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Some inequalities for (p, q) -mixed volume

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Abstract

Lutwak, Yang, and Zhang introduced the concept of (p, q) -mixed volume whose special cases contain the L_p -mixed volume and the L_p -dual mixed volume. In this article, associated with the (p, q) -mixed volumes, we establish related cyclic inequalities, monotonic inequalities, and product inequalities.

MSC: 52A20; 52A40; 52A39

Keywords: (p, q) -mixed volume; Cyclic inequality; Monotonic inequality; Product inequality

1 Introduction and main results

At the end of the nineteenth century, Brunn and Minkowski pioneered the classical Brunn–Minkowski theory of convex bodies, which is the product of Minkowski linear combination of vectors and volumes in the Euclidean space. The core of this theory are mixed volume, mixed area measure, and the basic Brunn–Minkowski inequality. In recent years, Brunn–Minkowski theory attracted wide attention (see [1, 2]).

By the 1960s, Firey put forward the concept of L_p -Minkowski combination of convex bodies (see [2]). In 1993, Lutwak [3] introduced the L_p -Minkowski linear combination of convex bodies to the classical Brunn–Minkowski theory, proposed the notions of L_p -mixed volume, L_p -mixed quermassintegrals, and L_p -surface area measure, and obtained the corresponding integral expression, which extended the classical Brunn–Minkowski theory to L_p space (called the L_p Brunn–Minkowski theory). This new theory has attracted a large number of researchers' interests in recent years (see [4–22]). Especially, the concept of L_p -mixed volume ($p \geq 1$) plays an important role in L_p Brunn–Minkowski theory (see [3, 23]).

The classical dual Brunn–Minkowski theory of star bodies was introduced by Lutwak [24] in 1975. In 1996, on the basis of L_p harmonic radial combination, Lutwak [23] put forward the concept of L_p -dual mixed volume ($p \geq 1$) and gave its integral expression. This means that the preliminary L_p dual Brunn–Minkowski theory has been established. Afterwards, Grinberg and Zhang defined the notion of L_p radial combination ($p > 0$). In 2002, Gardner improved $p > 0$ to $p \neq 0$ in L_p radial combination, and got a more extensive class of L_p -dual mixed volume ($p \neq 0$). For more information about the classical dual Brunn–Minkowski theory and L_p dual Brunn–Minkowski theory, please refer to [25–35].

Very recently, Huang et al. [10] constructed the dual curvature measure in dual Brunn–Minkowski theory. These measures are dual to Federer's curvature measures which are fundamental in the classical Brunn–Minkowski theory. In 2018, Lutwak, Yang, and Zhang

[36] made further work and introduced L_p dual curvature measures which include L_p surface area measure, dual curvature measures, and L_p integral curvatures. Using this new concept, they introduced the (p, q) -mixed volume, which unifies L_p -mixed volume and L_p -dual mixed volume. Thus, L_p Brunn–Minkowski theory and L_p dual Brunn–Minkowski theory are partially unified.

Let K be a convex body if K is a compact, convex subset in an n -dimensional Euclidean space \mathbb{R}^n with nonempty interior. The set of all convex bodies in \mathbb{R}^n is written as \mathcal{K}^n . Let \mathcal{K}_o^n denote the set of convex bodies containing the origin in their interiors. Let \mathcal{S}_o^n denote the set of star bodies (about the origin) in \mathbb{R}^n . We write u for the unit vector and B for the unit ball centered at the origin, the surface of B denoted by S^{n-1} . We shall use $V(K)$ for the n -dimensional volume of the body K in \mathbb{R}^n .

Suppose that \mathbb{R} is the set of real numbers. If $E \in \mathcal{K}^n$, the support function of E , $h_E = h(E, \cdot): \mathbb{R}^n \rightarrow \mathbb{R}$, is defined by (see [1, 2])

$$h(E, x) = \max\{x \cdot y : y \in E\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y in \mathbb{R}^n .

