl|\hat{\varphi}^{(j)}\bigl(2^{-j}\xi\bigr)\bigr|^2\,d\xi$$

$$\leq(1+\varepsilon)\biggl(\|P_j\chi\|_s^2-\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\bigl(1+\|\xi\|^2\bigr)^{2s}\bigl|\hat{\chi}(\xi)\bigr|^2\bigl|\hat{\varphi}^{(j)}\bigl(2^{-j}\xi\bigr)\bigr|^2\,d\xi\biggr)$$

$$+\biggl(1+\frac{2}{\varepsilon}\biggr)\|h-\chi\|_s^2+\biggl(1+\varepsilon-\frac{1}{(1+\varepsilon)}\biggr)\|\chi\|_s^2.$$

By the same way, we can obtain a similar lower bound. To prove that the left-hand side converges to 0 as $j$ converges to $+\infty$, we first take $\varepsilon$ sufficiently small. Then we choose $\chi$ approximating $h$ and finally $j$ large.

For the second part, we have to prove that, for every $h\in C_0^\infty(\mathbb{R}^d)$, $P_jh$ converges to zero in $H^s(\mathbb{R}^d)$ as $j\to-\infty$. We use the last expression of $\|P_jh\|_s$ obtained previously. We first estimate the sum over $q$ without the integral. By the Cauchy–Schwarz inequality and Proposition 2.1 we have

$$\sum_{q\in\mathbb{Z}^d}\bigl(1+\bigl\|\xi+2^j2\pi q\bigr\|^2\bigr)^s\bigl|\hat{h}\bigl(\xi+2^j2\pi q\bigr)\overline{\hat{\varphi}^{(j)}\bigl(2^{-j}\xi+2\pi q\bigr)}\bigr|$$

$$\leq\biggl(\sum_{q\in\mathbb{Z}^d}\bigl(1+\bigl\|\xi+2^j2\pi q\bigr\|^2\bigr)^s\bigl|\hat{h}\bigl(\xi+2^j2\pi q\bigr)\bigr|^2\biggr)^{1/2}.$$

We know that

$$\sum_{q\in\mathbb{Z}^d}\bigl(1+\bigl\|\xi+2^j2\pi q\bigr\|^2\bigr)^s\bigl|\hat{h}\bigl(\xi+2^j2\pi q\bigr)\bigr|^2\bigl(2^{j+1}\pi\bigr)^d\to\int_{\mathbb{R}^d}\bigl(1+\|\xi\|^2\bigr)^s\bigl|\hat{h}(\xi)\bigr|^2\,d\xi$$

if $j\leq-1$. It follows that

$$\|P_jh\|_s^2\leq\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\bigl(1+\|\xi\|^2\bigr)^s\bigl|\hat{h}(\xi)\hat{\varphi}^{(j)}\bigl(2^{-j}\xi\bigr)\bigr|2^{-jd}C\|h\|_s\,d\xi$$

$$\leq\frac{2^{-jd}C\|h\|_s}{(2\pi)^d}\biggl(\int_{\mathbb{R}^d}\bigl(1+2^{-j}\|\xi\|\bigr)^\alpha\bigl|\hat{\varphi}^{(j)}\bigl(2^{-j}\xi\bigr)\bigr|^2\,d\xi\biggr)^{1/2}$$

$$\times\biggl(\int_{\mathbb{R}^d}\bigl(1+2^{-j}\|\xi\|\bigr)^{-\alpha}\bigl(1+\|\xi\|^2\bigr)^{2s}\bigl|\hat{h}(\xi)\bigr|^2\,d\xi\biggr)^{1/2}$$

$$\leq\frac{C\sqrt{A}\|h\|_s}{(2\pi)^d}\biggl(\int_{\mathbb{R}^d}\bigl(1+2^{-j}\|\xi\|\bigr)^{-\alpha}\bigl(1+\|\xi\|^2\bigr)^{2s}\bigl|\hat{h}(\xi)\bigr|^2\,d\xi\biggr)^{1/2}.$$

The last expression converges to zero as $j$ converges to $-\infty$. □

Now we construct wavelets in $H^s(\mathbb{R}^d)$ with the help of previous propositions.

By definition, $V_j$ is the set of all $f\in H^s(\mathbb{R}^d)$ such that

$$\hat{f}(\xi)=m\bigl(2^{-j}\xi\bigr)\hat{\varphi}^{(j)}\bigl(2^{-j}\xi\bigr),$$

where $m\in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ is $2\pi\mathbb{Z}^d$-periodic. This follows immediately from the fact that the Fourier transform of $2^{jd/2}\varphi^{(j)}(2^jx-k)$ is

$$2^{-jd/2}e^{-i2^{-j}\langle k,\xi\rangle}\hat{\varphi}^{(j)}\bigl(2^{-j}\xi\bigr).$$

We have $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}^d$ iff there are $2\pi\mathbb{Z}^d$-periodic functions $m_0^{(j)} \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ such that the following scale relation holds:

$$\hat{\varphi}^{(j)}(2\xi) = m_0^{(j+1)}(\xi)\hat{\varphi}^{(j+1)}(\xi); \tag{2}$$

moreover, $\varphi^{(j)}$ and $\varphi^{(j+1)}$ satisfy the hypothesis of Proposition 2.1. Now, using our theorems and propositions, we develop the definition of MRA in $H^s(\mathbb{R}^d)$.

