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Shape-preserving properties of a new family of generalized Bernstein operators

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Abstract

In this paper, we introduce a new family of generalized Bernstein operators based on q integers, called (α, q)-Bernstein operators, denoted by $T_{n,q,\alpha}(f)$. We investigate a Kovovkin-type approximation theorem, and obtain the rate of convergence of $T_{n,q,\alpha}(f)$ to any continuous functions f. The main results are the identification of several shape-preserving properties of these operators, including their monotonicity- and convexity-preserving properties with respect to $f(x)$. We also obtain the monotonicity with n and q of $T_{n,q,\alpha}(f)$.

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Keywords: Bernstein operators; q-integers; Shape-preserving; Basis function; Monotonicity

1 Introduction

A generalization of Bernstein polynomials based on q-integers was proposed by Lupaş in 1987 in [1]. However, the Lupaş q-Bernstein operators are rational functions rather than polynomials. In 1997, Phillips [2] proposed the Phillips q-Bernstein polynomials, and for decades thereafter the application of q integers in positive linear operators became a hot topic in approximation theory, such as generalized q-Bernstein polynomials [3–6], Durrmeyer-type q-Bernstein operators [7–9], Kantorovich-type q-Bernstein operators [10–13], etc. As we know, q integers play important roles not only in approximation theory, but also in CAGD. Based on the Phillips q-Bernstein polynomials [2], which are generalizations of Bernstein polynomials, generalized Bézier curves and surfaces were introduced in [14–16]. In [14], Oruç and Phillips constructed q-Bézier curves using the basis functions of Phillips q-Bernstein polynomials. Dişibüyük and Oruç [15, 16] defined the q generalization of rational Bernstein–Bézier curves and tensor product q-Bernstein–Bézier surfaces. Moreover, Simeonov *et al.* [17] introduced a new variant of the blossom, the q blossom, which is specifically adapted to developing identities and algorithms for q-Bernstein bases and q-Bézier curves. In 2014, Han *et al.* [18] proposed a generalization of q-analog Bézier curves with one shape parameter, and established degree evaluation and de Casteljau algorithms and some other properties. In 2016, Han *et al.* [19] introduced a new generalization of weighted rational Bernstein–Bézier curves based on q integers, and investigated the generalized rational Bézier curve from a geometric point of view, obtaining degree evaluation and de Casteljau algorithms, etc.

Recently, Chen *et al.* [20] introduced a new family of α-Bernstein operators, and investigated some approximation properties, such as the rate of convergence, Voronovskaja-type asymptotic formulas, etc. They also obtained the monotonic and convex properties. For $f(x) \in [0,1]$, $n \in \mathbb{N}$, and any fixed real α, the α-Bernstein operators they introduced are defined as

$$T_{n,\alpha} = \sum_{i=0}^{n} f_i p_{n,i}^{(\alpha)}(x), \tag{1}$$

where $f_i = f(\frac{i}{n})$. For $i = 0, 1, \ldots, n$, the α-Bernstein polynomial $p_{n,i}^{\alpha}(x)$ of degree n is defined by $p_{1,0}^{(\alpha)}(x) = 1 - x$, $p_{1,1}^{(\alpha)}(x) = x$ and

$$\begin{aligned} p_{n,i}^{(\alpha)}(x) = {} & \left[\binom{n-2}{i}(1-\alpha)x + \binom{n-2}{i-2}(1-\alpha)(1-x) + \binom{n}{i}\alpha x(1-x)\right] \\ & \times x^{i-1}(1-x)^{n-1-i}, \end{aligned} \tag{2}$$

where $n \geq 2$.

Motivated by above research, in this paper we propose the q analogue of α-Bernstein operators, called (α, q)-Bernstein operators, which are defined as

$$T_{n,q,\alpha}(f;x) = \sum_{i=0}^{n} f_i p_{n,q,i}^{(\alpha)}(x), \tag{3}$$

where $q \in (0,1]$, $f_i = f(\frac{[i]_q}{[n]_q})$, $i = 0, 1, 2, \ldots, n$, $p_{1,q,0}^{(\alpha)}(x) = 1 - x$, $p_{1,q,1}^{(\alpha)}(x) = x$, and

$$\begin{aligned} p_{n,q,i}^{(\alpha)}(x) = {} & \left(\begin{bmatrix} n-2 \\ i \end{bmatrix}_q (1-\alpha)x + \begin{bmatrix} n-2 \\ i-2 \end{bmatrix}_q (1-\alpha)q^{n-i-2}\left(1-q^{n-i-1}x\right) \right. \\ & \left. + \begin{bmatrix} n \\ i \end{bmatrix}_q \alpha x\left(1-q^{n-i-1}x\right)\right) x^{i-1}(1-x)_q^{n-i-1} \quad (n \geq 2). \end{aligned} \tag{4}$$

By simple computations, we can also express the (α, q) operators (3) as

$$\begin{aligned} & T_{n,q,\alpha}(f;x) \\ & \quad = (1-\alpha)\sum_{i=0}^{n-1} g_i \begin{bmatrix} n-1 \\ i \end{bmatrix}_q x^i(1-x)_q^{n-1-i} + \alpha \sum_{i=0}^{n} f_i \begin{bmatrix} n \\ i \end{bmatrix}_q x^i(1-x)_q^{n-i}, \end{aligned} \tag{5}$$

where

$$g_i = \left(1 - \frac{q^{n-1-i}[i]_q}{[n-1]_q}\right) f_i + \frac{q^{n-1-i}[i]_q}{[n-1]_q} f_{i+1}. \tag{6}$$

Here, we mention some definitions based on q integers, the details of which can be found in [21, 22]. For any fixed real number $0 < q \leq 1$ and each non-negative integer k, we denote

q-integers by $[k]_q$, where

$$[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1, \\ k, & q = 1. \end{cases}$$

Also, q-factorial and q-binomial coefficients are defined as follows:

$$[k]_q! := \begin{cases} [k]_q[k-1]_q \cdots [1]_q, & k = 1, 2, \ldots, \\ 1, & k = 0, \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} \quad (n \geq k \geq 0).$$

The q-analog of $(1+x)^n$ is defined by $(1+x)_q^n := \prod_{s=0}^{n-1}(1+q^s x)$. The q derivative and q derivative of the product are defined as $D_q f(x) := \frac{d_q f(x)}{d_q x} = \frac{f(qx)-f(x)}{(q-1)x}$ and $D_q(f(x)g(x)) := f(qx)D_q g(x) + g(x)D_q f(x)$, respectively. We also have $D_q x^n = [n]_q x^{n-1}$ and $D_q(1-x)_q^n = -[n]_q(1-qx)_q^{n-1}$.