For a compact star-shaped (about the origin) E in \mathbb{R}^n , the radial function ρ_E of E , $\rho_E = \rho(E, \cdot): \mathbb{R}^n \setminus \{0\} \rightarrow [0, +\infty)$, is defined by (see [1, 2])

$$\rho(E, x) = \max\{\lambda \geq 0 : \lambda x \in E\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

If ρ_E is positive and continuous, then E is called a star body.

If $E \in \mathbb{R}^n$ is a nonempty subset, the polar set of E , E^* , is defined by (see [1, 2])

$$E^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in E\}.$$

From this, it is easy to get that $(E^*)^* = E$ for all $E \in \mathcal{K}_o^n$.

From the definition of polar, we know that if $E \in \mathcal{K}_o^n$, the support and radial function of E^* , the polar body of E , have the following relationships (see [1, 2]):

$$h(E^*, \cdot) = \frac{1}{\rho(E, \cdot)}, \quad \rho(E^*, \cdot) = \frac{1}{h(E, \cdot)}. \tag{1.1}$$

Very recently, Lutwak et al. defined a new concept (i.e., L_p dual curvature measures) as follows (see [36]): For $p, q \in \mathbb{R}$, $K \in \mathcal{K}_o^n$, and $L \in \mathcal{S}_o^n$, the L_p dual curvature measures $\tilde{C}_{p,q}(K, L, \cdot)$ on S^{n-1} is defined by

$$\int_{S^{n-1}} g(v) d\tilde{C}_{p,q}(K, L, v) = \frac{1}{n} \int_{S^{n-1}} g(\alpha_K(u)) h_K^{-p}(\alpha_K(u)) \rho_K^q(u) \rho_L^{n-q}(u) du \tag{1.2}$$

for each continuous $g : S^{n-1} \rightarrow \mathbb{R}$. Here α_K is the radial Gauss map (see [36]).

By (1.2), Lutwak, Yang, and Zhang [36] defined the (p, q) -mixed volumes as follows: For $K, L \in \mathcal{K}_o^n$, $M \in \mathcal{S}_o^n$, and $p, q \in \mathbb{R}$, the (p, q) -mixed volume $\tilde{V}_{p,q}(K, L, M)$ of K, L, M is defined by

$$\tilde{V}_{p,q}(K, L, M) = \int_{S^{n-1}} h_L^p(v) d\tilde{C}_{p,q}(K, M, v).$$

For the (p, q) -mixed volumes, the authors [36] gave the following integral formula:

$$\tilde{V}_{p,q}(K, L, M) = \frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^p (\alpha_K(u)) \rho_K^q(u) \rho_M^{n-q}(u) du. \tag{1.3}$$

By (1.3), Lutwak et al. introduced the L_p mixed volume for $p \in \mathbb{R}$. For $K, L \in \mathcal{K}_o^n$ and any real p , the L_p mixed volume $V_p(K, L)$ is given by

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u) dS_p(K, u).$$

Here $S_p(K, \cdot)$ denotes the L_p surface area measure (see [3]). The case of $p \geq 1$ is Lutwak’s L_p mixed volume (see [3]).

At the same time, for $K, L \in \mathcal{S}_o^n$ and $q \in \mathbb{R}$, they also defined the q th dual mixed volume $\tilde{V}_q(K, L)$ by

$$\tilde{V}_q(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^q(u) \rho_L^{n-q}(u) du.$$

In addition, they gave several special cases of (p, q) -mixed volume: For $p, q \in \mathbb{R}$, $K, L \in \mathcal{K}_o^n$, and $M \in \mathcal{S}_o^n$, then

$$\tilde{V}_{p,q}(K, K, K) = V(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^n(u) du, \tag{1.4}$$

$$\tilde{V}_{p,q}(K, K, M) = \tilde{V}_q(K, M), \tag{1.5}$$

$$\tilde{V}_{p,q}(K, L, K) = V_p(K, L), \tag{1.6}$$

$$\tilde{V}_{0,q}(K, L, M) = \tilde{V}_q(K, M), \tag{1.7}$$

$$\tilde{V}_{p,n}(K, L, M) = V_p(K, L). \tag{1.8}$$

In this paper, we further study the (p, q) -mixed volumes and establish some inequalities including cyclic inequalities, monotonic inequalities, and product inequalities. First, we give a class of cyclic inequalities as follows.