**Definition 2.3** Let $s$ be a real number. The MRA of $H^s(\mathbb{R}^d)$ is a sequence $V_j, j \in \mathbb{Z}$, of closed linear subspaces of $H^s(\mathbb{R}^d)$ such that

(a) $V_j \subset V_{j+1}$,
(b) $\bigcup_{j=-\infty}^{j=\infty} V_j = H^s(\mathbb{R}^d)$,
(c) $\bigcap_{j=-\infty}^{j=\infty} V_j = \{0\}^d$, and
(d) for every $j$, there is a function $\varphi^{(j)}$ such that the distributions $2^{jd/2}\varphi^{(j)}(2^j x - k)$, $k \in \mathbb{Z}^d$, form an orthonormal basis for $V_j$.

Before giving a necessary condition for the orthonormality, we define $E_d := \{0,1\}^d$ as the unit cube in the $d$-dimensional Euclidean space.

**Theorem 2.4** *If $\varphi^{(j)}$ and $\varphi^{(j+1)}$ satisfy the hypothesis of Proposition 2.1, then*

$$\sum_{q=0}^{2^d-1} \left| m_0^{(j+1)}(\xi + \gamma_q \pi) \right| = 1, \quad \gamma_q \in E_d, q = 1, 2, \ldots, 2^d - 1.$$

*Proof* We know from Proposition 2.1 that if the system is orthonormal, then

$$\begin{aligned}
\delta_{k,l} &= \langle \varphi_{j,k}^{(j)}, \varphi_{j,l}^{(j)} \rangle_s \\
&= \frac{2^{-jd}}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \hat{\varphi}^{(j)}(2^{-j}\xi) \right|^2 e^{-i2^{-j}\langle \xi, (k-l) \rangle} \left(1 + \|\xi\|^2\right)^s d\xi \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \hat{\varphi}^{(j)}(u) \right|^2 e^{-i\langle u, (k-l) \rangle} \left(1 + 2^{2j}\|u\|^2\right)^s du \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| m_0^{(j+1)}(u/2) \right|^2 \left| \hat{\varphi}^{(j+1)}(u/2) \right|^2 e^{-i\langle u, (k-l) \rangle} \left(1 + 2^{2j}\|u\|^2\right)^s du \\
&= \frac{2^d}{(2\pi)^d} \int_{\mathbb{R}^d} \left| m_0^{(j+1)}(\nu) \right|^2 \left| \hat{\varphi}^{(j+1)}(\nu) \right|^2 e^{-i2\langle \nu, (k-l) \rangle} \left(1 + 2^{2(j+1)}\|\nu\|^2\right)^s d\nu \\
&= \frac{1}{(\pi)^d} \int_{\mathbb{T}^d} \left| m_0^{(j+1)}(\nu) \right|^2 \sum_{r \in \mathbb{Z}^d} \left| \hat{\varphi}^{(j+1)}(\nu + 2\pi r) \right|^2 \\
&\quad \times \left(1 + 2^{2(j+1)}\|\nu + 2\pi r\|^2\right)^s e^{-i2\langle \nu, (k-l) \rangle} d\nu \\
&= \frac{1}{(\pi)^d} \int_{\mathbb{T}^d} \left| m_0^{(j+1)}(\nu) \right|^2 e^{-i2\langle \nu, (k-l) \rangle} d\nu \\
&= \frac{1}{(\pi)^d} \int_{[0,\pi)^d} \sum_{q=0}^{2^d-1} \left| m_0^{(j+1)}(\nu + \gamma_q \pi) \right|^2 e^{-i2\langle \nu, (k-l) \rangle} d\nu,
\end{aligned}$$

which implies that

$$\sum_{q=0}^{2^d-1} \left| m_0^{(j+1)}(\xi + \gamma_q \pi)\right|^2 = 1, \quad \gamma_q \in E_d,$$

if $k = l$. □

With the help of (2) and Theorem 2.4, we may define $\varphi^{(j)}$ by

$$\begin{aligned}
\hat{\varphi}^{(j)}(\xi) &= m_0^{(j+1)}(\xi/2)\hat{\varphi}^{(j+1)}(\xi/2) \\
&= \prod_{t=1}^{J} m_0^{(j+t)}(\xi/2^t)\hat{\varphi}^{(j+J)}(\xi/2^J) \\
&= \cdots = \frac{1}{(1+\|\xi\|^2)^{s/2}} \prod_{t=1}^{+\infty} m_0^{(j+t)}(\xi/2^t)
\end{aligned} \tag{3}$$