The rest of this paper is organized as follows. In the next section, we give some basic properties of the operators $T_{n,q,\alpha}(f)$, such as the moments and central moments for proving the convergence theorems, the forward difference form of $T_{n,q,\alpha}(f)$ for proving shape-preserving properties, etc. In Sect. 3, we obtain the convergence property and the rate of convergence theorem. In Sect. 4, we investigate some shape-preserving properties, such as monotonicity- and convexity-preserving properties with respect to $f(x)$, and also we study the monotonicity with n and q of $T_{n,q,\alpha}(f)$.

2 Auxiliary results

For proving the main results, we require the following lemmas.

Lemma 2.1 *We have the following equalities*:

$$T_{n,q,\alpha}(1;x) = 1, \qquad T_{n,q,\alpha}(t;x) = x. \tag{7}$$

Proof By (5), we have

$$\begin{aligned} T_{n,q,\alpha}(1;x) &= (1-\alpha)\sum_{i=0}^{n-1}\begin{bmatrix} n-1 \\ i \end{bmatrix}_q x^i(1-x)_q^{n-1-i} + \alpha\sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}_q x^i(1-x)_q^{n-i} \\ &= 1. \end{aligned}$$

However,

$$\begin{aligned} &T_{n,q,\alpha}(t;x) \\ &\quad = (1-\alpha)\sum_{i=0}^{n-1}\left[\left(1-\frac{q^{n-1-i}[i]_q}{[n-1]_q}\right)\frac{[i]_q}{[n]_q} + \frac{q^{n-1-i}[i]_q}{[n-1]_q}\frac{[i+1]_q}{[n]_q}\right]\begin{bmatrix} n-1 \\ i \end{bmatrix}_q x^i(1-x)_q^{n-1-i} \\ &\qquad + \alpha\sum_{i=0}^{n}\frac{[i]_q}{[n]_q}\begin{bmatrix} n \\ i \end{bmatrix}_q x^i(1-x)_q^{n-i} \end{aligned}$$

$$= (1-\alpha)\sum_{i=0}^{n-1}\frac{[i]_q}{[n-1]_q}\begin{bmatrix} n-1 \\ i \end{bmatrix}_q x^i(1-x)_q^{n-1-i} + \alpha\sum_{i=0}^{n}\frac{[i]_q}{[n]_q}\begin{bmatrix} n \\ i \end{bmatrix}_q x^i(1-x)_q^{n-i}$$

$$= (1-\alpha)x + \alpha x = x.$$

Lemma 2.1 is proved. □

Remark 2.2 From Lemma 2.1, we know that the (α,q)-Bernstein operators $T_{n,q,\alpha}(f;x)$ reproduce linear functions; that is,

$$T_{n,q,\alpha}(at+b;x) = ax+b,$$

for all real numbers a and b.

We immediately obtain Lemma 2.3 from (5) and Lemma 2.1.

Lemma 2.3 *For all functions f and g defined in $[0,1]$, $x \in [0,1]$, real numbers λ, μ defined in $[0,1]$, and $q \in (0,1]$, the following statements hold true.*

(i) *Endpoint interpolation*: $T_{n,q,\alpha}(f;0) = f(0)$ *and* $T_{n,q,\alpha}(f;1) = f(1)$.

(ii) *Linearity*: $T_{n,q,\alpha}(\lambda f + \mu g;x) = \lambda T_{n,q,\alpha}(f;x) + \mu T_{n,q,\alpha}(g;x)$.

(iii) *Non-negative*: *For* $0 \le \alpha \le 1$ *and* $0 < q < 1$, *if* f *is non-negative on* $[0,1]$, *so is* (α,q)-*Bernstein operators* $T_{n,q,\alpha}(f;x)$.

(iv) *Monotone*: *For fixed* $0 \le \alpha \le 1$ *and* $0 < q < 1$, *if* $f \ge g$, *then* $T_{n,q,\alpha}(f;x) \ge T_{n,q,\alpha}(g;x)$.

Lemma 2.4

(i) *The (α,q)-Bernstein operators may be expressed in the form*

$$T_{n,q,\alpha}(f;x) = \sum_{r=0}^{n}\left((1-\alpha)\begin{bmatrix} n-1 \\ r \end{bmatrix}_q \triangle_q^r g_0 + \alpha\begin{bmatrix} n \\ r \end{bmatrix}_q \triangle_q^r f_0\right)x^r, \tag{8}$$

where $\left[{n-1 \atop n}\right]_q = 0$, $\triangle_q^r f_j = \triangle_q^{r-1} f_{j+1} - q^{r-1}\triangle_q^{r-1} f_j$, $r \ge 1$, *with* $\triangle_q^0 f_j = f_j = f(\frac{[j]_q}{[n]_q})$.

(ii) *The higher-order forward difference of g_i may be expressed in the form*

$$\triangle_q^r g_i = \left(1 - \frac{q^{n-i-1}[i]_q}{[n-1]_q}\right)\triangle_q^r f_i + \frac{q^{n-i-1-r}[i+r]_q}{[n-1]_q}\triangle_q^r f_{i+1}, \tag{9}$$

where $\triangle_q^0 g_i = g_i$, *which is defined in* (6).