Theorem 1.1 *Suppose $p, q, r, s \in \mathbb{R}$ satisfy $1 \leq p < q < r \leq n$. If $K, L \in \mathcal{K}_o^n$ and $M \in \mathcal{S}_o^n$, then*

$$\tilde{V}_{q,s}(K, L, M)^{r-p} \leq \tilde{V}_{p,s}(K, L, M)^{r-q} \tilde{V}_{r,s}(K, L, M)^{q-p} \tag{1.9}$$

with equality if and only if K, L , and M are dilates.

Theorem 1.2 *Suppose $p, q, r, s \in \mathbb{R}$ satisfy $1 \leq p < q < r \leq n$. If $K, L \in \mathcal{K}_o^n$ and $M \in \mathcal{S}_o^n$, then*

$$\tilde{V}_{s,q}(K, L, M)^{r-p} \leq \tilde{V}_{s,p}(K, L, M)^{r-q} \tilde{V}_{s,r}(K, L, M)^{q-p} \tag{1.10}$$

with equality if and only if K, L , and M are dilates.

Then we obtain a type of monotonic inequalities as follows.

Theorem 1.3 *Suppose $p, q \in \mathbb{R}$ satisfy $1 \leq p < q < n$. If $K, L \in \mathcal{K}_o^n$ and $M \in \mathcal{S}_o^n$, then*

$$\left[\frac{\tilde{V}_{n-p,p}(K, L, M)}{V(K)} \right]^{\frac{1}{n-p}} \geq \left[\frac{\tilde{V}_{n-q,q}(K, L, M)}{V(K)} \right]^{\frac{1}{n-q}} \tag{1.11}$$

with equality if and only if K and M are dilates.

Theorem 1.4 *Suppose $p, q \in \mathbb{R}$ satisfy $1 \leq p < q < n$. If $K, L \in \mathcal{K}_o^n$ and $M \in \mathcal{S}_o^n$, then*

$$\left[\frac{\tilde{V}_{p,p}(K, L, M)}{V(M)} \right]^{\frac{1}{p}} \leq \left[\frac{\tilde{V}_{q,q}(K, L, M)}{V(M)} \right]^{\frac{1}{q}} \tag{1.12}$$

with equality if and only if K and M are dilates.

Finally, we set up a type of product inequalities as follows.

Theorem 1.5 *Suppose $p > 0, q \in \mathbb{R}$. If $K, L \in \mathcal{K}_o^n$ and $M \in \mathcal{S}_o^n$, then*

$$\tilde{V}_{p,q}(K, L, M) \tilde{V}_{p,q}(K, L^*, M) \geq \tilde{V}_{p,q}(K, B, M)^2 \tag{1.13}$$

with equality if and only if L is a ball centered at the origin.

Theorem 1.6 *Suppose $p, q \in \mathbb{R}$ and $q > n$. If $K, L, M \in \mathcal{K}_o^n$, then*

$$\tilde{V}_{p,q}(K, L, M) \tilde{V}_{p,q}(K, L, M^*) \geq \tilde{V}_{p,q}(K, L, B)^2 \tag{1.14}$$

with equality if and only if M is a ball centered at the origin.

The proofs of Theorems 1.1–1.6 will be completed in the next section.

2 Proofs of theorems

In this part, we give the proofs of Theorems 1.1–1.6.