for $j \in \mathbb{Z}$. For $V_j$, let $W_j$ be the orthogonal complement of $V_j$ in $V_{j+1}$. We have

$$\psi_{j,k,p}^{(j)} := 2^{jd/2}\psi_p^{(j)}(2^j x - k) \in V_{j+1} \tag{4}$$

if there are $2\pi\mathbb{Z}^d$-periodic functions $m_1^{(j)}, m_2^{(j)}, \ldots, m_{2^d-1}^{(j)} \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\hat{\psi}_p^{(j)}(2^{-j}\xi) = m_p^{(j+1)}(2^{-j-1}\xi)\hat{\varphi}^{(j+1)}(2^{-j-1}\xi), \quad p = 1, 2, \ldots, 2^d - 1.$$

**Theorem 2.5** *The distributions $\psi_{j,k,p}^{(j)}(x) = 2^{jd/2}\psi_p^{(j)}(2^j x - k)$ are orthonormal if*

$$\sum_{q=0}^{2^d-1} \left| m_p^{(j+1)}(\xi + \gamma_q \pi)\right| = 1, \quad \gamma_q \in E_d, \forall p = 1, 2, \ldots, 2^d - 1,$$

*and they are orthogonal to $V_j$ if*

$$\sum_{q=0}^{2^d-1} m_p^{(j+1)}(\xi + \gamma_q \pi)\overline{m_0^{(j+1)}(\xi + \gamma_q \pi)} = 0, \quad \gamma_q \in E_d, \forall p = 1, 2, \ldots, 2^d - 1. \tag{5}$$

*Proof*

$$\begin{aligned}
&\langle \psi_{j,k,p}^{(j)}, \psi_{j,l,p}^{(j)} \rangle_s \\
&\quad = \frac{2^{-jd}}{(2\pi)^d} \int_{\mathbb{R}^d} \left|\hat{\psi}_p^{(j)}(2^{-j}\xi)\right|^2 e^{-i2^{-j}\langle \xi,(k-l)\rangle} \left(1 + \|\xi\|^2\right)^s d\xi \\
&\quad = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left|\hat{\psi}_p^{(j)}(u)\right|^2 e^{-i\langle u,(k-l)\rangle} \left(1 + 2^{2j}\|u\|^2\right)^s du \\
&\quad = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left|m_p^{(j+1)}(u/2)\right|^2 \left|\hat{\varphi}^{(j+1)}(u/2)\right|^2 e^{-i\langle u,(k-l)\rangle} \left(1 + 2^{2j}\|u\|^2\right)^s du \\
&\quad = \frac{2^d}{(2\pi)^d} \int_{\mathbb{R}^d} \left|m_p^{(j+1)}(\nu)\right|^2 \left|\hat{\varphi}^{(j+1)}(\nu)\right|^2 e^{-i2\langle \nu,(k-l)\rangle} \left(1 + 2^{2(j+1)}\|\nu\|^2\right)^s d\nu
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(\pi)^d}\int_{\mathbb{T}^d}\left|m_p^{(j+1)}(\nu)\right|^2\sum_{r\in\mathbb{Z}^d}\left|\hat{\varphi}^{(j+1)}(\nu+2\pi r)\right|^2 \\
&\quad\times\left(1+2^{2(j+1)}\|\nu+2\pi r\|^2\right)^s e^{-i2\langle\nu,(k-l)\rangle}\,d\nu \\
&= \frac{1}{(\pi)^d}\int_{\mathbb{T}^d}\left|m_p^{(j+1)}(\nu)\right|^2 e^{-i2\langle\nu,(k-l)\rangle}\,d\nu \\
&= \frac{1}{(\pi)^d}\int_{[0,\pi)^d}\sum_{q=1}^{2^d-1}\left|m_p^{(j+1)}(\nu+\gamma_q\pi)\right|^2 e^{-i2\langle\nu,(k-l)\rangle}\,d\nu \\
&= \frac{1}{(\pi)^d}\int_{[0,\pi)^d} e^{-i2\langle\nu,(k-l)\rangle}\,d\nu.
\end{aligned}
$$

Therefore

$$
\left\langle\psi_{j,k,p}^{(j)},\psi_{j,l,p}^{(j)}\right\rangle_s = 1
$$

if $\sum_{q=0}^{2^d-1}|m_p^{(j+1)}(\xi+\gamma_q\pi)| = 1, \gamma_q\in E_d$, and $k=l$.