Proof We can obtain (8) easily by [2]. Next, in order to prove (9), we use induction on r. It is clear that (9) holds for $r = 0$. Let us assume that (9) holds for some $r = k \ge 0$. For $r = k+1$, we have

$$\triangle_q^{k+1} g_i$$

$$= \triangle_q^k g_{i+1} - q^k \triangle_q^k g_i$$

$$= \left(1 - \frac{q^{n-i-2}[i+1]_q}{[n-1]_q}\right)\triangle_q^k f_{i+1} + \frac{q^{n-i-2-k}[i+k+1]_q}{[n-1]_q}\triangle_q^k f_{i+2}$$

$$
\begin{aligned}
&-q^k\left[\left(1-\frac{q^{n-i-1}[i]_q}{[n-1]_q}\right)\triangle_q^k f_i+\frac{q^{n-i-k-1}[i+k]_q}{[n-1]_q}\triangle_q^k f_{i+1}\right]\\
&=\left[1-\frac{q^{n-i-2}(1+q[i]_q)}{[n-1]_q}\right]\triangle_q^k f_{i+1}-\left(1-\frac{q^{n-i-1}[i]_q}{[n-1]_q}\right)q^k\triangle_q^k f_i\\
&\quad-\frac{q^{n-i-1}[i+k]_q}{[n-1]_q}\triangle_q^k f_{i+1}+\frac{q^{n-i-2-k}[i+k]_q}{[n-1]_q}\triangle_q^k f_{i+2}\\
&=\left(1-\frac{q^{n-i-1}[i]_q}{[n-1]_q}\right)\triangle_q^{k+1} f_i-\frac{q^{n-i-2}}{[n-1]_q}\triangle_q^k f_{i+1}-\frac{q^{n-i-1}[i+k]_q}{[n-1]_q}\triangle_q^k f_{i+1}\\
&\quad+\frac{q^{n-i-2-k}[i+k+1]_q}{[n-1]_q}\triangle_q^k f_{i+2}\\
&=\left(1-\frac{q^{n-i-1}[i]_q}{[n-1]_q}\right)\triangle_q^{k+1} f_i-\frac{q^{n-i-2}[i+k+1]_q}{[n-1]_q}\triangle_q^k f_{i+1}+\frac{q^{n-i-1-k}[i+k+1]_q}{[n-1]_q}\triangle_q^k f_{i+2}\\
&=\left(1-\frac{q^{n-i-1}[i]_q}{[n-1]_q}\right)\triangle_q^{k+1} f_i+\frac{q^{n-i-k-2}[i+k+1]_q}{[n-1]_q}\left(\triangle_q^k f_{i+2}-q^k\triangle_q^k f_{i+1}\right)\\
&=\left(1-\frac{q^{n-i-1}[i]_q}{[n-1]_q}\right)\triangle_q^{k+1} f_i+\frac{q^{n-i-k-2}[i+k+1]_q}{[n-1]_q}\triangle_q^{k+1} f_{i+1}.
\end{aligned}
$$

This shows that (9) holds when k is replaced by $k+1$, and this completes the proof of Lemma 2.4. □

Since $f[\frac{[j]_q}{[n]_q},\frac{[j+1]_q}{[n]_q},\ldots,\frac{[j+k]_q}{[n]_q}]=\frac{[n]_q^k\triangle_q^k f_j}{q^{\frac{k(2j+k-1)}{2}}[k]_q!}=\frac{f^{(k)}(\xi)}{k!}$, where $\xi\in(\frac{[j]_q}{[n]_q},\frac{[j+k]_q}{[n]_q})$, the q differences of the monomial x^k of order greater than k are zero. We see from Lemma 2.4 that, for all $n\geq k$, $T_{n,q,\alpha}(t^k;x)$ is a polynomial of degree k. Actually, the (α,q)-Bernstein operators are degree-reducing on polynomials; that is, if f is a polynomial of degree m, and then $T_{n,q,\alpha}(f)$ is a polynomial of degree $\leq\min\{m,n\}$. In particular, we have the following results.

Lemma 2.5 *Letting $f(t)=t^k$, $n-1\geq k\geq 2$, we have*

$$
T_{n,q,\alpha}\left(t^k;x\right)=a_k x^k+a_{k-1}x^{k-1}+\cdots+a_1 x+a_0,
$$

where $a_k=\frac{q^{\frac{k(k-1)}{2}}[n-2]_q!}{[n-k]_q![n]_q^k}\{(1-\alpha)[n-k]_q[n-1+k]_q+\alpha[n]_q[n-1]_q\}$.

Proof Indeed, from (9) and $\triangle_q^k f_j=\frac{q^{\frac{k(2j+k-1)}{2}}[k]_q!f^{(k)}(\xi)}{k![n]_q^k}$, we have

$$
\triangle_q^k g_0=\triangle_q^k f_0+\frac{q^{n-1-k}[k]_q}{[n-1]_q}\triangle_q^k f_1,\qquad \triangle_q^k f_0=\frac{q^{\frac{k(k-1)}{2}}[k]_q!}{[n]_q^k},\qquad \triangle_q^k f_1=\frac{q^{\frac{k(k+1)}{2}}[k]_q!}{[n]_q^k}.
$$

Thus, we obtain

$$
\triangle_q^k g_0=\left(1+\frac{q^{n-1}[k]_q}{[n-1]_q}\right)\frac{q^{\frac{k(k-1)}{2}}[k]_q!}{[n]_q^k}=\frac{[n-1+k]_q}{[n-1]_q}\frac{q^{\frac{k(k-1)}{2}}[k]_q!}{[n]_q^k}.
$$

Hence, using (8), we have

$$a_k = \left[(1-\alpha)\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \frac{[n-1+k]_q}{[n-1]_q} + \alpha \begin{bmatrix} n \\ k \end{bmatrix}_q\right] \frac{q^{\frac{k(k-1)}{2}}[k]_q!}{[n]_q^k}.$$

We then obtain the proof of Lemma 2.5 by simple computations. □

Lemma 2.6 *The following equalities hold true*:

$$T_{n,q,\alpha}\left(t^2;x\right) = x^2 + \frac{x(1-x)}{[n]_q} + \frac{(1-\alpha)q^{n-1}[2]_q x(1-x)}{[n]_q^2}, \tag{10}$$

$$T_{n,q,\alpha}\left((t-x)^2;x\right) = \frac{x(1-x)}{[n]_q} + \frac{(1-\alpha)q^{n-1}[2]_q x(1-x)}{[n]_q^2}. \tag{11}$$

