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Almost sure central limit theorem for self-normalized products of the some partial sums of ρ^--mixing sequences

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Abstract

Let $\{X, X_n\}_{n\in N}$ be a strictly stationary ρ^--mixing sequence of positive random variables, under the suitable conditions, we get the almost sure central limit theorem for the products of the some partial sums $(\frac{\prod_{i=1}^k S_{k,i}}{(k-1)^n \mu^n})^{\frac{\mu}{\beta V_k}}$, where $\beta > 0$ is a constant, and $\mathrm{E}(X) = \mu$, $S_{k,i} = \sum_{j=1}^k X_j - X_i$, $1 \le i \le k$, $V_k^2 = \sum_{i=1}^k (X_i - \mu)^2$.

MSC: 60F15

Keywords: Almost sure central limit theorem; ρ^--Mixing sequence; Self-normalized; Products of the some partial sums

1 Introduction and main result

In 1988, Brosamler [1] and Schatte [2] proposed the almost sure central limit theorem (ASCLT) for the sequence of i.i.d. random variables. On the basis of i.i.d., Khurelbaatar and Grzegorz [3] got the ASCLT for the products of the some partial sums of random variables. In 2008, Miao [4] gave a new form of ASCLT for products of some partial sums.

Theorem A ([4]) *Let $\{X, X_n\}_{n\in N}$ be a sequence of i.i.d. positive square integrable random variables with $\mathrm{E}(X_1) = \mu$, $\mathrm{Var}(X_1) = \sigma^2 > 0$ and the coefficient of variation $\gamma = \frac{\sigma}{\mu}$. Denote the $S_{k,i} = \sum_{j=1}^k X_j - X_i$, $1 \le i \le k$. Then, for $\forall x \in R$,*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \mathrm{I}\left[\left(\frac{\prod_{k=1}^n S_{n,k}}{(n-1)^n \mu^n}\right)^{\frac{1}{\gamma\sqrt{n}}} \le x\right] = F(x) \quad a.s.,$$

where $F(\cdot)$ is the distribution function of the random variables $e^{\mathcal{N}}$, $\mathcal{N}$ is a standard normal random variable.

For random variables X, Y, define

$$\rho^-(X,Y) = 0 \vee \sup \frac{\mathrm{Cov}(f(X), g(Y))}{(\mathrm{Var} f(X))^{\frac{1}{2}} (\mathrm{Var}\, g(Y))^{\frac{1}{2}}},$$

where the sup is taken over all $f, g \in \mathscr{C}$ such that $\mathrm{E}(f(X))^2 < \infty$ and $\mathrm{E}(g(Y))^2 < \infty$, and $\mathscr{C}$ is a class of functions which are coordinatewise increasing.

Definition ([5]) A sequence $\{X, X_n\}_{n\in N}$ is called ρ^--mixing, if

$$\rho^-(s) = \sup\big\{\rho^-(S,T); S,T \subset N, \operatorname{dist}(S,T) \geq s\big\} \to 0, \quad s \to \infty,$$

where

$$\rho^-(S,T) = 0 \vee \sup\left\{\frac{\operatorname{Cov}\{f(X_i, i\in S), g(X_j, j\in T)\}}{\sqrt{\operatorname{Var}\{f(X_i, i\in S)\}\operatorname{Var}\{g(X_j, j\in T)\}}}, f,g \in \mathscr{C}\right\},$$

$\mathscr{C}$ is a class of functions which are coordinatewise increasing.

The precise definition of ρ^--mixing random variables was introduced initially by Zhang and Wang [5] in 1999. Obviously, ρ^--mixing random variables include NA and ρ^*-mixing random variables, which have a lot of applications, their limit properties have aroused wide interest recently, and a lot of results have been obtained by many authors. In 2005, Zhou [6] proved the almost central limit theorem of the ρ^--mixing sequence. The almost sure central limit theorem for products of the partial sums of ρ^--mixing sequences was given by Tan [7] in 2012. Because the denominator of the self-normalized partial sums contains random variables, this brings about difficulties to the study of the self-normalized form limit theorem of the ρ^--mixing sequence. At present, there are very few results of this kind. In this paper, we extend Theorem A, and get the almost sure central limit theorem for self-normalized products of the some partial sums of ρ^--mixing sequences.

Throughout this paper, $a_n \sim b_n$ means $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$, and C denotes a positive constant, which may take different values whenever it appears in different expressions, and $\log x = \ln(x \vee e)$. We assume $\{X, X_n\}_{n\in N}$ is a strictly stationary sequence of ρ^--mixing random variables, and we denote $Y_i = X_i - \mu$.

For every $1 \leq i \leq k \leq n$, define

$$\bar{Y}_{ni} = -\sqrt{n}\mathrm{I}(Y_i < -\sqrt{n}) + Y_i\mathrm{I}\big(|Y_i| \leq \sqrt{n}\big) + \sqrt{n}\mathrm{I}(Y_i > \sqrt{n}),$$

$$T_{k,n} = \sum_{i=1}^{k}\bar{Y}_{ni}, \qquad V_n^2 = \sum_{i=1}^{n}Y_i^2, \qquad \bar{V}_n^2 = \sum_{i=1}^{n}\bar{Y}_{ni}^2,$$

$$\bar{V}_{n,1}^2 = \sum_{i=1}^{n}\bar{Y}_{ni}^2\mathrm{I}(Y_i \geq 0), \qquad \bar{V}_{n,2}^2 = \sum_{i=1}^{n}\bar{Y}_{ni}^2\mathrm{I}(Y_i < 0),$$

$$\sigma_n^2 = \operatorname{Var}(T_{n,n}), \qquad \delta_n^2 = \mathrm{E}\big(\bar{Y}_{n1}^2\big), \qquad \delta_{n,1}^2 = \mathrm{E}\bar{Y}_{n1}^2\mathrm{I}(Y_1 \geq 0), \qquad \delta_{n,2}^2 = \mathrm{E}\bar{Y}_{n1}^2\mathrm{I}(Y_1 < 0),$$

apparently, $\delta_n^2 = \delta_{n,1}^2 + \delta_{n,2}^2$, $\mathrm{E}(\bar{V}_n^2) = n\delta_n^2 = n\delta_{n,1}^2 + n\delta_{n,2}^2$.

Our main theorem is as follows.