Now we prove second part of the theorem:

$$
\begin{aligned}
0 &= \left\langle\psi_{j,k,p}^{(j)},\varphi_{j,l,p}^{(j)}\right\rangle_s \\
&= \frac{2^{-jd}}{(2\pi)^d}\int_{\mathbb{R}^d}\left(1+\|\xi\|^2\right)^s\hat{\psi}_p^{(j)}\left(2^{-j}\xi\right)\overline{\hat{\varphi}^{(j)}\left(2^{-j}\xi\right)}e^{-i2^{-j}\langle\xi,(k-l)\rangle}\,d\xi \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(1+2^{2j}\|\xi\|^2\right)^s m_p^{(j+1)}(\xi/2)\overline{m_0^{(j+1)}(\xi/2)}\left|\hat{\varphi}^{(j+1)}(\xi/2)\right|^2 e^{-i\langle\xi,(k-l)\rangle}\,d\xi \\
&= \frac{2^d}{(2\pi)^d}\int_{\mathbb{R}^d}\left(1+2^{2(j+1)}\|\xi\|^2\right)^s m_p^{(j+1)}(\xi)\overline{m_0^{(j+1)}(\xi)}\left|\hat{\varphi}^{(j+1)}(\xi)\right|^2 e^{-i2\langle\xi,(k-l)\rangle}\,d\xi \\
&= \frac{1}{(\pi)^d}\int_{\mathbb{T}^d} m_p^{(j+1)}(\xi)\overline{m_0^{(j+1)}(\xi)} \\
&\quad\times\sum_{r\in\mathbb{Z}^d}\left(1+2^{2(j+1)}\|\xi+2r\pi\|^2\right)^s\left|\hat{\varphi}^{(j+1)}(\xi+2r\pi)\right|^2 e^{-i2\langle\xi,(k-l)\rangle}\,d\xi \\
&= \frac{1}{(\pi)^d}\int_{[0,\pi)^d}\sum_{q=1}^{2^d-1} m_p^{(j+1)}(\xi+\gamma_q\pi)\overline{m_0^{(j+1)}(\xi+\gamma_q\pi)}e^{-i2\langle\xi,(k-l)\rangle}\,d\xi,
\end{aligned}
$$

which implies

$$
\sum_{q=0}^{2^d-1} m_p^{(j+1)}(\xi+\gamma_q\pi)\overline{m_0^{(j+1)}(\xi+\gamma_q\pi)} = 0, \quad \gamma_q\in E_d, \forall p = 1,2,\ldots,2^d-1.
$$

□

Now we define unitary matrix with the help of our theorems,

$$
\begin{bmatrix}
m_0^{(j)}(\xi+\gamma_0\pi) & m_0^{(j)}(\xi+\gamma_1\pi) & \cdots & m_0^{(j)}(\xi+\gamma_{2^d-1}\pi) \\
m_1^{(j)}(\xi+\gamma_0\pi) & m_1^{(j)}(\xi+\gamma_1\pi) & \cdots & m_1^{(j)}(\xi+\gamma_{2^d-1}\pi) \\
\ddots & \ddots & \ddots & \ddots \\
m_{2^d-1}^{(j)}(\xi+\gamma_0\pi) & m_{2^d-1}^{(j)}(\xi+\gamma_1\pi) & \cdots & m_{2^d-1}^{(j)}(\xi+\gamma_{2^d-1}\pi)
\end{bmatrix}. \tag{6}
$$

**Theorem 2.6** *Suppose that the scaling function $\varphi^{(j)}, j \in \mathbb{Z}$, generate an MRA $\{V_j\}$ of $H^s(\mathbb{R}^d)$ and $\varphi^{(j)}_{j,k}, k \in \mathbb{Z}^d$, form an orthonormal basis for $V_j, j \in \mathbb{Z}$. Suppose that, for each $j \in \mathbb{Z}$, $m_p^{(j)}$ for $p = 1, 2, \ldots, 2^d - 1$ are such that matrix (6) is unitary. Define $\psi^{(j)}_{j,k,p}$ by (4) for $p = 1, 2, \ldots, 2^d - 1$ and $j \in \mathbb{Z}$. Then $W_j = W_{j,1} \oplus W_{j,2} \oplus \cdots \oplus W_{j,2^d-1}$ with $W_{j,p} = \overline{\operatorname{span}}\{2^{jd/2}\psi_p^{(j)}(2^j x - k) : k \in \mathbb{Z}\}$, $p = 1, 2, \ldots, 2^d - 1$, is perpendicular to $V_j$ in $V_{j+1}$, and $V_{j+1} = V_j \oplus W_j$. Therefore*

$$2^{jd/2}\psi_p^{(j)}\big(2^j x - k\big), \quad k \in \mathbb{Z}, p = 1, 2, \ldots, 2^d - 1,$$