Proof For $f(t) = t^2$, we have $\triangle_q^0 f_0 = f_0 = 0$, $\triangle_q^1 f_0 = f_1 - f_0 = \frac{1}{[n]_q^2}$, $\triangle_q^1 f_1 = f_2 - f_1 = \frac{2q+q^2}{[n]_q^2}$, $\triangle_q^2 f_0 = \triangle_q^1 f_1 - q\triangle_q^1 f_0 = f_2 - [2]_q f_1 + q f_0 = \frac{q[2]_q}{[n]_q^2}$, and $\triangle_q^2 f_1 = f_3 - [2]_q f_2 + q f_1 = \frac{q^3+q^4}{[n]_q^2}$. By (9), we have $\triangle_q^0 g_0 = 0$, and

$$\triangle_q^1 g_0 = \triangle_q^1 f_0 + \frac{q^{n-2}}{[n-1]_q}\triangle_q^1 f_1 = \frac{1}{[n]_q^2} + \frac{2q^{n-1}+q^n}{[n-1]_q[n]_q^2},$$

$$\triangle_q^2 g_0 = \triangle_q^2 f_0 + \frac{q^{n-3}[2]_q}{[n-1]_q}\triangle_q^2 f_1 = \frac{q[2]_q}{[n]_q^2} + \frac{[2]_q(q^n+q^{n+1})}{[n-1]_q[n]_q^2}.$$

From (8), we have

$$\begin{aligned}
&T_{n,q,\alpha}\left(t^2;x\right)\\
&= (1-\alpha)\triangle_q^0 g_0 + \alpha\triangle_q^0 f_0 + \left[(1-\alpha)[n-1]_q\triangle_q^1 g_0 + \alpha[n]_q\triangle_q^1 f_0\right]x\\
&\quad + \left[(1-\alpha)\frac{[n-1]_q[n-2]_q}{[2]_q}\triangle_q^2 g_0 + \alpha\frac{[n]_q[n-1]_q}{[2]_q}\triangle_q^2 f_0\right]x^2\\
&= \left[\frac{(1-\alpha)[n-1]_q}{[n]_q^2} + \frac{(1-\alpha)(2q^{n-1}+q^n)}{[n]_q^2} + \frac{\alpha}{[n]_q}\right]x\\
&\quad + \left[\frac{(1-\alpha)q[n-1]_q[n-2]_q}{[n]_q^2} + \frac{(1-\alpha)[n-2]_q(q^n+q^{n+1})}{[n]_q^2} + \frac{\alpha q[n-1]_q}{[n]_q}\right]x^2\\
&= \frac{[n]_q + (1-\alpha)q^{n-1}[2]_q}{[n]_q^2}x + \left(1 - \frac{1}{[n]_q} - \frac{(1-\alpha)q^{n-1}[2]_q}{[n]_q^2}\right)x^2\\
&= x^2 + \frac{x(1-x)}{[n]_q} + \frac{(1-\alpha)q^{n-1}[2]_q x(1-x)}{[n]_q^2}.
\end{aligned}$$

Hence, (10) is proved. Finally, using Lemma 2.1, we obtain

$$T_{n,q,\alpha}\left((t-x)^2;x\right) = T_{n,q,\alpha}\left(t^2;x\right) - 2xT_{n,q,\alpha}(t;x) + x^2 T_{n,q,\alpha}(1;x) = T_{n,q,\alpha}\left(t^2;x\right) - x^2.$$

Then (11) is proved by (10). This completes the proof of Lemma 2.6. □

3 Convergence properties

We now state the well-known Bohman–Korovkin theorem, followed by a proof based on that given by Cheney [23].

Theorem 3.1 *Let $\{L_n\}$ denote a sequence of monotone linear operators that map a function $f \in C[a,b]$ to a function $L_n f \in C[a,b]$, and let $L_n f \to f$ uniformly on $[a,b]$ for $f = 1, t$ and t^2. Then $L_n f \to f$ uniformly on $[a,b]$ for all $f \in C[a,b]$.*

Theorem 3.1 leads to the following theorem on the convergence of (α, q)-Bernstein operators.

Theorem 3.2 *Let $q := \{q_n\}$ denote a sequence such that $q_n \in (0,1)$ and $\lim_{n\to\infty} q_n = 1$. Then, for any $f \in C[0,1]$ and $\alpha \in [0,1]$, $T_{n,q,\alpha}(f;x)$ converges uniformly to $f(x)$ on $[0,1]$.*

Proof From Lemma 2.1, we see that $T_{n,q,\alpha}(f;x) = f(x)$ for $f(t) = 1$ and $f(t) = t$. Since $\lim_{n\to\infty} q_n = 1$, we see from (10) that $T_{n,q,\alpha}(f;x)$ converges uniformly to $f(x)$ for $f(t) = t^2$ as $n \to \infty$. It also follows that $T_{n,q,\alpha}$ is a monotone operator by Lemma 2.3; the proof is then completed by applying the Bohman–Korovkin theorem 3.1. □

As we know, the space $C[0,1]$ of all continuous functions on $[0,1]$ is a Banach space with sup-norm $\|f\| := \sup_{x\in[0,1]} |f(x)|$. Letting $f \in C[0,1]$, the Peetre K functional is defined by $K_2(f;\delta) := \inf_{g\in C^2[0,1]}\{\|f-g\| + \delta\|g''\|\}$, where $\delta > 0$ and $C^2[0,1] := \{g \in C[0,1] : g', g'' \in C[0,1]\}$. By [24], there exists an absolute constant $C > 0$, such that

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}), \tag{12}$$

where $\omega_2(f;\delta) := \sup_{0<h\leq\delta} \sup_{x,x+h,x+2h\in[0,1]} |f(x+2h) - 2f(x+h) + f(x)|$ is the second-order modulus of smoothness of $f \in C[0,1]$.

Theorem 3.3 *For $f \in C[0,1]$, $\alpha \in [0,1]$, $q \in (0,1)$, we have*

$$\left|T_{n,q,\alpha}(f;x) - f(x)\right| \leq C\omega_2\left(f; \frac{\sqrt{2[n]_q + (1-\alpha)2[2]_q q^{n-1}}}{4[n]_q}\right),$$

where C is a positive constant.