Theorem 1 *Let $\{X, X_n\}_{n\in N}$ be a strictly stationary ρ^--mixing sequence of positive random variables with $\mathrm{E}X = \mu > 0$, and for some $r > 2$, we have $0 < \mathrm{E}|X|^r < \infty$. Denote $S_{k,i} = \sum_{j=1}^{k}X_j - X_i$, $1 \leq i \leq k$ and $Y = X - \mu$. Suppose that*

(a$_1$) $\mathrm{E}\nu(Y^2\mathrm{I}(Y \geq 0)) > 0$, $\mathrm{E}(Y^2\mathrm{I}(Y < 0)) > 0$,

(a$_2$) $\sigma_1^2 = \mathrm{E}X_1^2 + 2\sum_{k=2}^{\infty}\operatorname{Cov}(X_1, X_k) > 0$, $\sum_{k=2}^{\infty}|\operatorname{Cov}(X_1, X_k)| < \infty$,

(a$_3$) $\sigma_k^2 \sim \beta^2 k\delta_k^2$, *for some* $\beta > 0$,

(a$_4$) $\rho^-(n) = O(\log^{-\delta} n)$, $\exists\delta > 1$.

Suppose $0 \le \alpha < \frac{1}{2}$, *and let*

$$d_k = \frac{\exp(\log^\alpha k)}{k}, \qquad D_n = \sum_{k=1}^n d_k, \tag{1}$$

then, for $\forall x \in R$, *we have*

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^n d_k \mathrm{I}\left[\left(\frac{\prod_{i=1}^k S_{k,i}}{(k-1)^k \mu^k}\right)^{\frac{\mu}{\beta V_k}} \le x\right] = F(x) \quad a.s., \tag{2}$$

where $F(\cdot)$ *is the distribution function of the random variables* $e^{\mathcal{N}}$, $\mathcal{N}$ *is a standard normal random variable.*

Corollary 1 *By* [8], (2) *remains valid if we replace the weight sequence* $\{d_k, k \ge 1\}$ *by any* $\{d_k^*, k \ge 1\}$ *such that* $0 \le d_k^* \le d_k$, $\sum_{k=1}^\infty d_k^* = \infty$.

Corollary 2 *If* $\{X_n, n \ge 1\}$ *is a sequence of strictly stationary independent positive random variables then one has* (a_3) *and* $\beta = 1$.

2 Some lemmas

We will need the following lemmas.

Lemma 2.1 ([7]) *Let* $\{X, X_n\}_{n\in N}$ *be a strictly stationary sequence of* ρ^-*-mixing random variables with* $\mathrm{E}X_1 = 0$, $0 < \mathrm{E}X_1^2 < \infty$, $\sigma_1^2 = \mathrm{E}X_1^2 + 2\sum_{k=2}^\infty \mathrm{Cov}(X_1, X_k) > 0$ *and* $\sum_{k=2}^\infty |\mathrm{Cov}(X_1, X_k)| < \infty$, *then, for* $0 < p < 2$, *we have*

$$\frac{S_n}{n^{\frac{1}{p}}} \to 0, \quad a.s., n \to \infty.$$

Lemma 2.2 ([9]) *Let* $\{X, X_n\}_{n\in N}$ *be a sequence of* ρ^-*-mixing random variables, with*

$$\mathrm{E}X_n = 0, \qquad \mathrm{E}|X_n|^q < \infty, \quad \forall n \ge 1, q \ge 2,$$

then there is a positive constant $C = C(q, \rho^-(\cdot))$ *only depending on* q *and* $\rho^-(\cdot)$ *such that*

$$\mathrm{E}\Big(\max_{1\le j\le n} |S_j|^q\Big) \le C\left\{\sum_{i=1}^n \mathrm{E}|X_i|^q + \left(\sum_{i=1}^n \mathrm{E}X_i^2\right)^{\frac{q}{2}}\right\}.$$

Lemma 2.3 ([10]) *Suppose that* $f_1(x)$ *and* $f_2(y)$ *are real, bounded, absolutely continuous functions on* R *with* $|f_1'(x)| \le C_1$ *and* $|f_2'(y)| \le C_2$, *then, for any random variables* X *and* Y,

$$\big|\mathrm{Cov}\big(f_1(X), f_2(Y)\big)\big| \le C_1C_2\big\{-\mathrm{Cov}(X, Y) + 8\rho^-(X, Y)\|X\|_{2,1}\|Y\|_{2,1}\big\},$$

where $\|X\|_{2,1} = \int_0^\infty (P(|X| > x))^{\frac{1}{2}}\, dx$.

Lemma 2.4 *Let $\{\xi, \xi_n\}_{n\in N}$ be a sequence of uniformly bounded random variables. If $\exists\delta > 1$, $\rho^-(n) = O(\log^{-\delta} n)$, there exist constants $C > 0$ and $\varepsilon > 0$, such that*

$$|\mathrm{E}\xi_k\xi_l| \le C\left(\rho^-(k) + \left(\frac{k}{l}\right)^{\varepsilon}\right), \quad 1 \le 2k < l, \tag{3}$$

then

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k\xi_k = 0, \quad a.s.$$

Proof See the proof of Theorem 1 in [7]. □

Lemma 2.5 *If the assumptions of Theorem 1 hold, then*

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k \mathrm{I}\left[\frac{T_{k,k} - \mathrm{E}(T_{k,k})}{\beta\delta_k\sqrt{k}} \le x\right] = \Phi(x) \quad a.s., \forall x \in R, \tag{4}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k\left[f\left(\frac{\bar{V}_{k,l}^2}{k\delta_{k,l}^2}\right) - \mathrm{E}f\left(\frac{\bar{V}_{k,l}^2}{k\delta_{k,l}^2}\right)\right] = 0 \quad a.s., l = 1, 2, \tag{5}$$

where d_k and D_k is defined as (1) and f is real, bounded, absolutely continuous function on R.