Proof of Theorem 1.1 For $p, q, r, s \in \mathbb{R}, K, L \in \mathcal{K}_o^n$, and $M \in \mathcal{S}_o^n$. Since $1 \leq p < q < r \leq n$, then $\frac{r-p}{r-q} > 1$. From (1.3) and Hölder’s integral inequality, we get that for $u \in S^{n-1}$

$$\begin{aligned} & \tilde{V}_{p,s}(K, L, M)^{\frac{r-q}{r-p}} \tilde{V}_{r,s}(K, L, M)^{\frac{q-p}{r-p}} \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^p (\alpha_K(u)) \rho_K^s(u) \rho_M^{n-s}(u) du \right]^{\frac{r-q}{r-p}} \\ & \quad \cdot \left[\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^r (\alpha_K(u)) \rho_K^s(u) \rho_M^{n-s}(u) du \right]^{\frac{q-p}{r-p}} \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \left(\left(\frac{h_L}{h_K} \right)^{\frac{p(r-q)}{r-p}} (\alpha_K(u)) \rho_K^{\frac{s(r-q)}{r-p}}(u) \rho_M^{\frac{(n-s)(r-q)}{r-p}}(u) \right)^{\frac{r-p}{r-q}} du \right]^{\frac{r-q}{r-p}} \\ & \quad \cdot \left[\frac{1}{n} \int_{S^{n-1}} \left(\left(\frac{h_L}{h_K} \right)^{\frac{r(q-p)}{r-p}} (\alpha_K(u)) \rho_K^{\frac{s(q-p)}{r-p}}(u) \rho_M^{\frac{(n-s)(q-p)}{r-p}}(u) \right)^{\frac{r-p}{q-p}} du \right]^{\frac{q-p}{r-p}} \end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K}\right)^q (\alpha_K(u)) \rho_K^s(u) \rho_M^{n-s}(u) \, du \\ &= \tilde{V}_{q,s}(K, L, M), \end{aligned}$$

i.e.,

$$\tilde{V}_{q,s}(K, L, M)^{r-p} \leq \tilde{V}_{p,s}(K, L, M)^{r-q} \tilde{V}_{r,s}(K, L, M)^{q-p}. \tag{2.1}$$

This yields (1.9). According to the equality condition of Hölder’s integral inequality, we see that equality holds in (2.1) if and only if K, L , and M are dilates. \square

In (2.1), if $M = K$ or $s = n$, by (1.6) or (1.8), we can get the following result (see [23]).

Corollary 2.1 *Suppose that $p, q, r \in \mathbb{R}$ satisfy $1 \leq p < q < r \leq n$. If $K, L \in \mathcal{K}_o^n$, then*

$$V_q(K, L)^{r-p} \leq V_p(K, L)^{r-q} V_r(K, L)^{q-p}$$

with equality if and only if K and L are dilates.

Proof of Theorem 1.2 For $p, q, r, s \in \mathbb{R}, K, L \in \mathcal{K}_o^n$, and $M \in \mathcal{S}_o^n$. Since $1 \leq p < q < r \leq n$, then $\frac{r-p}{r-q} > 1$. From (1.3) and Hölder’s integral inequality, we get that for $u \in S^{n-1}$

$$\begin{aligned} &\tilde{V}_{s,p}(K, L, M)^{\frac{r-q}{r-p}} \tilde{V}_{s,r}(K, L, M)^{\frac{q-p}{r-p}} \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K}\right)^s (\alpha_K(u)) \rho_K^p(u) \rho_M^{n-p}(u) \, du \right]^{\frac{r-q}{r-p}} \\ &\quad \cdot \left[\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K}\right)^s (\alpha_K(u)) \rho_K^r(u) \rho_M^{n-r}(u) \, du \right]^{\frac{q-p}{r-p}} \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \left(\left(\frac{h_L}{h_K}\right)^{\frac{s(r-q)}{r-p}} (\alpha_K(u)) \rho_K^{\frac{p(r-q)}{r-p}}(u) \rho_M^{\frac{(n-p)(r-q)}{r-p}}(u)\right)^{\frac{r-p}{r-q}} \, du \right]^{\frac{r-q}{r-p}} \\ &\quad \cdot \left[\frac{1}{n} \int_{S^{n-1}} \left(\left(\frac{h_L}{h_K}\right)^{\frac{s(q-p)}{r-p}} (\alpha_K(u)) \rho_K^{\frac{r(q-p)}{r-p}}(u) \rho_M^{\frac{(n-r)(q-p)}{r-p}}(u)\right)^{\frac{r-p}{q-p}} \, du \right]^{\frac{q-p}{r-p}} \\ &\geq \frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K}\right)^s (\alpha_K(u)) \rho_K^q(u) \rho_M^{n-q}(u) \, du \\ &= \tilde{V}_{s,q}(K, L, M). \end{aligned}$$