Proof Letting $g \in C^2[0,1]$, $x, t \in [0,1]$, by Taylor's expansion we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\,du.$$

Using Lemma 2.1, we obtain

$$T_{n,q,\alpha}(g;x) = g(x) + T_{n,q,\alpha}\left(\int_x^t (t-u)g''(u)\,du; x\right).$$

Thus, we have

$$\left|T_{n,q,\alpha}(g;x) - g(x)\right| = \left|T_{n,q,\alpha}\left(\int_x^t (t-u)g''(u)\,du; x\right)\right|$$

$$\begin{aligned}&\le T_{n,q,\alpha}\left(\left|\int_x^t (t-u)\left|g''(u)\right|du\right|;x\right)\\&\le T_{n,q,\alpha}\left((t-x)^2;x\right)\left\|g''\right\|\\&\le \frac{[n]_q+(1-\alpha)q^{n-1}[2]_q}{4[n]_q^2}\left\|g''\right\|.\end{aligned} \tag{13}$$

However, using Lemma 2.1, we have

$$\left|T_{n,q,\alpha}(f;x)\right| \le \|f\|. \tag{14}$$

Now, (13) and (14) imply

$$\begin{aligned}\left|T_{n,q,\alpha}(f;x)-f(x)\right| &\le \left|T_{n,q,\alpha}(f-g;x)-(f-g)(x)\right|+\left|T_{n,q,\alpha}(g;x)-g(x)\right|\\&\le 2\|f-g\|+\frac{[n]_q+(1-\alpha)q^{n-1}[2]_q}{4[n]_q^2}\left\|g''\right\|.\end{aligned}$$

Hence, taking the infimum on the right-hand side over all $g \in C^2[0,1]$, we obtain

$$\left|T_{n,q,\alpha}(f;x)-f(x)\right| \le 2K_2\left(f;\frac{[n]_q+(1-\alpha)q^{n-1}[2]_q}{8[n]_q^2}\right).$$

By (12), we obtain

$$\left|T_{n,q,\alpha}(f;x)-f(x)\right| \le C\omega_2\left(f;\frac{\sqrt{2[n]_q+(1-\alpha)2[2]_q q^{n-1}}}{4[n]_q}\right),$$

where C is a positive constant. Theorem 3.3 is proved. □

Remark 3.4 Letting $q := \{q_n\}$ denote a sequence such that $q_n \in (0,1)$ and $\lim_{n\to\infty} q_n = 1$, we know that, under the conditions of theorem 3.3, the convergence rate of the operators $T_{n,q,\alpha}(f)$ to f is $1/\sqrt{[n]_q}$ as $n \to \infty$. This convergence rate can be improved depending on the choice of q, at least as fast as $1/\sqrt{n}$.

Example 3.5 Letting $f(x) = 1-\cos(4e^x)$, the graphs of $f(x)$ and $T_{n,q,0.9}(f;x)$ with different values of n and q are shown in Fig. 1. Figure 2 shows the graphs of $f(x)$ and $T_{10,0.9,\alpha}(f;x)$ with $\alpha = 0.6$ and $\alpha = 0.9$.

4 Shape-preserving properties

The (α,q)-Bernstein operators $T_{n,q,\alpha}(f;x)$ have a monotonicity-preserving property.

Theorem 4.1 *Let $f \in C[0,1]$. If f is a monotonically increasing or monotonically decreasing function on $[0,1]$, so are all its (α,q)-Bernstein operators for fixed $q \in (0,1)$ and $\alpha \in [0,1]$.*

Proof From (5), we have

$$T_{n+1,q,\alpha}(f;x) = (1-\alpha)\sum_{i=0}^{n} g_i \begin{bmatrix} n \\ i \end{bmatrix}_q x^i(1-x)_q^{n-i} + \alpha\sum_{i=0}^{n+1} f_i \begin{bmatrix} n+1 \\ i \end{bmatrix}_q x^i(1-x)_q^{n+1-i},$$