Proof Firstly, we prove (4), by the property of ρ^--mixing sequence, we know that $\{\bar{Y}_{ni}\}_{n\ge1,i\le n}$ is a ρ^--mixing sequence; using Lemma 2.1 in [7], the condition (a_2), (a_3), and $\beta > 0$, $\delta_k^2 \to \mathrm{E}Y^2 > 0$, it follows that

$$\frac{T_{k,k} - \mathrm{E}(T_{k,k})}{\beta\delta_k\sqrt{k}} \xrightarrow{\mathrm{d}} \mathscr{N}, \quad k \to \infty,$$

hence, for any $g(x)$ which is a bounded function with bounded continuous derivative, we have

$$\mathrm{E}g\left(\frac{T_{k,k} - E(T_{k,k})}{\beta\delta_k\sqrt{k}}\right) \to \mathrm{E}g(\mathscr{N}), \quad k \to \infty,$$

by the Toeplitz lemma, we get

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k \mathrm{E}\left[g\left(\frac{T_{k,k} - \mathrm{E}(T_{k,k})}{\beta\delta_k\sqrt{k}}\right)\right] = \mathrm{E}\big(g(\mathscr{N})\big).$$

On the other hand, from Theorem 7.1 of [11] and Sect. 2 of [12], we know that (4) is equivalent to

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k g\left(\frac{T_{k,k} - \mathrm{E}(T_{k,k})}{\beta\delta_k\sqrt{k}}\right) = \mathrm{E}\big(g(\mathscr{N})\big) \quad \text{a.s.},$$

hence, to prove (4), it suffices to prove

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k\left[g\left(\frac{T_{k,k}-\mathrm{E}(T_{k,k})}{\beta\delta_k\sqrt{k}}\right)-\mathrm{E}\left(g\frac{T_{k,k}-\mathrm{E}(T_{k,k})}{\beta\delta_k\sqrt{k}}\right)\right]=0\quad\text{a.s.,}\tag{6}$$

noting that

$$\xi_k=g\left(\frac{T_{k,k}-\mathrm{E}(T_{k,k})}{\beta\delta_k\sqrt{k}}\right)-\mathrm{E}\left(g\left(\frac{T_{k,k}-\mathrm{E}(T_{k,k})}{\beta\delta_k\sqrt{k}}\right)\right),$$

for every $1\le 2k<l$, we have

$$\begin{aligned}|\mathrm{E}\xi_k\xi_l| &= \left|\mathrm{Cov}\left(g\left(\frac{T_{k,k}-\mathrm{E}T_{k,k}}{\beta\delta_k\sqrt{k}}\right),g\left(\frac{T_{l,l}-\mathrm{E}T_{l,l}}{\beta\delta_l\sqrt{l}}\right)\right)\right| \\ &\le \left|\mathrm{Cov}\left(g\left(\frac{T_{k,k}-\mathrm{E}T_{k,k}}{\beta\delta_k\sqrt{k}}\right),g\left(\frac{T_{l,l}-\mathrm{E}T_{l,l}}{\beta\delta_l\sqrt{l}}\right)-g\left(\frac{T_{l,l}-\mathrm{E}T_{l,l}-(T_{2k,l}-\mathrm{E}T_{2k,l})}{\beta\delta_l\sqrt{l}}\right)\right)\right| \\ &\quad+\left|\mathrm{Cov}\left(g\left(\frac{T_{k,k}-\mathrm{E}T_{k,k}}{\beta\delta_k\sqrt{k}}\right),g\left(\frac{T_{l,l}-\mathrm{E}T_{l,l}-(T_{2k,l}-\mathrm{E}T_{2k,l})}{\beta\delta_l\sqrt{l}}\right)\right)\right| \\ &= I_1+I_2.\end{aligned}\tag{7}$$

First we estimate I_1; we know that g is a bounded Lipschitz function, i.e., there exists a constant C such that

$$\left|g(x)-g(y)\right|\le C|x-y|$$

for any $x,y\in R$, since $\{\bar{Y}_{ni}\}_{n\ge1,i\le n}$ also is a ρ^--mixing sequence; we use the condition $\delta_l^2\to\mathrm{E}(Y^2)<\infty$, $l\to\infty$, and Lemma 2.2, to get

$$\begin{aligned}I_1&\le C\frac{\mathrm{E}|T_{2k,l}-\mathrm{E}T_{2k,l}|}{\sqrt{l}}\le C\frac{\sqrt{\mathrm{E}(T_{2k,l}-\mathrm{E}T_{2k,l})^2}}{\sqrt{l}}\\ &\le\frac{C}{\sqrt{l}}\sqrt{\sum_{i=1}^{2k}\mathrm{E}\bar{Y}_{l,i}^2}\le\frac{C}{\sqrt{l}}\sqrt{\sum_{i=1}^{2k}\mathrm{E}Y^2}\le C\left(\frac{k}{l}\right)^{\frac{1}{2}}.\end{aligned}\tag{8}$$

Next we estimate I_2; by Lemma 2.2, we have

$$\begin{aligned}\mathrm{Var}\left(\frac{T_{k,k}-\mathrm{E}T_{k,k}}{\beta\delta_k\sqrt{k}}\right)&\le\frac{C}{k}\mathrm{Var}(T_{k,k}-\mathrm{E}T_{k,k})\\ &\le\frac{C}{k}\sum_{i=1}^{k}\mathrm{E}(\bar{Y}_{ki}-\mathrm{E}\bar{Y}_{ki})^2\le\frac{C}{k}\sum_{i=1}^{k}\mathrm{E}(\bar{Y}_{ki})^2\le\frac{C}{k}\cdot k\le C\end{aligned}$$

and

$$\begin{aligned}\operatorname{Var}\left(\frac{T_{l,l}-\mathrm{E}T_{l,l}-(T_{2k,l}-\mathrm{E}T_{2k,l})}{\beta\delta_l\sqrt{l}}\right) &\le \frac{C}{l}\operatorname{Var}\bigl(T_{l,l}-\mathrm{E}T_{l,l}-(T_{2k,l}-\mathrm{E}T_{2k,l})\bigr)\\ &\le \frac{C}{l}\sum_{i=2k+1}^{l}\mathrm{E}(\bar{Y}_{li}-\mathrm{E}\bar{Y}_{li})^2 \le \frac{C}{l}\left(\sum_{i=1}^{l}\mathrm{E}\bar{Y}_{li}^2\right)\\ &\le \frac{C}{l}\cdot l \le C.\end{aligned}$$