Thus, we get

$$\tilde{V}_{s,q}(K, L, M)^{r-p} \leq \tilde{V}_{s,p}(K, L, M)^{r-q} \tilde{V}_{s,r}(K, L, M)^{q-p}. \tag{2.2}$$

This yields (1.10). According to the equality condition of Hölder’s integral inequality, we see that equality holds in (2.2) if and only if K, L , and M are dilates. \square

Combined with (1.5) and (1.7), taking $L = K$ or $s = 0$ in (2.2), we obtain the following corollary.

Corollary 2.2 *Suppose $p, q, r \in \mathbb{R}$ satisfy $1 \leq p < q < r \leq n$. If $K \in \mathcal{K}_o^n$ and $M \in \mathcal{S}_o^n$, then*

$$\tilde{V}_q(K, M)^{r-p} \leq \tilde{V}_p(K, M)^{r-q} \tilde{V}_r(K, M)^{q-p}$$

with equality if and only if K and M are dilates.

Proof of Theorem 1.3 For $p, q \in \mathbb{R}$, $K, L \in \mathcal{K}_o^n$, and $M \in \mathcal{S}_o^n$. Since $1 \leq p < q$, then $\frac{n-q}{n-p} < 1$. From (1.3), (1.4), and Hölder’s integral inequality, we obtain that for $u \in S^{n-1}$

$$\begin{aligned} & \tilde{V}_{n-q,q}(K, L, M)^{\frac{n-p}{n-q}} V(K)^{\frac{p-q}{n-q}} \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^{n-q} (\alpha_K(u)) \rho_K^q(u) \rho_M^{n-q}(u) du \right]^{\frac{n-p}{n-q}} \cdot \left[\frac{1}{n} \int_{S^{n-1}} \rho_K^n(u) du \right]^{\frac{p-q}{n-q}} \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \left(\left(\frac{h_L}{h_K} \right)^{n-p} (\alpha_K(u)) \rho_K^{\frac{q(n-p)}{n-q}}(u) \rho_M^{n-p}(u) \right)^{\frac{n-q}{n-p}} du \right]^{\frac{n-p}{n-q}} \\ & \quad \cdot \left[\frac{1}{n} \int_{S^{n-1}} (\rho_K^{\frac{n(p-q)}{n-q}}(u))^{\frac{n-q}{p-q}} du \right]^{\frac{p-q}{n-q}} \\ & \leq \frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^{n-p} (\alpha_K(u)) \rho_K^p(u) \rho_M^{n-p}(u) du \\ & = \tilde{V}_{n-p,p}(K, L, M), \end{aligned}$$

i.e.,

$$\left[\frac{\tilde{V}_{n-p,p}(K, L, M)}{V(K)} \right]^{\frac{1}{n-p}} \geq \left[\frac{\tilde{V}_{n-q,q}(K, L, M)}{V(K)} \right]^{\frac{1}{n-q}}. \tag{2.3}$$

This gives (1.11). According to the equality condition of Hölder’s integral inequality, we know that equality holds in (2.3) if and only if K and M are dilates. □

