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Functional version for Furuta parametric relative operator entropy

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Abstract

Functional version for the so-called Furuta parametric relative operator entropy is here investigated. Some related functional inequalities are also discussed. The theoretical results obtained by our functional approach immediately imply those of operator versions in a simple, fast, and nice way.

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1 Introduction

Let H be a complex Hilbert space. We denote by $\mathcal{B}(H)$ the $\mathbb{C}^*$-algebra of bounded linear operators acting on H and by $\mathcal{B}^{+*}(H)$ the open cone of all (self-adjoint) positive invertible operators in $\mathcal{B}(H)$. Let $A, B \in \mathcal{B}^{+*}(H)$ and $p \in [0, 1]$ be a real number. The expressions

$$A\nabla_p B = (1-p)A + pB,$$

$$A!_p B = \left((1-p)A^{-1} + pB^{-1}\right)^{-1},$$

$$A\sharp_p B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^p A^{1/2}$$

are known in the literature as the weighted arithmetic mean, weighted harmonic mean, and weighted geometric mean of A and B, respectively. If $p = 1/2$ they are simply denoted by $A\nabla B$, $A!B$, and $A\sharp B$, respectively. The previous operator means satisfy the following relationships:

$$A\nabla_p B = B\nabla_{1-p}A, \qquad A!_p B = B!_{1-p}A, \qquad A\sharp_p B = B\sharp_{1-p}A. \tag{1.1}$$

It is well known that the double inequality

$$A!_p B \leq A\sharp_p B \leq A\nabla_p B \tag{1.2}$$

holds for any $A, B \in \mathcal{B}^{+*}(H)$ and $p \in [0, 1]$. Here, the notation $T \leq S$ means that $T, S \in \mathcal{B}(H)$ are self-adjoint and $S - T$ is positive semi-definite.

Otherwise, the relative operator entropy $S(A|B)$ and the Tsallis relative operator entropy $T_p(A|B)$ are, respectively, defined by (see [2, 3, 6])

$$S(A|B) = A^{1/2} \log\big(A^{-1/2}BA^{-1/2}\big)A^{1/2}, \qquad T_p(A|B) = \frac{A\sharp_p B - A}{p}, \quad p \neq 0.$$

The following double inequality is known in the literature:

$$A - AB^{-1}A \leq S(A|B) \leq B - A. \tag{1.3}$$

In [5], Furuta introduced a parametric extension of $S(A|B)$ as follows:

$$S_p(A|B) = A^{1/2}\big(A^{-1/2}BA^{-1/2}\big)^p \log\big(A^{-1/2}BA^{-1/2}\big)A^{1/2}. \tag{1.4}$$

In fact, $S_p(A|B)$ was introduced in [5] for any real number p, but here we restrict ourselves to the case $p \in [0, 1]$.

As pointed out in [5], it is not hard to see that

$$S_0(A|B) = S(A|B), \qquad S_1(A|B) = -S(B|A) \quad \text{and} \quad S_p(A|B) = -S_{1-p}(B|A).$$

The fundamental goal of this paper is to give an extension of $S_p(A|B)$ when the operator variables A and B are (convex) functionals. Some functional relationships and inequalities are provided as well. The related operator versions are deduced in a fast and nice way.

2 Functional extensions

The previous operator concepts have been extended from the case that the variables are positive operators to the case that the variables are convex functionals, see [9].

Let $\tilde{\mathbb{R}}^H$ be the extended space of all functionals defined from H into $\mathbb{R} \cup \{+\infty\}$. Let $f, g \in \tilde{\mathbb{R}}^H$ be two given functionals (convex or not) and $p \in (0, 1)$. The expressions

$$\begin{aligned}
\mathcal{A}_p(f, g) &= (1-p)f + pg, \\
\mathcal{H}_p(f, g) &= \big((1-p)f^* + pg^*\big)^*, \\
\mathcal{G}_p(f, g) &= \frac{\sin(p\pi)}{\pi} \int_0^1 \frac{t^{p-1}}{(1-t)^p} \mathcal{H}_t(f, g)\, dt
\end{aligned} \tag{2.1}$$

are called, by analogy, the weighted functional arithmetic mean, the weighted harmonic mean, and the weighted geometric mean of f and g, respectively. Here, the notation f^* refers to the Fenchel conjugate of f defined by

$$\forall x^* \in H \quad f^*\big(x^*\big) = \sup_{x \in H}\big\{\Re e\langle x^*, x\rangle - f(x)\big\}. \tag{2.2}$$