By the definition of a ρ^--mixing sequence, $\mathrm{E}Y^2<\infty$, and Lemma 2.3, we have

$$\begin{aligned}I_2 &\le \Biggl(-\operatorname{Cov}\left(\frac{T_{k,k}-\mathrm{E}T_{k,k}}{\beta\delta_k\sqrt{k}},\frac{T_{l,l}-\mathrm{E}T_{l,l}-(T_{2k,l}-\mathrm{E}T_{2k,l})}{\beta\delta_l\sqrt{l}}\right)\\ &\quad+8\rho^-\left(\frac{T_{k,k}-\mathrm{E}T_{k,k}}{\beta\delta_k\sqrt{k}},\frac{T_{l,l}-\mathrm{E}T_{l,l}-(T_{2k,l}-\mathrm{E}T_{2k,l})}{\beta\delta_l\sqrt{l}}\right)\\ &\quad\cdot\left\|\frac{T_{k,k}-\mathrm{E}T_{k,k}}{\beta\delta_k\sqrt{k}}\right\|_{2,1}\cdot\left\|\frac{T_{l,l}-\mathrm{E}T_{l,l}-(T_{2k,l}-\mathrm{E}T_{2k,l})}{\beta\delta_l\sqrt{l}}\right\|_{2,1}\Biggr)\\ &\le C\rho^-(k)\left(\operatorname{Var}\left(\frac{T_{k,k}-\mathrm{E}T_{k,k}}{\beta\delta_k\sqrt{k}}\right)\right)^{\frac{1}{2}}\cdot\left(\operatorname{Var}\left(\frac{T_{l,l}-\mathrm{E}T_{l,l}-(T_{2k,l}-\mathrm{E}T_{2k,l})}{\beta\delta_l\sqrt{l}}\right)\right)^{\frac{1}{2}}\\ &\quad+8\rho^-(k)\cdot\left\|\frac{T_{k,k}-\mathrm{E}T_{k,k}}{\beta\delta_k\sqrt{k}}\right\|_{2,1}\cdot\left\|\frac{T_{l,l}-\mathrm{E}T_{l,l}-(T_{2k,l}-\mathrm{E}T_{2k,l})}{\beta\delta_l\sqrt{l}}\right\|_{2,1}.\end{aligned}$$

By $\|X\|_{2,1}\le r/(r-2)\|X\|_r$, $r>2$ (see p. 254 of [10] or p. 251 of [13]), Minkowski inequality, Lemma 2.2, and the Hölder inequality, we get

$$\begin{aligned}\left\|\frac{T_{k,k}-\mathrm{E}T_{k,k}}{\beta\delta_k\sqrt{k}}\right\|_{2,1} &\le \frac{r}{r-2}\left\|\frac{T_{k,k}-\mathrm{E}T_{k,k}}{\beta\delta_k\sqrt{k}}\right\|_r\\ &= \frac{r}{r-2}\frac{1}{\beta\delta_k\sqrt{k}}\bigl(\mathrm{E}|T_{k,k}-\mathrm{E}T_{k,k}|^r\bigr)^{\frac{1}{r}}\\ &\le \frac{C}{\sqrt{k}}\left(\sum_{i=1}^{k}\mathrm{E}|\bar{Y}_{ki}|^r+\left(\sum_{i=1}^{k}\mathrm{E}\bar{Y}_{ki}^2\right)^{r/2}\right)^{1/r}\\ &\le \frac{C}{\sqrt{k}}\bigl(k+k^{r/2}\bigr)^{1/r}\le C,\end{aligned}$$

similarly

$$\left\|\frac{T_{l,l}-\mathrm{E}T_{l,l}-(T_{2k,l}-\mathrm{E}T_{2k,l})}{\beta\delta_l\sqrt{l}}\right\|_{2,1}\le C.$$

Hence

$$I_2\le C\rho^-(k). \tag{9}$$

Combining with (7)–(9), (3) holds, and by (a_4), Lemma 2.4, (6) holds, then (4) is true.

Secondly, we prove (5); for $\forall k \geq 1$, $\eta_k = f(\bar{V}_{k,1}^2/(k\delta_{k,1}^2)) - \mathrm{E}(f(\bar{V}_{k,1}^2/(k\delta_{k,1}^2)))$, we have

$$\begin{aligned}
|\mathrm{E}\eta_k\eta_l| &= \left|\mathrm{Cov}\left(f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right), f\left(\frac{\bar{V}_{l,1}^2}{l\delta_{l,1}^2}\right)\right)\right| \\
&\leq \left|\mathrm{Cov}\left(f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right), f\left(\frac{\bar{V}_{l,1}^2}{l\delta_{l,1}^2}\right) - f\left(\frac{\sum_{i=2k+1}^{l}\bar{Y}_{l,i}^2 I(Y_i \geq 0)}{l\delta_{l,1}^2}\right)\right)\right| \\
&\quad + \left|\mathrm{Cov}\left(f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right), f\left(\frac{\sum_{i=2k+1}^{l}\bar{Y}_{l,i}^2 I(Y_i \geq 0)}{l\delta_{l,1}^2}\right)\right)\right| \\
&= J_1 + J_2,
\end{aligned} \tag{10}$$

by the property of f, we know

$$J_1 \leq C\left(\mathrm{E}\left(\sum_{i=1}^{2k}\bar{Y}_{ki}^2 \mathrm{I}(Y_i \geq 0)\right)\Big/ l\right) \leq C\left(\frac{k}{l}\right). \tag{11}$$

Now we estimate J_2,

$$\begin{aligned}
\mathrm{Var}\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right) &= \mathrm{Var}\left(\frac{\sum_{i=1}^{k}\bar{Y}_{ki}^2\mathrm{I}(Y_i \geq 0)}{k\delta_{k,1}^2}\right) \\
&\leq \frac{C}{k^2}\mathrm{E}\left(\sum_{i=1}^{k}\bar{Y}_{ki}^2\mathrm{I}(Y_i \geq 0)\right)^2 \\
&= \frac{C}{k^2}\mathrm{E}\left(\sum_{i=1}^{k}\bar{Y}_{ki}^2\mathrm{I}(Y_i \geq 0) - \mathrm{E}\left(\sum_{i=1}^{k}\bar{Y}_{ki}^2\mathrm{I}(Y_i \geq 0)\right) + \mathrm{E}\left(\sum_{i=1}^{k}\bar{Y}_{ki}^2\mathrm{I}(Y_i \geq 0)\right)\right)^2 \\
&\leq \frac{C}{k^2}\mathrm{E}\left(\sum_{i=1}^{k}\left(\bar{Y}_{ki}^2\mathrm{I}(Y_i \geq 0) - \mathrm{E}\left(\bar{Y}_{ki}^2\mathrm{I}(Y_i \geq 0)\right)\right)\right)^2 \\
&\quad + \frac{C}{k^2}\left(\sum_{i=1}^{k}\mathrm{E}\left(\bar{Y}_{ki}^2\mathrm{I}(Y_i \geq 0)\right)\right)^2 \\
&\leq \frac{C}{k^2}\sum_{i=1}^{k}\mathrm{E}\bar{Y}_{ki}^4\mathrm{I}(Y_i \geq 0) + \frac{C}{k^2}\left(k\mathrm{E}\left(\bar{Y}_{k1}^2\mathrm{I}(Y_1 \geq 0)\right)\right)^2 \\
&\leq \frac{C}{k^2}\sum_{i=1}^{k}\mathrm{E}k(Y_i)^2 \leq C,
\end{aligned}$$