Proof of Theorem 1.4 For $p, q \in \mathbb{R}$, $K, L \in \mathcal{K}_o^n$, and $M \in \mathcal{S}_o^n$. Since $1 \leq p < q$, then $\frac{q}{p} > 1$. From (1.3), (1.4), and Hölder’s integral inequality, we obtain that for $u \in S^{n-1}$

$$\begin{aligned} & \tilde{V}_{q,q}(K, L, M)^{\frac{p}{q}} V(M)^{\frac{q-p}{q}} \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^q (\alpha_K(u)) \rho_K^q(u) \rho_M^{n-q}(u) du \right]^{\frac{p}{q}} \cdot \left[\frac{1}{n} \int_{S^{n-1}} \rho_M^n(u) du \right]^{\frac{q-p}{q}} \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \left(\left(\frac{h_L}{h_K} \right)^p (\alpha_K(u)) \rho_K^p(u) \rho_M^{\frac{p(n-q)}{q}}(u) \right)^{\frac{q}{p}} du \right]^{\frac{p}{q}} \\ & \quad \cdot \left[\frac{1}{n} \int_{S^{n-1}} (\rho_M^{\frac{n(q-p)}{q}}(u))^{\frac{q}{q-p}} du \right]^{\frac{q-p}{q}} \\ & \geq \frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^p (\alpha_K(u)) \rho_K^p(u) \rho_M^{n-p}(u) du \\ & = \tilde{V}_{p,p}(K, L, M), \end{aligned}$$

i.e.,

$$\left[\frac{\tilde{V}_{p,p}(K, L, M)}{V(M)} \right]^{\frac{1}{p}} \leq \left[\frac{\tilde{V}_{q,q}(K, L, M)}{V(M)} \right]^{\frac{1}{q}}. \tag{2.4}$$

This gives (1.12). According to the equality condition of Hölder’s integral inequality, we know that equality holds in (2.4) if and only if K and M are dilates. \square

We can get the following corollary by (1.5) and (1.7) in (2.4).

Corollary 2.3 *Suppose $p, q \in \mathbb{R}$ satisfy $1 \leq p < q$. If $K \in \mathcal{K}_o^n$ and $M \in \mathcal{S}_o^n$, then*

$$\left[\frac{\tilde{V}_p(K, M)}{V(M)} \right]^{\frac{1}{p}} \leq \left[\frac{\tilde{V}_q(K, M)}{V(M)} \right]^{\frac{1}{q}}$$

with equality if and only if K and M are dilates.

Proof of Theorem 1.5 For $p > 0, q \in \mathbb{R}, K, L \in \mathcal{K}_o^n$, and $M \in \mathcal{S}_o^n$. From the definitions of support function and radial function, we know

$$\rho_{L^*}(u) \leq h_{L^*}(u) \tag{2.5}$$

with equality if and only if L is a ball centered at the origin.

From (1.3), (1.1), (2.5), and Cauchy’s integral inequality, and noticing that $h(B, \cdot) = 1$, we have

$$\begin{aligned} & \tilde{V}_{p,q}(K, L, M)^{\frac{1}{2}} \tilde{V}_{p,q}(K, L^*, M)^{\frac{1}{2}} \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^p (\alpha_K(u)) \rho_K^q(u) \rho_M^{n-q}(u) du \right]^{\frac{1}{2}} \\ & \quad \cdot \left[\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_{L^*}}{h_K} \right)^p (\alpha_K(u)) \rho_K^q(u) \rho_M^{n-q}(u) du \right]^{\frac{1}{2}} \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \rho_{L^*}^{-p}(\alpha_K(u)) h_K^{-p}(\alpha_K(u)) \rho_K^q(u) \rho_M^{n-q}(u) du \right]^{\frac{1}{2}} \\ & \quad \cdot \left[\frac{1}{n} \int_{S^{n-1}} h_{L^*}^p(\alpha_K(u)) h_K^{-p}(\alpha_K(u)) \rho_K^q(u) \rho_M^{n-q}(u) du \right]^{\frac{1}{2}} \\ &\geq \frac{1}{n} \int_{S^{n-1}} \left(\frac{h_{L^*}}{\rho_{L^*}} \right)^{\frac{p}{2}} (\alpha_K(u)) h_K^{-p}(\alpha_K(u)) \rho_K^q(u) \rho_M^{n-q}(u) du \\ &\geq \frac{1}{n} \int_{S^{n-1}} h_K^{-p}(\alpha_K(u)) \rho_K^q(u) \rho_M^{n-q}(u) du \\ &= \frac{1}{n} \int_{S^{n-1}} \left(\frac{h_B}{h_K} \right)^p (\alpha_K(u)) \rho_K^q(u) \rho_M^{n-q}(u) du \\ &= \tilde{V}_{p,q}(K, B, M), \end{aligned}$$