For $p = 1/2$, we will denote the previous functional means by $\mathcal{A}(f, g)$, $\mathcal{H}(f, g)$ and $\mathcal{G}(f, g)$, respectively. We extend these means on the whole interval $[0, 1]$ by setting:

$$\mathcal{A}_0(f, g) = \mathcal{H}_0(f, g) = \mathcal{G}_0(f, g) = f, \qquad \mathcal{A}_1(f, g) = \mathcal{H}_1(f, g) = \mathcal{G}_1(f, g) = g. \tag{2.3}$$

We mention that here we adopt the conventions $0.(+\infty) = +\infty$ and $(+\infty) - (+\infty) = +\infty$, as usual in convex analysis [1, 8]. With this, relations (2.3) are not immediate from their related functional means (2.1) since the involved functionals f and/or g can take the value $+\infty$.

For the same reason, analogous relationships of (1.1) for the previous functional means are also valid, i.e.,

$$\mathcal{A}_p(f,g) = \mathcal{A}_{1-p}(g,f), \qquad \mathcal{H}_p(f,g) = \mathcal{H}_{1-p}(g,f), \qquad \mathcal{G}_p(f,g) = \mathcal{G}_{1-p}(g,f).$$

In fact, the first two relations are immediate from their definitions, and for the third one, there is a detailed proof in [12]. Also, the analog of (1.2), i.e.,

$$\mathcal{H}_p(f,g) \leq \mathcal{G}_p(f,g) \leq \mathcal{A}_p(f,g), \tag{2.4}$$

holds for any $f, g \in \tilde{\mathbb{R}}^H$ and $p \in [0, 1]$. Here the notation $f \leq g$ refers to the point-wise order between $f \in \tilde{\mathbb{R}}^H$ and $g \in \tilde{\mathbb{R}}^H$ defined by: $f \leq g$ if and only if $g(x) - f(x) \geq 0$ for all $x \in H$, with the convention $+\infty - (+\infty) = +\infty$ as already pointed before. The double inequality (2.4) implies that the three involved functional means are with finite values whenever f and g are so.

In the earlier papers [9] and [10] we extended $S(A|B)$ and $T_p(A|B)$ from operators to (convex) functionals, respectively, as follows:

$$\mathcal{S}(f|g) = \int_0^1 \frac{\mathcal{H}_t(f,g) - f}{t}\, dt,$$

$$\mathcal{T}_p(f|g) = \frac{\mathcal{G}_p(f,g) - f}{p}, \quad p \neq 0.$$

The previous functional concepts were constructed as extensions of their related operator versions in the following sense: if $\mathcal{O}(A, B)$ is one of the previous operator concepts, its functional extension $\mathcal{F}(f, g)$ is such that

$$\mathcal{F}(f_A, f_B) = f_{\mathcal{O}(A,B)}, \tag{2.5}$$

where the notation f_T, for any $T \in \mathcal{B}(H)$, refers to the quadratic function generated by the operator T, i.e., $f_T(x) = (1/2)\langle Tx, x \rangle$ for all $x \in H$.

3 Needed tools

Let $f \in \tilde{\mathbb{R}}^H$. We denote by $\mathrm{dom} f := \{x \in H : f(x) < +\infty\}$ the so-called effective domain of f. The notation $\mathrm{int}(\mathrm{dom} f)$ refers to the topological interior of $\mathrm{dom} f$ in H. The Fenchel conjugate f^* of f defined by (2.2) satisfies

$$f^*(x^*) := \sup_{x \in \mathrm{dom} f} \{\Re e\langle x^*, x \rangle - f(x)\}$$

for any $x^* \in H$. As supremum of a family of affine (so convex) functions, f^* is always convex even if f is not. The conjugate map $f \longmapsto f^*$ is point-wise decreasing and convex. That is,

$f \leq g$ implies $g^* \leq f^*$, and the inequality

$$\big((1-p)f + pg\big)^* \leq (1-p)f^* + pg^*$$

holds for any $f, g \in \tilde{\mathbb{R}}^H$ and $p \in [0, 1]$.