*is an orthonormal basis for $H^s(\mathbb{R}^d)$.*

*Proof* First,we show that $\psi^{(j)}_{j,k,p} \perp V_j$, for all $k \in \mathbb{Z}^d$ and $p = 1, 2, \ldots, 2^d - 1$. Indeed,

$$\begin{aligned}
&(2\pi)^d \big\langle \psi^{(j)}_{j,k,p}(x), \varphi^{(j)}_{j,k}(x)\big\rangle_s \\
&\quad = (2\pi)^d \big\langle 2^{jd/2}\psi_p^{(j)}\big(2^j x - k_1\big), 2^{jd/2}\varphi^{(j)}_{j,k}\big(2^j x - k_2\big)\big\rangle_s \\
&\quad = \int_{\mathbb{R}^d} \big(1 + \|\xi\|^2\big)^s \mathcal{F}\big(2^{jd/2}\psi_p^{(j)}\big(2^j\xi - k_1\big)\big)\overline{\mathcal{F}\big(2^{jd/2}\varphi^{(j)}_{j,k}\big(2^j\xi - k_2\big)\big)}\, d\xi \\
&\quad = 2^{-jd}\int_{\mathbb{R}^d} \big(1 + \|\xi\|^2\big)^s \hat{\psi}_p^{(j)}\big(2^{-j}\xi\big)\overline{\hat{\varphi}^{(j)}\big(2^{-j}\xi\big)} e^{2^{-j}\langle \xi,(k_2-k_1)\rangle}\, d\xi \\
&\quad = \int_{\mathbb{R}^d} \big(1 + 2^{2j}\|\xi\|^2\big)^s m_p^{(j+1)}(\xi/2)\hat{\varphi}^{(j+1)}(\xi/2) \\
&\qquad \times \overline{m_0^{(j+1)}(\xi/2)\hat{\varphi}^{(j+1)}(\xi/2)} e^{2^{-j}\langle \xi,(k_2-k_1)\rangle}\, d\xi \\
&\quad = \int_{\mathbb{T}^d} \sum_{l \in \mathbb{Z}^d} \big(1 + 2^{2j}\|\xi + 2\pi l\|^2\big)^s m_p^{(j+1)}(\xi/2 + \pi l)\hat{\varphi}^{(j+1)}(\xi/2 + \pi l) \\
&\qquad \times \overline{m_0^{(j+1)}(\xi/2 + \pi l)\hat{\varphi}^{(j+1)}(\xi/2 + \pi l)} e^{2^{-j}\langle \xi,(k_2-k_1)\rangle}\, d\xi \\
&\quad = \int_{\mathbb{T}^d} \left[\sum_{q=0}^{2^d-1} m_p^{(j+1)}(\xi/2 + \gamma_q\pi)\overline{m_0^{(j+1)}(\xi/2 + \gamma_q\pi)}\right] e^{2^{-j}\langle \xi,(k_2-k_1)\rangle}\, d\xi
\end{aligned}$$

by Proposition 2.1. This expression is equal to zero because matrix (6) is unitary. Similarly, we can show that $W_{j,p_1} \perp W_{j,p_2}$ for all $p_1, p_2 \in \{1, 2, \ldots, 2^d - 1\}$.

We know show that $V_{j+1} = V_j \oplus W_{j,1} \oplus W_{j,2} \oplus \cdots \oplus W_{j,2^d-1}$ for any $f \in V_{j+1}$. We write

$$\hat{f}(\xi) = B\big(2^{-j-1}\xi\big)\hat{\varphi}^{(j+1)}\big(2^{-j-1}\xi\big).$$

We will demonstrate that there exist $2\pi\mathbb{Z}^d$-periodic functions $G(2^{-j}\xi)$ and $H_p(2^{-j}\xi)$ such that

$$\hat{f}(\xi) = G\big(2^{-j}\xi\big)\hat{\varphi}^{(j)}\big(2^{-j}\xi\big) + \sum_{p=1}^{2^d-1} H_p\big(2^{-j}\xi\big)\hat{\psi}_p^{(j)}\big(2^{-j}\xi\big).$$

Now, we have

$$B(\xi/2)\hat{\varphi}^{(j+1)}(\xi/2) = G(\xi)\hat{\varphi}^{(j)}(\xi) + \sum_{p=1}^{2^d-1} H_p(\xi)\hat{\psi}_p^{(j)}(\xi)$$

$$= G(\xi)m_0^{(j+1)}(\xi/2)\hat{\varphi}^{(j+1)}(\xi/2) + \sum_{p=1}^{2^d-1} H_p(\xi)m_p^{(j+1)}(\xi/2)\hat{\varphi}^{(j+1)}(\xi/2).$$

It follows that

$$B(\xi/2) = G(\xi)m_0^{(j+1)}(\xi/2) + \sum_{p=1}^{2^d-1} H_p(\xi)m_p^{(j+1)}(\xi/2).$$

By the periodicity ($2\pi\mathbb{Z}^d$-periodic) of $G$ and $H_p$ we have

$$B(\xi/2 + \gamma_q\pi) = G(\xi)m_0^{(j+1)}(\xi/2 + \gamma_q\pi) + \sum_{p=1}^{2^d-1} H_p(\xi)m_p^{(j+1)}(\xi/2 + \gamma_q\pi)$$

for $q = 0, 1, \ldots, 2^d - 1$. This completes proof. □

## 3 Multivariate box spline

Now we give an example of multivariate box splines in a Sobolev space. Using them, we construct a wavelet in $H^s(\mathbb{R}^d)$.