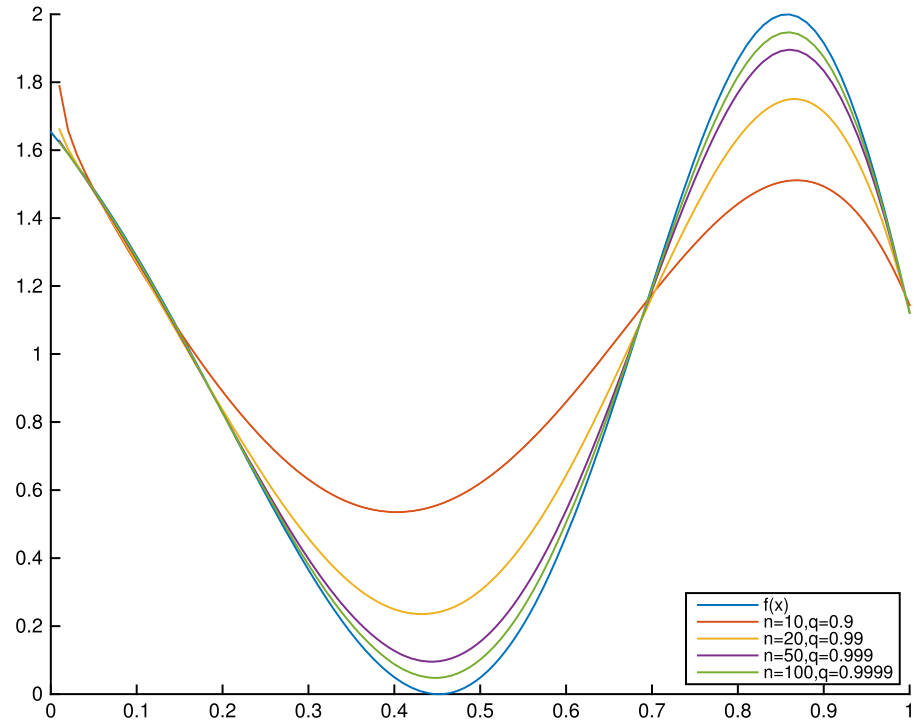

Figure 1 Convergence of $T_{n,q,\alpha}(f;x)$ to $f(x)$ for fixed $\alpha = 0.9$

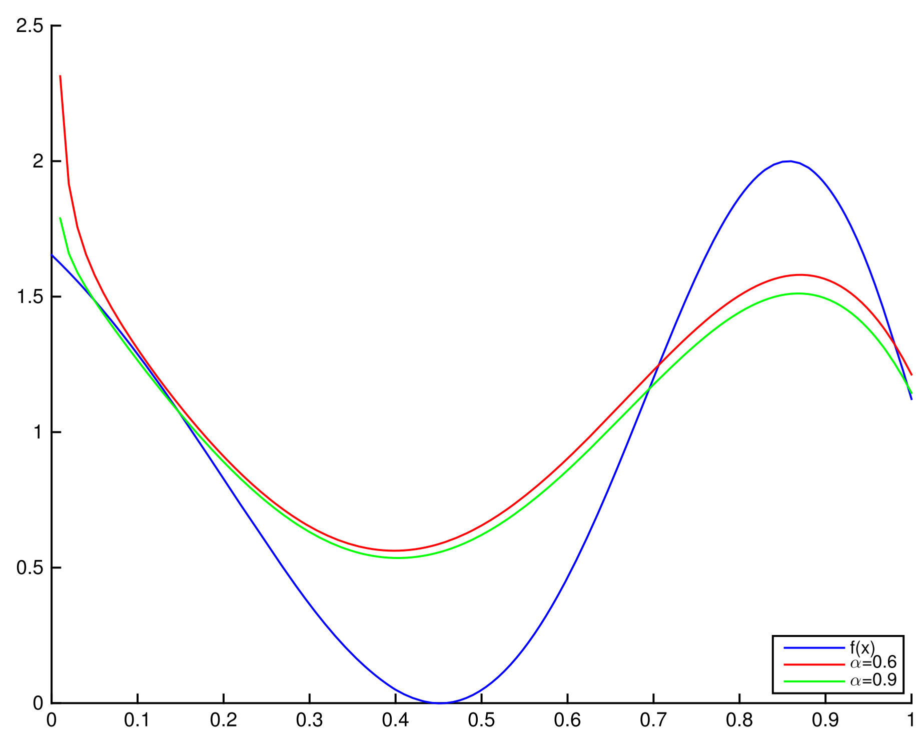

Figure 2 Convergence of $T_{n,q,\alpha}(f;x)$ to $f(x)$ for fixed $q = 0.9$

where $f_i = \frac{[i]_q}{[n+1]_q}$, $g_i = (1 - \frac{q^{n-i}[i]_q}{[n]_q})f_i + \frac{q^{n-i}[i]_q}{[n]_q}f_{i+1}$. Then the q derivative of $T_{n+1,q,\alpha}(f;x)$ is

$$\begin{aligned}
&D_q\big[T_{n+1,q,\alpha}(f;x)\big] \\
&\quad = (1-\alpha)\sum_{i=0}^{n} g_i \begin{bmatrix} n \\ i \end{bmatrix}_q D_q\big[x^i(1-x)_q^{n-i}\big] + \alpha\sum_{i=0}^{n+1} f_i \begin{bmatrix} n+1 \\ i \end{bmatrix}_q D_q\big[x^i(1-x)_q^{n+1-i}\big],
\end{aligned}$$

and we denote the first and second parts of the right-hand side of the last equation by Λ_1 and Λ_2, respectively. We then have

$$\begin{aligned}
&\Lambda_1 \\
&\quad = (1-\alpha)\sum_{i=0}^{n} g_i \begin{bmatrix} n \\ i \end{bmatrix}_q \big[[i]_q x^{i-1}(1-qx)_q^{n-i} - [n-i]_q x^i(1-qx)_q^{n-i-1}\big] \\
&\quad = (1-\alpha)[n]_q \left[\sum_{i=1}^{n} g_i \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q x^{i-1}(1-qx)_q^{n-i} - \sum_{i=0}^{n-1} g_i \begin{bmatrix} n-1 \\ i \end{bmatrix}_q x^i(1-qx)_q^{n-i-1}\right] \\
&\quad = (1-\alpha)[n]_q \sum_{i=0}^{n-1} \begin{bmatrix} n-1 \\ i \end{bmatrix}_q x^i(1-qx)_q^{n-i-1}\triangle_q^1 g_i.
\end{aligned}$$

Using (9), we obtain

$$\triangle_q^1 g_i = \left(1 - \frac{q^{n-i}[i]_q}{[n]_q}\right)\triangle_q^1 f_i + \frac{q^{n-i-1}[i+1]}{[n]_q}\triangle_q^1 f_{i+1}.$$

Thus, we have

$$\begin{aligned}
\Lambda_1 &= (1-\alpha)\sum_{i=0}^{n-1}\big[\big([n]_q - q^{n-i}[i]_q\big)\triangle_q^1 f_i + q^{n-i-1}[i+1]_q\triangle_q^1 f_{i+1}\big] \begin{bmatrix} n-1 \\ i \end{bmatrix}_q \\
&\quad \times x^i(1-qx)_q^{n-i-1}.
\end{aligned} \tag{15}$$

Similarly, we can obtain

$$\Lambda_2 = \alpha[n+1]_q \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q x^i(1-qx)_q^{n-i}\triangle_q^1 f_i. \tag{16}$$

Therefore, by using (15) and (16), the derivative of (α, q)-Bernstein operators $T_{n,q,\alpha}(f;x)$ may be expressed in the form

$$\begin{aligned}
&D_q\big[T_{n,q,\alpha}(f;x)\big] \\
&\quad = (1-\alpha)\sum_{i=0}^{n-1}\big[\big([n]_q - q^{n-i}[i]_q\big)\triangle_q^1 f_i + q^{n-i-1}[i+1]_q\triangle_q^1 f_{i+1}\big] \begin{bmatrix} n-1 \\ i \end{bmatrix}_q \\
&\qquad \times x^i(1-qx)_q^{n-i-1} + \alpha[n+1]_q \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q x^i(1-qx)_q^{n-i}\triangle_q^1 f_i.
\end{aligned}$$