The sub-differential of f at $x \in \text{dom} f$ is the set $\partial f(x)$ defined by

$$\partial f(x) = \big\{ x^* \in H; \forall z \in H, \quad f(z) \geq f(x) + \Re e\langle x^*, z - x\rangle \big\}.$$

As it is well known, $\partial f(x)$ is a (possibly empty) convex and closed set. If $x \in \text{int}(\text{dom} f)$, then $\partial f(x) \neq \emptyset$. In the case where $\partial f(x) \neq \emptyset$, we have the equivalence:

$$x^* \in \partial f(x) \quad \Longleftrightarrow \quad f(x) + f^*\big(x^*\big) = \Re e\langle x^*, x\rangle.$$

As usual we denote by $\Gamma_0(H)$ the cone of all functionals $f \in \tilde{\mathbb{R}}^H$ that are convex, lower semi-continuous, and proper (i.e., not identically equal to $+\infty$). It is well known that $f^{**} := (f^*)^* \leq f$ for any $f \in \tilde{\mathbb{R}}^H$ and $f \in \Gamma_0(H)$ if and only if $f = f^{**} := (f^*)^*$. Moreover, $x^* \in \partial f(x)$ always implies $x \in \partial f^*(x^*)$, with reversed implication provided that $f \in \Gamma_0(H)$.

The function f is called Gâteaux-differentiable (in short G-differentiable) at x if the directional derivative

$$f'(x, d) = \lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t}$$

of f at x exists in every direction $d \in H$ and the map $d \longmapsto f'(x, d)$ is linear and continuous. In this case we write $f'(x, d) = \nabla f(x).d$ and $\nabla f(x)$ is called the G-derivative of f at x. It is well known that if f is convex and G-differentiable at x, then $\partial f(x) = \{\nabla f(x)\}$.

For the sake of clearness and simplicity for the reader, we state the following example illustrating the previous concepts.

Example 3.1 Let $A \in \mathcal{B}(H)$ and let f_A be the quadratic function associated to A, i.e., $f_A(x) = (1/2)\langle Ax, x\rangle$ for all $x \in H$.

(i) Assume that $A \in \mathcal{B}^{+*}(H)$. Then f_A is convex and G-differentiable on H, and so

$$\forall x \in H \quad \partial f_A(x) = \big\{\nabla f_A(x)\big\} = \{Ax\}.$$

The coefficient $1/2$ appearing in f_A enjoys a symmetry role in the aim to have

$$(f_A)^*\big(x^*\big) = (1/2)\langle A^{-1}x^*, x^*\rangle \quad \text{for all } x^* \in H, \text{ or in short } (f_A)^* = f_{A^{-1}}.$$

(ii) For any $A, B \in \mathcal{B}(H)$, it is easy to check that $f_A \pm f_B = f_{A \pm B}$ and $f_A(Bx) = f_{BAB}(x)$ for any $x \in H$.

The following result, which will be needed later, has been proved in [11].

Theorem 3.2 *Let* $f \in \Gamma_\circ(H)$ *be such that* $\text{int}(\text{dom} f)$ *is nonempty. Then*

(i) *The inequality*

$$\sup_{x^* \in \partial f(x)} \left(f^* - g^*\right)\left(x^*\right) \leq \mathcal{S}(f/g)(x) \leq (g - f)(x) \tag{3.1}$$

holds true for all $x \in \text{int}(\text{dom} f)$.

(ii) *If f is moreover G-differentiable at x, then we have*

$$f^*\left(\nabla f(x)\right) - g^*\left(\nabla f(x)\right) \leq \mathcal{S}(f/g)(x) \leq (g - f)(x). \tag{3.2}$$

As explained in [11], (3.1), as well as (3.2), is a functional extension of (1.3) from positive operators to convex functionals.

For the sake of simplicity for the reader, we need to introduce an auxiliary notation. For $f, g \in \tilde{\mathbb{R}}^H$ and $p \in [0, 1]$, we set

$$\mathcal{T}_p^*(f|g) = \frac{(\mathcal{G}_p(f, g))^* - f^*}{p}, \quad p \neq 0. \tag{3.3}$$

We have the following result summarizing the elementary properties of $\mathcal{T}_p^*(f|g)$.