Let $D$ be the direction matrix of order $d \times \sum_{i=1}^{d+1} m_i, m_i \in \mathbb{N}_0, \forall i$, whose column vectors consist of $(m_1, m_2, \ldots, m_{d+1})$ copies of the following $d + 1$ column vectors: $(1, 0, \ldots, 0)^T, (0, 1, 0, \ldots, 0)^T, \ldots, (0, 0, \ldots, 1)^T$, and $(1, 1, \ldots, 1)^T$.

Fix $s \geq 0$ and the natural numbers $(m_1, m_2, \ldots, m_{d+1})$ such that

$$\bigl\{m[D] := \min\{m_i + m_j : i \neq j \text{ for all } i, j = 1, 2, \ldots, d + 1\}\bigr\} + \frac{1}{2} > s.$$

Let $M_{m_1,m_2,\ldots,m_{d+1}}$ be a multivariate box spline function defined in terms of the Fourier transform by

$$\widehat{M}_{m_1,m_2,\ldots,m_{d+1}}(\xi) = \prod_{j=1}^{d+1} \left(\frac{1 - e^{-i\langle k_j,\xi\rangle}}{i\langle k_j, \xi\rangle}\right)^{m_j}, \quad k_j \in D, m_j \in \mathbb{N}_0, \forall j.$$

The multivariate box spline $M_{m_1,m_2,\ldots,m_{d+1}}$ belongs to $C^{m[D]-1}$, where $m[D] + 1$ is the minimum number of columns that can be discarded from $D$ to obtain a matrix of rank $< d$ (see [15]).

For

$$W^{(j)}_{m_1,m_2,\ldots,m_{d+1}}(\xi) := \sum_{l\in\mathbb{Z}^d} \bigl(1 + 2^{2j}\|\xi + 2\pi l\|^2\bigr)^s \bigl|\widehat{M}_{m_1,m_2,\ldots,m_{d+1}}(\xi + 2\pi l)\bigr|^2,$$

it is known that there exist $c, C \geq 0$ such that

$$0 \leq c \leq \sum_{l\in\mathbb{Z}^d} \bigl|\widehat{M}_{m_1,m_2,\ldots,m_{d+1}}(\xi + 2\pi l)\bigr|^2 \leq C < \infty.$$

Considering $\xi := (\xi_1, \xi_2, \ldots, \xi_d)$ and $l := (l_1, l_2, \ldots, l_d)$, we have

$$\sum_{l\in\mathbb{Z}^d} \bigl(1 + 2^{2j}\|\xi + 2\pi l\|^2\bigr)^s \bigl|\widehat{M}_{m_1,m_2,\ldots,m_{d+1}}(\xi + 2\pi l)\bigr|^2$$

$$= \sum_{(l_1,l_2,\dots,l_d)\in\mathbb{Z}^d} \left(1 + 2^{2j} \sum_{i=1}^{d} |\xi_i + 2\pi l_i|^2\right)^s \left|\widehat{M}_{m_1,m_2,\dots,m_{d+1}}(\xi + 2\pi l)\right|^2. \tag{7}$$

By mathematical induction we know that, for positive real numbers $x_i$, $i = 1,\dots,d$,

$$\left(\sum_{i=1}^{d} x_i\right)^m \leq d^m \left(\sum_{i=1}^{d} (x_i)^m\right), \quad x_i \in \mathbb{R}_+. \tag{8}$$

From (7) and (8) we have

$$\begin{aligned}
&\sum_{(l_1,l_2,\dots,l_d)\in\mathbb{Z}^d} \left(1 + 2^{2j} \sum_{i=1}^{d} |\xi_i + 2\pi l_i|^2\right)^s \left|\widehat{M}_{m_1,m_2,\dots,m_{d+1}}(\xi + 2\pi l)\right|^2 \\
&\leq (d+1)^s \left(c + 2^{2js} \sum_{(l_1,l_2,\dots,l_d)\in\mathbb{Z}^d} \left(\sum_{i=1}^{d} |\xi_i + 2\pi l_i|^{2s}\right) \left|\widehat{M}_{m_1,m_2,\dots,m_{d+1}}(\xi + 2\pi l)\right|^2\right) \\
&\leq (d+1)^s \left(c + 2^{2js} C' \sum_{(l_1,l_2,\dots,l_d)\in\mathbb{Z}^d} \left(\sum_{i=1}^{d} \left|\widehat{M}_{m_i-s,m_{d+1}}(\xi + 2\pi l)\right|^2\right)\right) \\
&\leq C_j < +\infty,
\end{aligned}$$

where $C', C_j > 0$, and $m_i - s, m_{d+1}$ is the $i$th term subtracted by $s$. Hence we have the following:

**Lemma 3.1** *There exists two constants $c_j$ and $C_j$ such that*

$$0 < c_j \leq W^{(j)}_{m_1,m_2,\dots,m_{d+1}}(\xi) \leq C_j < +\infty.$$

Now, for every $j \in \mathbb{Z}$, we define

$$\hat{\varphi}^{(j)}(\xi) = \frac{\widehat{M}_{m_1,m_2,\dots,m_{d+1}}(\xi)}{\sqrt{W^{(j)}_{m_1,m_2,\dots,m_{d+1}}(\xi)}}. \tag{9}$$

Now we find a $2\pi\mathbb{Z}^d$-periodic function $m_0^{(j)} \in L^2(\mathbb{Z}^d)$ for which the scaling relation (5) holds:

$$\varphi^{(j)}(2\xi) = m_0^{(j+1)}(\xi)\varphi^{(j+1)}(\xi).$$

From (9) we get

$$\begin{aligned}
m_0^{(j+1)}(\xi) &= \frac{\widehat{M}_{m_1,m_2,\dots,m_{d+1}}(2\xi)}{\widehat{M}_{m_1,m_2,\dots,m_{d+1}}(\xi)} \sqrt{\frac{W^{(j+1)}_{m_1,m_2,\dots,m_{d+1}}(\xi)}{W^{(j)}_{m_1,m_2,\dots,m_{d+1}}(2\xi)}} \\
&= \prod_{i=1}^{d+1} \left(\frac{1 + e^{-i\langle k_i,\xi\rangle}}{2}\right)^{m_i} \sqrt{\frac{W^{(j+1)}_{m_1,m_2,\dots,m_{d+1}}(\xi)}{W^{(j)}_{m_1,m_2,\dots,m_{d+1}}(2\xi)}}.
\end{aligned}$$

Finally, let us construct wavelets associated with the scaling function $\varphi^{(j)}, j \in \mathbb{Z}$. We define the $2\pi\mathbb{Z}^d$-periodic functions $m_p^{(j)}$, $p = 1, 2, \ldots, 2^{d-1}$, by

$$m_p^{(j)}(\xi) = e^{-i\langle\gamma_p,\xi\rangle}\mathcal{L}_p^{(j)}(2\xi)\overline{m_0^{(j+1)}(\xi + \gamma_p\pi)},$$

where the trigonometric polynomial $\mathcal{L}_p^{(j)}$ is to be chosen such that $m_p^{(j)}$ satisfies (6) for all $p$.

**Proposition 3.2** *Suppose $\varphi^{(j)}$ is a scaling function for an MRA $V_j, j \in \mathbb{Z}$, of $H^s(\mathbb{R}^d)$ and $m_0^{(j)}$ is the associated low pass filter. Then the distributions $2^{j/2}\psi^{(j)}(2^j x - k), j \in \mathbb{Z}, k \in \mathbb{Z}^d$, are an orthonormal basis for $H^s(\mathbb{R}^d)$ if and only if*

$$\hat{\psi}_p^{(j)}(2\xi) = e^{-i\langle\gamma_p,\xi\rangle}\mathcal{L}_p^{(j)}(2\xi)\overline{m_0^{(j+1)}(\xi + \gamma_p\pi)}\hat{\varphi}^{(j+1)}(\xi), \quad \forall p = 1, 2, \ldots, 2^{d-1},$$

*a.e. on $\mathbb{R}^d$ for some $2\pi\mathbb{Z}^d$-periodic function $\mathcal{L}_p^{(j)}$ such that*

$$\left|\mathcal{L}_p^{(j)}(\xi)\right| = 1, \quad \forall p = 1, 2, \ldots, 2^{d-1}, a.e.\ \xi \in \mathbb{T}^d.$$

*Proof* From Proposition 2.1 we get

$$\sum_{k\in\mathbb{Z}^d}\left(1 + 2^{2j}\|\xi + 2k\pi\|^2\right)^s\left|\hat{\psi}_p^{(j)}(\xi + 2k\pi)\right|^2 = 1, \quad \forall p = 1, 2, \ldots, 2^{d-1}. \tag{10}$$

Now, we have only to verify the density condition

$$\lim_{j\to+\infty}\left|\varphi^{(j)}\left(2^{-j}\xi\right)\right| = \left(1 + \|\xi\|^2\right)^{-s/2}.$$

By definition,

$$W_{m_1,m_2,\ldots,m_{d+1}}^{(j)}\left(2^{-j}\xi\right) = \sum_{k\in\mathbb{Z}^d}\left(1 + \left\|\xi + 2^{j+1}\pi k\right\|^2\right)^s\left|\widehat{M}_{m_1,m_2,\ldots,m_{d+1}}\left(2^{-j}\xi + 2\pi k\right)\right|^2.$$