Since if f is monotonically increasing on $[0,1]$, the forward differences $\triangle_q^1 f_i$ and $\triangle_q^1 f_{i+1}$ are non-negative, and so is $D_q[T_{n,q,\alpha}(f;x)]$. Hence, (α,q)-Bernstein operators $T_{n,q,\alpha}(f;x)$ are monotonically increasing on $[0,1]$ for fixed $q\in(0,1)$ and $\alpha\in[0,1]$. On the contrary, if f is monotonically decreasing on $[0,1]$, then operators $T_{n,q,\alpha}(f;x)$ are monotonically decreasing on $[0,1]$ for fixed $q\in(0,1)$ and $\alpha\in[0,1]$. Theorem 4.1 is proved. □

The (α,q)-Bernstein operators $T_{n,q,\alpha}(f;x)$ have a convexity-preserving property

Theorem 4.2 *Let $f\in C[0,1]$. If f is convex on $[0,1]$, so are all of its (α,q)-Bernstein operators $T_{n,q,\alpha}(f;x)$ for fixed $q\in(0,1)$ and $\alpha\in[0,1]$.*

Proof From (5), we obtain

$$\begin{aligned} &T_{n+2,q,\alpha}(f;x)\\ &\quad=(1-\alpha)\sum_{i=0}^{n+1}g_i\begin{bmatrix}n+1\\ i\end{bmatrix}_q x^i(1-x)_q^{n-i+1}+\alpha\sum_{i=0}^{n+2}f_i\begin{bmatrix}n+2\\ i\end{bmatrix}_q x^i(1-x)_q^{n+2-i},\end{aligned}$$

where $f_i=\frac{[i]_q}{[n+2]_q}$, $g_i=(1-\frac{q^{n-i+1}[i]_q}{[n+1]_q})f_i+\frac{q^{n-i+1}[i]_q}{[n+1]_q}f_{i+1}$. The q-derivative of $T_{n+2,q,\alpha}(f;x)$ can easily obtained by the proof theorem 4.1, which may be expressed as

$$\begin{aligned} D_q\bigl[T_{n+2,q,\alpha}(f;x)\bigr]&=(1-\alpha)[n+1]_q\sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}_q x^i(1-qx)_q^{n-i}(g_{i+1}-g_i)\\ &\quad+\alpha[n+2]_q\sum_{i=0}^{n+1}\begin{bmatrix}n+1\\ i\end{bmatrix}_q x^i(1-qx)_q^{n-i+1}(f_{i+1}-f_i).\end{aligned}$$

Then we have

$$\begin{aligned} &D_q^2\bigl[T_{n+2,q,\alpha}(f;x)\bigr]\\ &\quad=(1-\alpha)[n+1]_q\sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}_q(g_{i+1}-g_i)D_q\bigl[x^i(1-qx)_q^{n-i}\bigr]\\ &\qquad+\alpha[n+2]_q\sum_{i=0}^{n+1}\begin{bmatrix}n+1\\ i\end{bmatrix}_q(f_{i+1}-f_i)D_q\bigl[x^i(1-qx)_q^{n-i-1}\bigr].\end{aligned}$$

By some easy computations, we obtain

$$\begin{aligned} D_q^2\bigl[T_{n+2,q,\alpha}(f;x)\bigr]&=(1-\alpha)[n+1]_q[n]_q\sum_{i=0}^{n-1}\begin{bmatrix}n-1\\ i\end{bmatrix}_q x^i\bigl(1-q^2x\bigr)_q^{n-i-1}\triangle_q^2 g_i\\ &\quad+\alpha[n+2]_q[n+1]_q\sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}_q x^i\bigl(1-q^2x\bigr)_q^{n-i}\triangle_q^2 f_i,\end{aligned}$$

where $\triangle_q^2 g_i=(1-\frac{q^{n-i+1}[i]_q}{[n+1]_q})\triangle_q^2 f_i+\frac{q^{n-i-1}[i+2]_q}{[n+1]_q}\triangle_q^2 f_{i+1}$. By the connection between the second-order q differences and convexity, we know that $\triangle_q^2 f_i$ and $\triangle_q^2 f_{i+1}$ are all non-negative since

f is convex on $[0,1]$. Hence, we obtain $D_q^2[T_{n+2,q,\alpha}(f;x)] \geq 0$, and then the convexity-preserving property of $T_{n,q,\alpha}(f;x)$. Theorem 4.2 is proved. □

Next, if $f(x)$ is convex, the (α,q)-Bernstein operators $T_{n,q,\alpha}(f;x)$, for n fixed, are monotonic in q.

Theorem 4.3 *For* $0 < q_1 \leq q_2 \leq 1, \alpha \in [0,1]$ *and for* $f(x)$ *convex on* $[0,1]$, *then* $T_{n,q_2,\alpha}(f;x) \leq T_{n,q_1,\alpha}(f;x)$.

Proof In the following main proof of our results, we must introduce a linear polynomial function:

$$g(x) = \frac{f_{i+1} - f_i}{\frac{[i+1]_q}{[n]_q} - \frac{[i]_q}{[n]_q}} \left(x - \frac{[i]_q}{[n]_q} \right) + f_i, \tag{17}$$

where $\frac{[i]_q}{[n]_q} \leq x < \frac{[i+1]_q}{[n]_q}$, $f_i = f(\frac{[i]_q}{[n]_q})$, $i = 0, \dots, n-1$. Then it is straightforward to check that $g_i = g(\frac{[i]_q}{[n-1]_q})$. Since f is convex on $[0,1]$, the intrinsic linear polynomial function $g(x)$ must be convex on $[0,1]$ as well. Therefore, by the classical results of q-Bernstein operators (see [3]), we note that

$$T_{n,q,\alpha}(f;x) = (1-\alpha)B_{n-1}^q(g;x) + \alpha B_n^q(f;x). \tag{18}$$

We have $B_{n-1}^{q_2}(g;x) \leq B_{n-1}^{q_1}(g;x)$ and $B_n^{q_2}(f;x) \leq B_n^{q_1}(f;x)$, and the desired result is obvious. Theorem 4.3 is proved. □

Finally, if $f(x)$ is convex, we give the monotonicity of (α,q)-Bernstein operators $T_{n,q,\alpha}(f;x)$ with n.