i.e.,

$$\tilde{V}_{p,q}(K, L, M) \tilde{V}_{p,q}(K, L^*, M) \geq \tilde{V}_{p,q}(K, B, M)^2. \tag{2.6}$$

Obviously, equality holds in (2.6) if and only if L is a ball centered at the origin. \square

If we take $M = K$ or $q = n$ in (2.6) and associate with (1.6) and (1.8), the following corollary can be obtained (see [37]).

Corollary 2.4 *Suppose $p > 0$. If $K, L \in \mathcal{K}_o^n$, then*

$$V_p(K, L)V_p(K, L^*) \geq V_p(K, B)^2$$

with equality if and only if L is a ball centered at the origin.

Proof of Theorem 1.6 For $p, q \in \mathbb{R}$ and $q > n$, $K, L, M \in \mathcal{K}_o^n$. From the definitions of support function and radial function, we know

$$\rho_M(u) \leq h_M(u) \tag{2.7}$$

with equality if and only if M is a ball centered at the origin.

From (1.3), (1.1), (2.7), and Cauchy’s integral inequality, and together with $\rho(B, \cdot) = 1$, we obtain

$$\begin{aligned} & \tilde{V}_{p,q}(K, L, M)\tilde{V}_{p,q}(K, L, M^*) \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^p (\alpha_K(u)) \rho_K^q(u) \rho_M^{n-q}(u) du \right] \\ & \quad \cdot \left[\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^p (\alpha_K(u)) \rho_K^q(u) \rho_{M^*}^{n-q}(u) du \right] \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^p (\alpha_K(u)) \rho_K^q(u) \rho_M^{n-q}(u) du \right] \\ & \quad \cdot \left[\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^p (\alpha_K(u)) \rho_K^q(u) h_M^{q-n}(u) du \right] \\ &\geq \left[\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^p (\alpha_K(u)) \rho_K^q(u) \left(\frac{h_M}{\rho_M} \right)^{\frac{q-n}{2}}(u) du \right]^2 \\ &\geq \left[\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^p (\alpha_K(u)) \rho_K^q(u) du \right]^2 \\ &= \left[\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^p (\alpha_K(u)) \rho_K^q(u) \rho_B^{n-q}(u) du \right]^2 \\ &= \tilde{V}_{p,q}(K, L, B)^2, \end{aligned}$$

This gives (1.14). Obviously, according to the equality of (2.7), we know that equality holds in (1.14) if and only if M is a ball centered at the origin. □

By (1.5) and (1.7), taking $L = K$ or $p = 0$ in Theorem 1.6, we also obtain the following corollary.

Corollary 2.5 *Suppose $q \in \mathbb{R}$ and $q > n$. If $K, M \in \mathcal{K}_o^n$, then*

$$\tilde{V}_q(K, M)\tilde{V}_q(K, M^*) \geq \tilde{V}_q(K, B)^2$$

with equality if and only if M is a ball centered at the origin.

Acknowledgements

The authors want to express earnest thankfulness for the referees who provided extremely precious and helpful comments and suggestions. This made the article more accurate and readable.

Funding

Research is supported in part by the Natural Science Foundation of China (Grant No. 11371224) and Innovation Foundation of Graduate Student of China Three Gorges University (Grant No. 2018SSPY136).

Competing interests

The authors state that they have no competitive interests.

Authors' contributions

The authors were devoted equally to the writing of this article. The authors read and approved the final manuscript.

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Received: 27 May 2018 Accepted: 6 September 2018 Published online: 19 September 2018

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