Proposition 3.3 *The following assertions hold*:

(i) *For any* $p \in [0, 1)$, *one has*

$$\mathcal{T}_{1-p}^*(g|f) = \frac{(\mathcal{G}_p(f, g))^* - g^*}{1 - p}.$$

(ii) *For all* $p \in (0, 1]$, *the left-hand side of the inequality*

$$\frac{(\mathcal{A}_p(f, g))^*(x^*) - f^*(x^*)}{p} \leq \mathcal{T}_p^*(f|g)\left(x^*\right) \leq g^*\left(x^*\right) - f^*\left(x^*\right)$$

holds for any $x^* \in H$, *while the right-hand side holds for* x^* *such that* $g^*(x^*) = +\infty$ *or* $x^* \in \text{dom} f^*$.

Proof (i) Follows from (3.3) with the relation $\mathcal{G}_p(f, g) = \mathcal{G}_{1-p}(g, f)$.

(ii) From (2.4) we obtain by taking the conjugate side by side

$$\left(\mathcal{A}_p(f, g)\right)^* \leq \left(\mathcal{G}_p(f, g)\right)^* \leq \left(\mathcal{H}_p(f, g)\right)^*.$$

Remarking that

$$\left(\mathcal{H}_p(f, g)\right)^* \leq (1 - p)f^* + pg^*,$$

we then deduce the desired result. □

Proposition 3.4 *For any* $A, B \in \mathcal{B}^{+*}(H)$ *and* $p \in (0, 1]$, *there holds*

$$\mathcal{T}_p^*(f_A|f_B) = f_{T_p(A^{-1}|B^{-1})}.$$

Proof First, if for fixed $p \in [0, 1]$ we take $\mathcal{F} = \mathcal{G}_p$ and $\mathcal{O}(A, B) = A \sharp_p B$ in (2.5), then we have

$$\mathcal{G}_p(f_A, f_B) = f_{A \sharp_p B}.$$

Now, by (3.3) we have

$$\mathcal{T}_p^*(f_A | f_B) = \frac{(\mathcal{G}_p(f_A, f_B))^* - f_A^*}{p} = \frac{f_{A \sharp_p B}^* - f_A^*}{p} = \frac{f_{(A \sharp_p B)^{-1}} - f_{A^{-1}}}{p} = \frac{f_{A^{-1} \sharp_p B^{-1}} - f_{A^{-1}}}{p}.$$

This, with the fact that $\alpha f_T = f_{\alpha T}$ and $f_T - f_S = f_{T-S}$ for any $\alpha \in \mathbb{R}$ and $T, S \in \mathcal{B}(H)$, immediately yields the desired result. □

4 Functional version of $S_p(A|B)$

As already pointed out before, our aim here is to give an analog of $S_p(A|B)$ when the operator arguments A and B are (convex) functionals f and g, respectively. Such an analog seems to be hard to define from (1.4) since (1.4) involves the product of operators whose analogs for functionals are not known yet. For this, we need to state the following result.

Theorem 4.1 *The equalities*

$$S_p(A|B) = -\frac{S(A \sharp_p B | A)}{p} = \frac{S(A \sharp_p B | B)}{1 - p} \tag{4.1}$$

hold for any $A, B \in \mathcal{B}^{+*}(H)$ and $p \in (0, 1)$.

Proof Indeed, we have the property

$$T^* S(A|B) T = S(T^* A T | T^* B T)$$

for any $A, B \in \mathcal{B}^*(H)$ and any invertible operator $T \in \mathcal{B}(H)$ by using Kubo–Ando theory [7] and the integral form

$$S(A|B) = \int_0^1 \frac{A !_t B - A}{t} \, dt.$$

We thus have the first equality as

$$\begin{aligned} S(A \sharp_p B | A) &= A^{1/2} S(I \sharp_p A^{-1/2} B A^{-1/2} | I) A^{1/2} \\ &= A^{1/2} S((A^{-1/2} B A^{-1/2})^p | I) A^{1/2} \\ &= -A^{1/2} (A^{-1/2} B A^{-1/2})^p \log (A^{-1/2} B A^{-1/2})^p A^{1/2} \\ &= -p S_p(A|B), \end{aligned}$$

since $S(A|I) = -A \log A$ for any $A \in \mathcal{B}^{+*}(H)$.

The second equality can be proved in a similar manner. □

Now, to give a functional version of $S_p(A|B)$, we use (4.1) which is more appropriate for our aim since (4.1) involves only operator concepts (relative operator entropy and operator geometric mean) whose functional extensions are already done. Taking into account

a symmetric character between p and $1 - p$ in our desired definition, we then put the following.