and similarly $\mathrm{Var}(\sum_{i=2k+1}^{l}\bar{Y}_{li}^2\mathrm{I}(Y_i \geq 0)/(l\delta_{l,1}^2)) \leq C$. On the other hand, we have

$$\begin{aligned}
\left\|\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right\|_{2,1} &\leq \frac{r}{r-2}\cdot\frac{C}{k}\left(\mathrm{E}\left|\bar{V}_{k,1}^2\right|^r\right)^{1/r} \\
&\leq \frac{C}{k}\left(\mathrm{E}\left|\sum_{i=1}^{k}\left(\bar{Y}_{ki}^2\mathrm{I}(Y_i \geq 0) - \mathrm{E}\left(\bar{Y}_{ki}^2\mathrm{I}(Y_i \geq 0)\right)\right)\right|^r + \left|\sum_{i=1}^{k}\mathrm{E}\left(\bar{Y}_{ki}^2\mathrm{I}(Y_i \geq 0)\right)\right|^r\right)^{1/r} \\
&\leq \frac{C}{k}\left(\sum_{i=1}^{k}\mathrm{E}\left|\left(\bar{Y}_{ki}^2\mathrm{I}(Y_i \geq 0) - \mathrm{E}\left(\bar{Y}_{ki}^2\mathrm{I}(Y_i \geq 0)\right)\right)\right|^r\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \left(\sum_{i=1}^{k} \mathrm{E}\big(\bar{Y}_{ki}^2 \mathrm{I}(Y_i \geq 0) - \mathrm{E}\big(\bar{Y}_{ki}^2 \mathrm{I}(Y_i \geq 0)\big)\big)^2 \right)^{r/2} \Bigg)^{1/r} \\
&+ \frac{C}{k} \left| \sum_{i=1}^{k} \mathrm{E}\big(\bar{Y}_{ki}^2 \mathrm{I}(Y_i \geq 0)\big) \right| \\
&\leq \frac{C}{k} \left(\sum_{i=1}^{k} \mathrm{E}\big|\bar{Y}_{ki}^2 \mathrm{I}(Y_i \geq 0)\big|^r + \left(\sum_{i=1}^{k} \mathrm{E}\big|\bar{Y}_{ki}^2 \mathrm{I}(Y_i \geq 0)\big|^2 \right)^{r/2} \right)^{1/r} \\
&+ \frac{C}{k} \big| k\mathrm{E}\big(\bar{Y}_{k1}^2 \mathrm{I}(Y_1 \geq 0)\big) \big| \\
&\leq \frac{C}{k} \left(\sum_{i=1}^{k} \mathrm{E}|\sqrt{k} Y_i|^r + \left(\sum_{i=1}^{k} \mathrm{E}|\sqrt{k} Y_i|^2 \right)^{r/2} \right)^{1/r} + C_1 \\
&\leq \frac{C}{k} \big(k^{1+r/2} + k^r\big)^{1/r} + C_1 \leq C,
\end{aligned}
$$

similarly

$$
\left\| \sum_{i=2k+1}^{l} \bar{Y}_{li}^2 \mathrm{I}(Y_i \geq 0) / \big(l\delta_{l,1}^2\big) \right\|_{2,1} \leq C.
$$

Thus, by Lemma 2.3, we have

$$
\begin{aligned}
J_2 &\leq C \Bigg\{ -\mathrm{Cov}\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}, \frac{\sum_{i=2k+1}^{l} \bar{Y}_{li}^2 \mathrm{I}(Y_i \geq 0)}{l\delta_{l,1}^2} \right) \\
&\quad + 8\rho^-\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}, \frac{\sum_{i=2k+1}^{l} \bar{Y}_{li}^2 \mathrm{I}(Y_i \geq 0)}{l\delta_{l,1}^2} \right) \cdot \left\| \frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2} \right\|_{2,1} \cdot \left\| \frac{\sum_{i=2k+1}^{l} \bar{Y}_{li}^2 \mathrm{I}(Y_i \geq 0)}{l\delta_{l,1}^2} \right\|_{2,1} \Bigg\} \\
&\leq C \Bigg\{ \rho^-(k) \left(\mathrm{Var}\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2} \right) \right)^{1/2} \cdot \mathrm{Var}\left(\frac{\sum_{i=2k+1}^{l} \bar{Y}_{li}^2 \mathrm{I}(Y_i \geq 0)}{l\delta_{l,1}^2} \right)^{1/2} \\
&\quad + \rho^-(k) \cdot \left\| \frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2} \right\|_{2,1} \cdot \left\| \frac{\sum_{i=2k+1}^{l} \bar{Y}_{li}^2 \mathrm{I}(Y_i \geq 0)}{l\delta_{l,1}^2} \right\|_{2,1} \Bigg\} \\
&\leq C\rho^-(k),
\end{aligned}
\tag{12}
$$

hence, combining with (11) and (12), (3) holds, and by Lemma 2.4, (5) holds. □

3 Proof of Theorem 1

Let $C_{k,i} = \frac{S_{k,i}}{(k-1)\mu}$, hence, (2) is equivalent to

$$
\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \mathrm{I}\left(\frac{\mu}{\beta V_k} \sum_{i=1}^{k} \log C_{k,i} \leq x \right) = \Phi(x) \quad \text{a.s.} \tag{13}
$$

So we only need to prove (13), for a fixed k, $1 \leq k \leq n$ and $\forall \varepsilon > 0$; we have

$$
\lim_{k\to\infty} P\left\{ \bigcup_{m=k}^{\infty} \left(\left| \frac{X_i}{m} \right| \geq \varepsilon \right) \right\} = \lim_{k\to\infty} P\left\{ \left| \frac{X_i}{k} \right| \geq \varepsilon \right\} = \lim_{k\to\infty} P\big\{ |X_1| \geq \varepsilon k \big\} = 0,
$$

therefore, by Theorem 1.5.2 in [14], we have

$$\frac{X_i}{k} \to 0 \quad \text{a.s. } k \to \infty,$$

on the unanimous establishment of i.