Theorem 4.4 *If* $f(x)$ *is convex on* $[0,1]$, *for fixed* $q \in (0,1)$ *and* $\alpha \in [0,1]$, *we have*

$$T_{n-1,q,\alpha}(f;x) - T_{n,q,\alpha}(f;x) \geq 0 \quad (n \geq 2).$$

Proof Combining (17) and (18), and the fact that if f and g are convex on $[0,1]$, then

$$B_{n-2}^q(g;x) \geq B_{n-1}^q(g;x), \qquad B_{n-1}^q(f;x) \geq B_n^q(f;x)$$

(see [25]). The desired result is obvious. □

Example 4.5 Letting the convex function $f(x) = 1 - \sin(\pi x)$, $x \in [0,1]$, the graphs of $f(x)$ and $T_{n,0.9,0.9}(f;x)$ with different values of $n = 10, 15, 20, 30$ are shown in Fig. 3. Figure 4 shows the graphs of $f(x) = 1 - \sin(\pi x)$ and $T_{10,q,0.9}(f;x)$ with $q = 0.6, 0.7, 0.8, 0.9$.

5 Conclusion

In this paper, we proposed a new family of generalized Bernstein operators, named (α,q)-Bernstein operators, and denoted by $T_{n,q,\alpha}(f)$. We study the rate of convergence of these operators, investigate their monotonicity-, convexity-preserving properties with respect to $f(x)$, and also obtain their monotonicity with n and q of $T_{n,q,\alpha}(f)$.

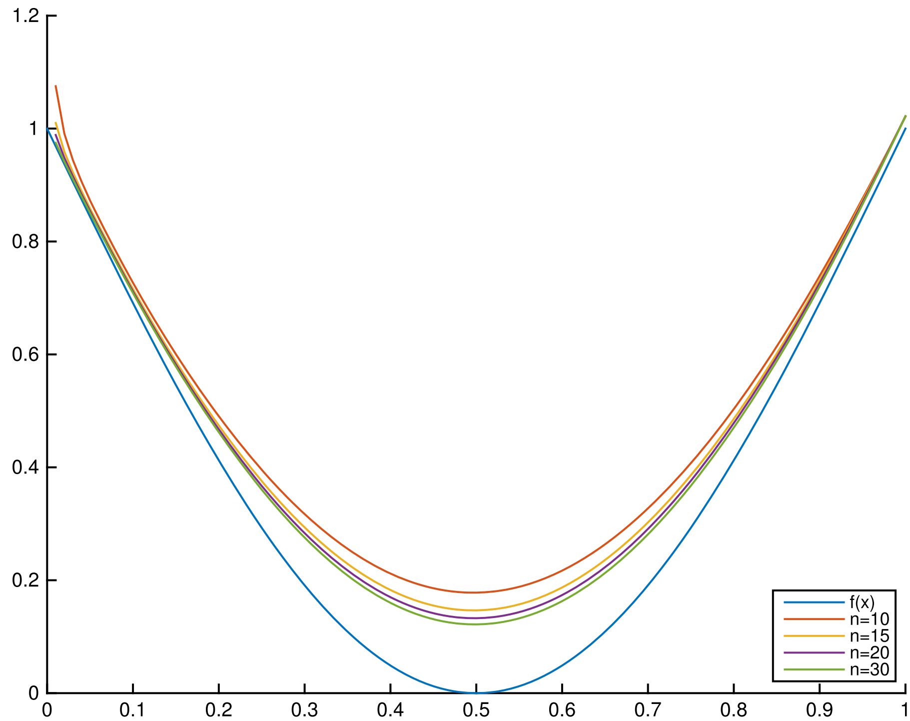

Figure 3 Monotonicity of $T_{n,q,\alpha}(f;x)$ in the parameter n

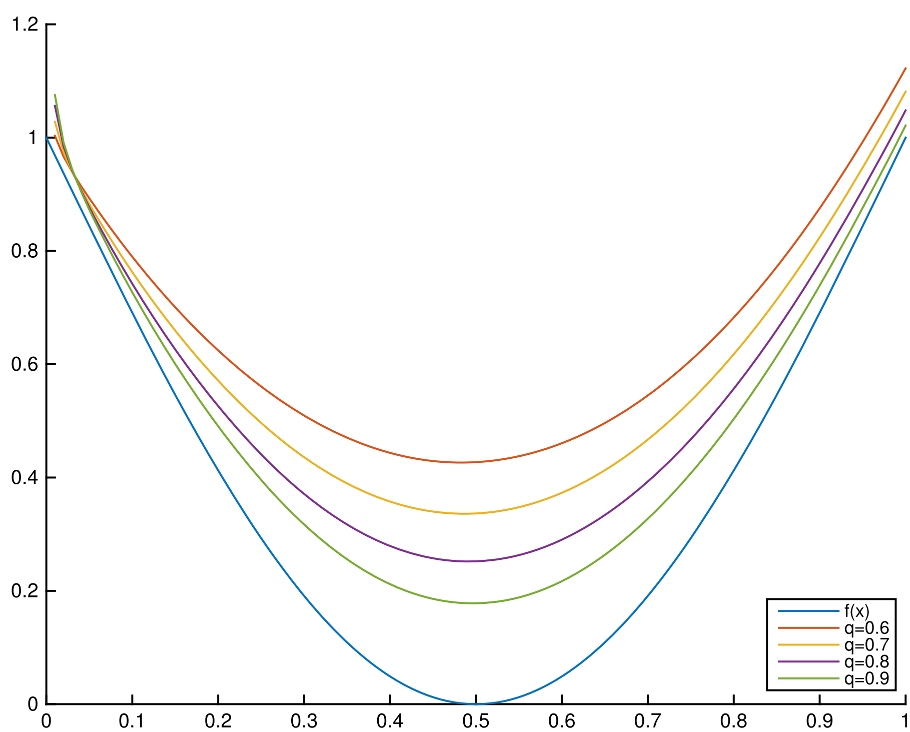

Figure 4 Monotonicity of $T_{n,q,\alpha}(f;x)$ in the parameter q

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Competing interests
The authors declare that there have no competing interests.

Authors' contributions
The authors carried out the whole manuscript. All authors read and approved the final manuscript.

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