The term associated with $k = 0$ converges to $(1 + \|\xi\|^2)^s$. Using the estimates

$$\left|\widehat{M}_{m_1,m_2,\ldots,m_{d+1}}\left(2^{-j}\xi + 2\pi k\right)\right| = \left|\prod_{j'=1}^{d+1}\left(\frac{1 - e^{-i\langle k_{j'},2^{-j}\xi\rangle}}{i\langle k_{j'}, 2^{-j}\xi + 2\pi k\rangle}\right)^{m_{j'}}\right|$$

for $\xi = (\xi_1, \xi_2, \ldots, \xi_d)$ and

$$\begin{aligned}
&\left|\prod_{j'=1}^{d+1}\left(\frac{1 - e^{-i\langle k_{j'},2^{-j}\xi\rangle}}{i\langle k_{j'}, 2^{-j}\xi + 2\pi k\rangle}\right)^{m_{j'}}\right| \\
&\quad\le \left(\prod_{j'=1}^{d+1}\left|\frac{\sin(2^{-(j+1)}\xi_{j'})}{2^{-j-1}\xi_{j'} + \pi k}\right|^{m_{j'}}\right)\left(\left|\frac{\sin(2^{-(j+1)}\sum_{j'=1}^{d}\xi_{j'})}{2^{-j-1}\sum_{j'=1}^{d}\xi_{j'} + d\pi k}\right|^{m_{d+1}}\right) \\
&\quad\le \frac{2^{-(j+1)(\sum_{j'=1}^{d+1}m_{j'})}(\prod_{j'=1}^{d}|\xi_{j'}|^{m_{j'}})(|\sum_{j'=1}^{d}\xi_{j'}|^{m_{d+1}})}{|k|^{\sum_{j'=1}^{d+1}m_{j'}}}
\end{aligned}$$

for $2^{-(j+1)}(\prod_{j'=1}^{d}|\xi_{j'}|^{m_{j'}})(|\sum_{j'=1}^{d}\xi_{j'}|^{m_{d+1}}) < 1$ and $k = 0$, we see that, as $j \to +\infty$, the sum of the other terms converges to 0. The conclusion follows easily.

If $\psi_p^{(j)}$ is an orthonormal wavelet, then the orthonormality of $\{2^{j/2}\psi_p^{(j)}(2^j \cdot -k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d, p = 1, 2, \dots, 2^{d-1}\}$ gives us

$$\begin{aligned}
1 &= \sum_{k\in\mathbb{Z}^d}\bigl(1 + 2^{2j}\|\xi + 2k\pi\|^2\bigr)^s\bigl|\hat{\psi}_p^{(j)}(\xi + 2k\pi)\bigr|^2 \\
&= \sum_{k\in\mathbb{Z}^d}\bigl(1 + 2^{2j}\|\xi + 2k\pi\|^2\bigr)^s\bigl|\mathcal{L}_p^{(j)}(\xi)\bigr|^2\bigl|\hat{\varphi}^{(j+1)}(\xi/2 + k\pi)\bigr|^2 \\
&\quad\times\bigl|m_0^{(j+1)}(\xi/2 + k\pi + \gamma_p\pi)\bigr|^2 \\
&= \bigl|\mathcal{L}_p^{(j)}(\xi)\bigr|^2\Biggl(\sum_{l\in\mathbb{Z}^d}\bigl(1 + 2^{2(j+1)}\|\xi/2 + 2l\pi\|^2\bigr)^s\bigl|\hat{\varphi}^{(j+1)}(\xi/2 + 2l\pi)\bigr|^2 \\
&\quad\times\sum_{q=1}^{2^d-1}\bigl|m_0^{(j+1)}(\xi/2 + \gamma_q\pi)\bigr|^2 + \sum_{l\in\mathbb{Z}^d}\bigl(1 + 2^{2(j+1)}\|\xi/2 + 2l\pi + \gamma_q\pi\|^2\bigr)^s \\
&\quad\times\bigl|\hat{\varphi}^{(j+1)}(\xi/2 + 2l\pi + \gamma_q\pi)\bigr|^2\bigl|m_0^{(j+1)}(\xi/2)\bigr|^2\Biggr) \\
&= \bigl|\mathcal{L}_p^{(j)}(\xi)\bigr|^2\Biggl(\sum_{q=0}^{2^d-1}\bigl|m_0^{(j+1)}(\xi/2 + \gamma_q\pi)\bigr|^2\Biggr) = \bigl|\mathcal{L}_p^{(j)}(\xi)\bigr|^2, \quad p = 1, 2, \dots, 2^{d-1},
\end{aligned}$$

for a.e. $\xi \in \mathbb{T}^d$ and $\gamma_q, \gamma_p \in E_d$, which finishes our proof. □

## 4 Conclusion

In this paper, we have successfully generalized MRA over higher-dimensional Sobolev spaces by giving orthonormality and density conditions. Further, we constructed nonseparable orthonormal wavelets in a higher-dimensional Sobolev space by using multivariate box splines. The main obstacle in constructing wavelets is constructing low-pass and high-pass filters with the help of multivariate box splines, which satisfy the condition of orthonormality in $H^s(\mathbb{R}^d)$ for every scale $j$ (because the $H^s$-norm is not dilation invariant).

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**Authors' contributions**
Both authors contributed equally to the manuscript. Both authors read and approved the final manuscript.

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