By Lemma 2.1, for some $\frac{4}{3} < p < 2$, and enough large k, we have

$$\begin{aligned} \sup_{1\le i\le k} |C_{k,i} - 1| &\le \left|\frac{\sum_{j=1}^{k}(X_j - \mu)}{(k-1)\mu}\right| + \sup_{1\le i\le k}\left|\frac{X_i}{(k-1)\mu}\right| + \frac{1}{k-1} \\ &\le \left|\frac{S_k - k\mu}{k^{\frac{1}{p}}} \cdot \frac{k^{\frac{1}{p}}}{(k-1)\mu}\right| \le Ck^{\frac{1}{p}-1}, \end{aligned}$$

by $\log(1 + x) = x + O(x^2)$, $x \to 0$, we get

$$\begin{aligned} &\left|\frac{\mu}{\beta\delta_k\sqrt{(1\pm\varepsilon)k}}\sum_{i=1}^{k}\ln C_{k,i} - \frac{\mu}{\beta\delta_k\sqrt{(1\pm\varepsilon)k}}\sum_{i=1}^{k}(C_{k,i} - 1)\right| \\ &\quad\le \frac{C\mu}{\beta\delta_k\sqrt{(1\pm\varepsilon)k}}\sum_{i=1}^{k}(C_{k,i} - 1)^2 \\ &\quad\le \frac{C}{\sqrt{k}}k^{\frac{2}{p}-1} \to 0 \quad \text{a.s.}, k \to \infty, \end{aligned}$$

and then, for $\delta > 0$ and every ω, there exists $k_0 = k_0(\omega, \delta, x)$; when $k > k_0$, we have

$$\begin{aligned} &\mathrm{I}\left\{\frac{\mu}{\beta\delta_k\sqrt{(1\pm\varepsilon)k}}\sum_{i=1}^{k}(C_{k,i} - 1) \le x - \delta\right\} \\ &\quad\le \mathrm{I}\left\{\frac{\mu}{\beta\delta_k\sqrt{(1\pm\varepsilon)k}}\sum_{i=1}^{k}\log C_{k,i} \le x\right\} \\ &\quad\le \mathrm{I}\left\{\frac{\mu}{\beta\delta_k\sqrt{(1\pm\varepsilon)k}}\sum_{i=1}^{k}(C_{k,i} - 1) \le x + \delta\right\}, \end{aligned} \tag{14}$$

under the condition $|X_i - \mu| \le \sqrt{k}$, $1 \le i \le k$, we have

$$\mu\sum_{i=1}^{k}(C_{k,i} - 1) = \sum_{i=1}^{k}\frac{S_{k,i} - (k-1)\mu}{k-1} = \sum_{i=1}^{k}Y_i = \sum_{i=1}^{k}\bar{Y}_{ki} = T_{k,i}, \tag{15}$$

furthermore, by (14) and (15), for any given $0 < \varepsilon < 1$, $\delta > 0$, when $k > k_0$, we obtain

$$\begin{aligned} &\mathrm{I}\left(\frac{\mu}{\beta V_k}\sum_{i=1}^{k}\log C_{k,i} \le x\right) \\ &\quad\le \mathrm{I}\left(\frac{T_{k,i}}{\delta_k\beta\sqrt{k(1+\varepsilon)}} \le x + \delta\right) + \mathrm{I}\left(\bar{V}_k^2 > (1+\varepsilon)k\delta_k^2\right) \\ &\qquad + \mathrm{I}\left(\bigcup_{i=1}^{k}\left(|X_i - \mu| > \sqrt{k}\right)\right), \quad x \ge 0, \end{aligned}$$

$$\begin{aligned}
&\mathrm{I}\left(\frac{\mu}{\beta V_k}\sum_{i=1}^{k}\log C_{k,i}\le x\right)\\
&\quad\le \mathrm{I}\left(\frac{T_{k,i}}{\delta_k\beta\sqrt{k(1-\varepsilon)}}\le x+\delta\right)+\mathrm{I}\bigl(\bar{V}_k^2<(1-\varepsilon)k\delta_k^2\bigr)\\
&\qquad+\mathrm{I}\left(\bigcup_{i=1}^{k}\bigl(|X_i-\mu|>\sqrt{k}\bigr)\right),\quad x<0,\\
&\mathrm{I}\left(\frac{\mu}{\beta V_k}\sum_{i=1}^{k}\log C_{k,i}\le x\right)\\
&\quad\ge \mathrm{I}\left(\frac{T_{k,i}}{\delta_k\beta\sqrt{k(1-\varepsilon)}}\le x-\delta\right)-\mathrm{I}\bigl(\bar{V}_k^2<(1-\varepsilon)k\delta_k^2\bigr)\\
&\qquad-\mathrm{I}\left(\bigcup_{i=1}^{k}\bigl(|X_i-\mu|>\sqrt{k}\bigr)\right),\quad x\ge 0,\\
&\mathrm{I}\left(\frac{\mu}{\beta V_k}\sum_{i=1}^{k}\log C_{k,i}\le x\right)\\
&\quad\ge \mathrm{I}\left(\frac{T_{k,i}}{\delta_k\beta\sqrt{k(1+\varepsilon)}}\le x-\delta\right)-\mathrm{I}\bigl(\bar{V}_k^2>(1+\varepsilon)k\delta_k^2\bigr)\\
&\qquad-\mathrm{I}\left(\bigcup_{i=1}^{k}\bigl(|X_i-\mu|>\sqrt{k}\bigr)\right),\quad x<0.
\end{aligned}$$

Therefore, to prove (13), for any $0<\varepsilon<1$, $\delta_1>0$, it suffices to prove

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k\mathrm{I}\left(\frac{T_{k,i}}{\beta\delta_k\sqrt{k}}\le\sqrt{1\pm\varepsilon}x\pm\delta_1\right)=\Phi(\sqrt{1\pm\varepsilon}x\pm\delta_1)\quad\text{a.s.},\tag{16}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k\mathrm{I}\left(\bigcup_{i=1}^{k}\bigl(|X_i-\mu|>\sqrt{k}\bigr)\right)=0\quad\text{a.s.},\tag{17}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k\mathrm{I}\bigl(\bar{V}_k^2>(1+\varepsilon)k\delta_k^2\bigr)=0\quad\text{a.s.},\tag{18}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k\mathrm{I}\bigl(\bar{V}_k^2<(1-\varepsilon)k\delta_k^2\bigr)=0\quad\text{a.s.}\tag{19}$$