Definition 4.2 Let $f, g \in \tilde{\mathbb{R}}^H$ and $p \in [0, 1]$. We set

$$\mathcal{S}_p(f|g) = \frac{\mathcal{S}((\mathcal{G}_p(f,g))|g)}{2(1-p)} - \frac{\mathcal{S}((\mathcal{G}_p(f,g))|f)}{2p}, \tag{4.2}$$

with

$$\mathcal{S}_0(f|g) = \mathcal{S}(f|g) \quad \text{and} \quad \mathcal{S}_1(f|g) = -\mathcal{S}(g|f).$$

As a first result we state the following.

Proposition 4.3 *Let $f, g \in \tilde{\mathbb{R}}^H$. Then we have*

$$\mathcal{S}_{1/2}(f|g) = \mathcal{S}\big((\mathcal{G}(f,g))|g\big) - \mathcal{S}\big((\mathcal{G}(f,g))|f\big). \tag{4.3}$$

Further, if $\operatorname{dom} f = \operatorname{dom} g = H$, then the equality

$$\mathcal{S}_p(f|g) = -\mathcal{S}_{1-p}(g|f) \tag{4.4}$$

holds for any $p \in (0, 1)$.

Proof Equality (4.3) is immediate from (4.2). However, we mention that (4.4) is not immediate from (4.2) since our involved functionals could take the value $+\infty$. Indeed, we pay attention to the fact that, if $\phi, \psi \in \tilde{\mathbb{R}}^H$, the equality $\phi - \psi = -(\psi - \phi)$ is not always true unless $\operatorname{dom} \phi \cup \operatorname{dom} \psi = H$. For this reason, we have assumed in our statement that $\operatorname{dom} f = \operatorname{dom} g = H$ in the aim to guarantee that $\mathcal{H}_t(\mathcal{G}_p(f,g),g)$ or $\mathcal{H}_t(\mathcal{G}_p(f,g),f)$ is with finite values. With this, (4.4) can be deduced from (4.2) when we refer to the relationship $\mathcal{G}_p(\phi, \psi) = \mathcal{G}_{1-p}(\psi, \phi)$ valid for any $\phi, \psi \in \tilde{\mathbb{R}}^H$ and $p \in [0, 1]$. □

A connection between the functional parametric entropy $\mathcal{S}_p(f|g)$ and the operator parametric entropy $S_p(A|B)$ is expressed by the following result.

Proposition 4.4 *Let $A, B \in \mathcal{B}^{+*}(H)$ and $p \in [0, 1]$. Then we have*

$$\mathcal{S}_p(f_A|f_B) = f_{S_p(A|B)}. \tag{4.5}$$

Proof By (4.2), with (2.5) and (4.1), we have

$$\mathcal{S}_p(f_A|f_B) = \frac{\mathcal{S}(f_{A\sharp_p B}|f_B)}{2(1-p)} - \frac{\mathcal{S}(f_{A\sharp_p B}|f_A)}{2p} = \frac{f_{S(A\sharp_p B|B)}}{2(1-p)} - \frac{f_{S(A\sharp_p B|A)}}{2p}.$$

This, with similar arguments as in the proof of Proposition 3.4, implies the desired result. □

Relationship (4.5) justifies that $\mathcal{S}_p(f|g)$ is a reasonable extension of $S_p(A|B)$, from operators to functionals, in the sense of (2.5).

For the sake of simplicity, we use in the next theorem and in its proof the following notations:

$$\mathcal{G}_p := \mathcal{G}_p(f,g), \qquad \mathcal{G}_p^* := \big(\mathcal{G}_p(f,g)\big)^*, \qquad \nabla\mathcal{G}_p := \nabla\big(\mathcal{G}_p(f,g)\big), \qquad \partial\mathcal{G}_p := \partial\big(\mathcal{G}_p(f,g)\big).$$

We now are in a position to state the following main result.