Firstly, we prove (16), by $\mathrm{E}(Y^2)<\infty$, we know $\lim_{x\to\infty}x^2P(|Y|>x)=0$, and by $\mathrm{E}(Y)=0$, it follows that

$$\begin{aligned}
\bigl|\mathrm{E}(T_{k,i})\bigr| &= \left|\mathrm{E}\left(\sum_{i=1}^{k}\bar{Y}_{ki}\right)\right| = |k\mathrm{E}\bar{Y}_{k1}|\\
&\le k\bigl|\mathrm{E}\bigl(Y\mathrm{I}\bigl(|Y|>\sqrt{k}\bigr)\bigr)\bigr|+k^{\frac{3}{2}}\mathrm{E}\bigl(\mathrm{I}\bigl(|Y|>\sqrt{k}\bigr)\bigr)\\
&\le \sqrt{k}\mathrm{E}\bigl(Y^2\mathrm{I}\bigl(|Y|>\sqrt{k}\bigr)\bigr)+k^{\frac{3}{2}}P\bigl(|Y|>\sqrt{k}\bigr)=o(\sqrt{k}),
\end{aligned}$$

so, combining with $\delta_k^2 \to \mathrm{E}(Y^2) < \infty$, for any $\alpha > 0$, when $k \to \infty$, we have

$$
\begin{aligned}
&\mathrm{I}\left(\frac{T_{k,i} - ET_{k,i}}{\beta\delta_k\sqrt{k}} \le \sqrt{1 \pm \varepsilon}x \pm \delta_1 - \alpha\right) \\
&\quad \le \mathrm{I}\left(\frac{T_{k,i}}{\beta\delta_k\sqrt{k}} \le \sqrt{1 \pm \varepsilon}x \pm \delta_1\right) \\
&\quad \le \mathrm{I}\left(\frac{T_{k,i} - ET_{k,i}}{\beta\delta_k\sqrt{k}} \le \sqrt{1 \pm \varepsilon}x \pm \delta_1 + \alpha\right),
\end{aligned}
$$

thus, by (4), we get

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \mathrm{I}\left(\frac{T_{k,i}}{\beta\delta_k\sqrt{k}} \le \sqrt{1 \pm \varepsilon}x \pm \delta_1\right) \\
&\quad \ge \lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \mathrm{I}\left(\frac{T_{k,i} - ET_{k,i}}{\beta\delta_k\sqrt{k}} \le \sqrt{1 \pm \varepsilon}x \pm \delta_1 - \alpha\right) \\
&\quad \to \Phi(\sqrt{1 \pm \varepsilon}x \pm \delta_1 - \alpha), \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \mathrm{I}\left(\frac{T_{k,i}}{\beta\delta_k\sqrt{k}} \le \sqrt{1 \pm \varepsilon}x \pm \delta_1\right) \\
&\quad \le \lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \mathrm{I}\left(\frac{T_{k,i} - ET_{k,i}}{\beta\delta_k\sqrt{k}} \le \sqrt{1 \pm \varepsilon}x \pm \delta_1 + \alpha\right) \\
&\quad \to \Phi(\sqrt{1 \pm \varepsilon}x \pm \delta_1 + \alpha) \quad \text{a.s.}, \qquad (21)
\end{aligned}
$$

letting $\alpha \to 0$ in (20) and (21), (16) holds.

Now, we prove (17); by $\mathrm{E}(Y^2) < \infty$, we know $\lim_{x\to\infty} x^2 P(|Y| > x) = 0$, such that

$$
\mathrm{EI}\left(\bigcup_{i=1}^{k}(|Y_i| > \sqrt{k})\right) \le \sum_{i=1}^{k} P(|Y_i| > \sqrt{k}) \le kP(|Y| > \sqrt{k}) \to 0, \quad k \to \infty,
$$

by the Toeplitz lemma, we get

$$
\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \mathrm{EI}\left(\bigcup_{i=1}^{k}(|Y_i| > \sqrt{k})\right) \to 0 \quad \text{a.s.}, \qquad (22)
$$

hence, to prove (17), it suffices to prove

$$
\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \left(\mathrm{I}\left(\bigcup_{i=1}^{k}(|Y_i| > \sqrt{k})\right) - \mathrm{E}\left[\mathrm{I}\left(\bigcup_{i=1}^{k}(|Y_i| > \sqrt{k})\right)\right]\right) \to 0 \quad \text{a.s.}, \qquad (23)
$$

writing

$$
\mathscr{Z}_k = \mathrm{I}\left(\bigcup_{i=1}^{k}(|Y_i| > \sqrt{k})\right) - \mathrm{E}\left[\mathrm{I}\left(\bigcup_{i=1}^{k}(|Y_i| > \sqrt{k})\right)\right],
$$

for every $0 \le 2k < l$, so by the definition of ρ^--mixing sequence, we have

$$\begin{aligned}
\mathrm{E}|\mathscr{Z}_k\mathscr{Z}_l| &= \left|\mathrm{Cov}\left(\mathrm{I}\left(\bigcup_{i=1}^{k}(|Y_i| > \sqrt{k})\right), \mathrm{I}\left(\bigcup_{i=1}^{l}(|Y_i| > \sqrt{l})\right)\right)\right| \\
&\le \left|\mathrm{Cov}\left(\mathrm{I}\left(\bigcup_{i=1}^{k}(|Y_i| > \sqrt{k})\right), \mathrm{I}\left(\bigcup_{i=1}^{l}(|Y_i| > \sqrt{l})\right) - \mathrm{I}\left(\bigcup_{i=2k+1}^{l}(|Y_i| > \sqrt{l})\right)\right)\right| \\
&\quad + \left|\mathrm{Cov}\left(\mathrm{I}\left(\bigcup_{i=1}^{k}(|Y_i| > \sqrt{k})\right), \mathrm{I}\left(\bigcup_{i=2k+1}^{l}(|Y_i| > \sqrt{l})\right)\right)\right| \\
&\le \mathrm{E}\left|\mathrm{I}\left(\bigcup_{i=1}^{l}(|Y_i| > \sqrt{l})\right) - \mathrm{I}\left(\bigcup_{i=2k+1}^{l}(|Y_i| > \sqrt{l})\right)\right| \\
&\quad + \rho^-(k)\sqrt{\mathrm{Var}\left(\mathrm{I}\left(\bigcup_{i=1}^{k}(|Y_i| > \sqrt{k})\right)\right)\mathrm{Var}\left(\mathrm{I}\left(\bigcup_{i=2k+1}^{l}(|Y_i| > \sqrt{l})\right)\right)} \\
&\le \mathrm{E}\left[\mathrm{I}\left(\bigcup_{i=1}^{2k}(|Y_i| > \sqrt{l})\right)\right] + C\rho^-(k) \\
&\le \sum_{i=1}^{k} P(|Y_i| > \sqrt{l}) + C\rho^-(k) \\
&\le kP(|Y| > \sqrt{l}) + C\rho^-(k) \\
&\le C\left(\frac{k}{l} + \rho^-(k)\right),
\end{aligned}$$

so by Lemma 2.4, (23) holds. And combining with (22), we know that (17) holds.