Theorem 4.5 *Let $f,g \in \tilde{\mathbb{R}}^H$ be such that* $\mathrm{int}(\mathrm{dom}\,\mathcal{G}_p(f,g)) \neq \emptyset$. *Then the following double inequality*

$$\frac{1}{2}\bigg(\sup_{x^* \in \partial\mathcal{G}_p(x)} \mathcal{T}_{1-p}^*(g|f)\big(x^*\big) + \mathcal{T}_p(f|g)(x)\bigg)$$

$$\leq \mathcal{S}_p(f|g)(x)$$

$$\leq \frac{1}{2}\bigg(-\mathcal{T}_{1-p}(g|f)(x) - \sup_{x^* \in \partial\mathcal{G}_p(x)} \mathcal{T}_p^*(f|g)\big(x^*\big)\bigg) \qquad (4.6)$$

holds for any $x \in \mathrm{int}(\mathrm{dom}\,\mathcal{G}_p(f,g))$ and $p \in (0,1)$.

Proof Since $\mathrm{int}(\mathrm{dom}\,\mathcal{G}_p(f,g)) \neq \emptyset$, then $\partial\mathcal{G}_p(x) \neq \emptyset$ for any $x \in \mathrm{int}(\mathrm{dom}\,\mathcal{G}_p(f,g))$.

Now, according to Theorem 3.2, we have, for $x \in \mathrm{int}(\mathrm{dom}\,\mathcal{G}_p(f,g))$,

$$\sup_{x^* \in \partial\mathcal{G}_p(x)} \big(\mathcal{G}_p^* - f^*\big)\big(x^*\big) \leq \mathcal{S}(\mathcal{G}_p|f)(x) \leq (f - \mathcal{G}_p)(x), \qquad (4.7)$$

and

$$\sup_{x^* \in \partial\mathcal{G}_p(x)} \big(\mathcal{G}_p^* - g^*\big)\big(x^*\big) \leq \mathcal{S}(\mathcal{G}_p|g)(x) \leq (g - \mathcal{G}_p)(x). \qquad (4.8)$$

Multiplying (4.7) by $-1/p$ and (4.8) by $1/(1-p)$ and then summing side by side, we obtain the desired inequalities after simple manipulations with the help of Proposition 3.3. The details are simple and therefore omitted. □

Remark 4.6 It is worth mentioning that the condition $\mathrm{int}(\mathrm{dom}\,\mathcal{G}_p(f,g)) \neq \emptyset$ is satisfied if $\mathrm{int}(\mathrm{dom}\,f \cap \mathrm{dom}\,g) \neq \emptyset$ since $\mathrm{dom}\,f \cap \mathrm{dom}\,g \subset \mathrm{dom}\,\mathcal{G}_p(f,g)$.

Corollary 4.7 *Let $f,g \in \tilde{\mathbb{R}}^H$ be such that $\mathcal{G}_p(f,g)$ is G-differentiable at $x \in H$. Then the inequalities* (*in the point-wise order sense*)

$$\frac{1}{2}\big(\mathcal{T}_{1-p}^*(g|f)\big(\nabla\mathcal{G}_p(f,g)\big) + \mathcal{T}_p(f|g)\big)$$

$$\leq \mathcal{S}_p(f|g)$$

$$\leq \frac{1}{2}\big(-\mathcal{T}_{1-p}(g|f) - \mathcal{T}_p^*(f|g)\big(\nabla\mathcal{G}_p(f,g)\big)\big)$$

hold for any $p \in (0,1)$.

Proof Since $\mathcal{G}_p(f,g)$ is G-differentiable at x, then $\partial\mathcal{G}_p(f,g)(x) = \{\nabla\mathcal{G}_p(f,g)(x)\}$. Substituting this in (4.6) and using the definition of the point-wise order, we immediately obtain the desired inequalities.   □

The operator version of the above theorem (and corollary) reads as follows.

Corollary 4.8 *Let $A, B \in \mathcal{B}^{+*}(H)$ and $p \in (0,1)$. Then we have*

$$\frac{1}{2}\big((A\sharp_p B)T_{1-p}\big(B^{-1}|A^{-1}\big)(A\sharp_p B) + T_p(A|B)\big)$$

$$\leq S_p(A|B)$$

$$\leq \frac{1}{2}\big(-T_{1-p}(B|A) - (A\sharp_p B)T_p\big(A^{-1}|B^{-1}\big)(A\sharp_p B)\big).$$

Proof Combining Corollary 4.7, Proposition 3.4, and Example 3.1,(ii), we obtain the desired operator inequalities after simple manipulations. The details are simple and therefore omitted.   □

Corollary 4.8 gives the relation between Furuta parametric relative operator entropy and Tsallis relative operator entropy in a more general setting than the result in [4, Theorem 2.3].

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