Next, we prove (18); by $\mathrm{E}(\bar{V}_k^2) = k\delta_k^2$, $\bar{V}_k^2 = \bar{V}_{k,1}^2 + \bar{V}_{k,2}^2$, $\mathrm{E}(\bar{V}_{k,l}^2) = k\delta_{k,l}^2$, and $\delta_{k,1}^2 \le \delta_k^2$, $l = 1, 2$, we have

$$\begin{aligned}
\mathrm{I}(\bar{V}_k^2 > (1+\varepsilon)k\delta_k^2) &= \mathrm{I}(\bar{V}_k^2 - \mathrm{E}(\bar{V}_k^2) > \varepsilon k\delta_k^2) \\
&\le \mathrm{I}(\bar{V}_{k,1}^2 - \mathrm{E}(\bar{V}_{k,1}^2) > \varepsilon k\delta_k^2/2) + \mathrm{I}(\bar{V}_{k,2}^2 - \mathrm{E}(\bar{V}_{k,2}^2) > \varepsilon k\delta_k^2/2) \\
&\le \mathrm{I}\left(\bar{V}_{k,1}^2 > \left(1+\frac{\varepsilon}{2}\right)k\delta_{k,1}^2\right) + \mathrm{I}\left(\bar{V}_{k,2}^2 > \left(1+\frac{\varepsilon}{2}\right)k\delta_{k,2}^2\right),
\end{aligned}$$

therefore, by the arbitrariness of $\varepsilon > 0$, to prove (18), it suffices to prove

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k\mathrm{I}\left(\bar{V}_{k,l}^2 > \left(1+\frac{\varepsilon}{2}\right)k\delta_{k,l}^2\right) = 0 \quad \text{a.s. } l = 1, 2, \tag{24}$$

when $l = 1$, for given $\varepsilon > 0$, let f be a bounded function with bounded continuous derivative such that

$$\mathrm{I}(x > 1+\varepsilon) \le f(x) \le \mathrm{I}\left(x > 1+\frac{\varepsilon}{2}\right), \tag{25}$$

under the condition

$$\mathrm{E}\big(\bar{V}_{k,1}^2\big) = k\delta_{k,1}^2, \qquad \mathrm{E}\big(Y^2\big) < \infty, \qquad \mathrm{E}\big(Y^2\mathrm{I}(Y \ge 0)\big) > 0,$$

by the Markov inequality, and Lemma 2.2, we get

$$\begin{aligned} &P\left(\bar{V}_{k,1}^2 > \left(1+\frac{\varepsilon}{2}\right)k\delta_{k,1}^2\right) \\ &\quad = P\left(\bar{V}_{k,1}^2 - \mathrm{E}\big(\bar{V}_{k,1}^2\big) > \frac{\varepsilon}{2}k\delta_{k,1}^2\right) \\ &\quad \le C\frac{\mathrm{E}(\bar{V}_{k,1}^2 - \mathrm{E}(\bar{V}_{k,1}^2))^2}{k^2} \le C\frac{\sum_{i=1}^k \mathrm{E}(\bar{Y}_{ki}^2\mathrm{I}(\bar{Y}_{ki} \ge 0))^2}{k^2} \\ &\quad \le C\frac{\mathrm{E}\bar{Y}_{k1}^4\mathrm{I}(\bar{Y}_{k1} \ge 0)}{k} \le C\frac{\mathrm{E}Y^4\mathrm{I}(0 \le Y \le \sqrt{k}) + k^2P(Y > \sqrt{k})}{k}, \end{aligned} \tag{26}$$

because $\mathrm{E}(Y^2) < \infty$ implies $\lim_{x\to\infty} x^2P(|Y| > x) = 0$, we have

$$\begin{aligned} \mathrm{E}Y^4\mathrm{I}(0 \le Y \le \sqrt{k}) &= \int_0^\infty P\big(|Y|\mathrm{I}(0 \le Y \le \sqrt{k}) \ge t\big)4t^3\,dt \\ &\le C\int_0^{\sqrt{k}} P\big(|Y| \ge t\big)t^3\,dt \\ &= \int_0^{\sqrt{k}} o(1)t\,dt = o(1)k, \end{aligned}$$

thus, combining with (26),

$$P\left(\bar{V}_{k,1}^2 > \left(1+\frac{\varepsilon}{2}\right)k\delta_{k,1}^2\right) \to 0, \quad k \to \infty.$$

Therefore, from (5), (25) and the Toeplitz lemma

$$\begin{aligned} 0 &\le \frac{1}{D_n}\sum_{k=1}^n d_k\mathrm{I}\left(\bar{V}_{k,1}^2 > \left(1+\frac{\varepsilon}{2}\right)k\delta_{k,1}^2\right) \\ &\le \frac{1}{D_n}\sum_{k=1}^n d_k f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right) \\ &= \frac{1}{D_n}\sum_{k=1}^n d_k\mathrm{E}\left(f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right)\right) + \frac{1}{D_n}\sum_{k=1}^n d_k\left(f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right) - \mathrm{E}\left(f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right)\right)\right) \\ &\le \frac{1}{D_n}\sum_{k=1}^n d_k\mathrm{E}\left(\mathrm{I}\left(\bar{V}_{k,1}^2 > \left(1+\frac{\varepsilon}{2}\right)k\delta_{k,1}^2\right)\right) + \frac{1}{D_n}\sum_{k=1}^n d_k\left(f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right) - \mathrm{E}\left(f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right)\right)\right) \\ &= \frac{1}{D_n}\sum_{k=1}^n d_kP\left(\bar{V}_{k,1}^2 > \left(1+\frac{\varepsilon}{2}\right)k\delta_{k,1}^2\right) + \frac{1}{D_n}\sum_{k=1}^n d_k\left(f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right) - \mathrm{E}\left(f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right)\right)\right) \\ &\to 0 \quad \text{a.s.}, k \to \infty, \end{aligned}$$

hence, (24) holds for $l = 1$. Similarly, we can prove (24) for $l = 2$, so (18) is true. By similar methods used to prove (18), we can prove (19), this completes the proof of Theorem 1.

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Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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