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Equivalent conditions and applications of a class of Hilbert-type integral inequalities involving multiple functions with quasi-homogeneous kernels

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Abstract

Let $K(x_1,\ldots,x_n)$ satisfy

$$K(x_1,\ldots,tx_i,\ldots,x_n)=t^{\lambda\lambda_i}K(t^{-\frac{\lambda_i}{\lambda_1}}x_1,\ldots,x_i,\ldots,t^{-\frac{\lambda_i}{\lambda_n}}x_n)$$

for $t>0$. With this integral kernel, by using the method and technique of weight coefficients, the equivalent conditions and the best constant factors for the validity of Hilbert-type integral inequalities involving multiple functions are discussed. Finally, the applications of the integral inequalities are considered.

MSC: 26D15; 47A07

Keywords: Hilbert-type integral inequality; Quasi-homogeneous kernel; Equivalent conditions; Best constant factor

1 Introduction

Let $x=(x_1,\ldots,x_n)$, $\mathbf{R}_+^n=\{x=(x_1,\ldots,x_n):x_i>0\ (i=1,\ldots,n)\}$, $r>1$, $f(t)\geq 0$, and α be a constant. Set

$$L_\alpha^r(0,+\infty)=\left\{f(t):\|f\|_{r,\alpha}=\left(\int_0^{+\infty}t^\alpha f^r(t)\,dt\right)^{1/r}<+\infty\right\}.$$

If $\sum_{i=1}^n\frac{1}{p_i}=1$ $(p_i>1, i=1,\ldots,n)$, $\alpha_i\in\mathbf{R}$, $f_i(x_i)\in L_{\alpha_i}^{p_i}(0,+\infty)$ $(i=1,\ldots,n)$, $K(x_1,\ldots,x_n)\geq 0$, M is a constant, then we name the following inequality a Hilbert-type integral inequality:

$$\int_{\mathbf{R}_+^n}K(x_1,\ldots,x_n)\prod_{i=1}^n f_i(x_i)\,dx_1\cdots dx_n\leq M\prod_{i=1}^n\|f_i\|_{p_i,\alpha_i}.$$

An integral kernel $K(x_1,\ldots,x_n)$ is said to be a quasi-homogeneous function with parameters $(\lambda,\lambda_1,\ldots,\lambda_n)$ if, for $t>0$,

$$K(x_1,\ldots,tx_i,\ldots,x_n)=t^{\lambda\lambda_i}K\bigl(t^{-\frac{\lambda_i}{\lambda_1}}x_1,\ldots,x_i,\ldots,t^{-\frac{\lambda_i}{\lambda_n}}x_n\bigr)\quad(i=1,\ldots,n).$$

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Obviously, $K(x_1,\dots,x_n)$ becomes a homogeneous function of order $\lambda\lambda_0$ when $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda_0$.

So far, many good results have been obtained in the study of Hilbert-type inequalities (cf. [1–24]). What are the necessary and sufficient conditions for the validity of a Hilbert-type inequality? What is the best constant factor when the inequality holds? The research on such problems is undoubtedly of great significance to the study and applications of Hilbert-type inequality theory, but unfortunately, the research on this type of problems is rarely seen.

In this paper, we focus on the quasi-homogeneous integral kernels, discuss the equivalent conditions for the validity of Hilbert-type integral inequalities involving multiple functions, and obtain the expressions of the best constant factors when the inequalities are established. Finally, we discuss their applications.

2 Some lemmas

Lemma 1 *Let integer* $n \geq 2, \sum_{i=1}^{n} \frac{1}{p_i} = 1$ $(p_i > 1, i = 1,\dots,n), \lambda \in \mathbf{R}, \alpha_i \in \mathbf{R}, \lambda_i > 0$ (*or* $\lambda_i < 0$) $(i = 1,\dots,n)$, *and* $K(x_1,\dots,x_n)$ *be a nonnegative measurable function with parameters* $(\lambda,\lambda_1,\dots,\lambda_n)$, $\sum_{i=1}^{n} \frac{\alpha_i+1}{\lambda_i p_i} = \lambda + \sum_{i=1}^{n} \frac{1}{\lambda_i}$. *Set*

$$W_j = \int_{\mathbf{R}_+^{n-1}} K(u_1,\dots,u_{j-1},1,u_{j+1},\dots,u_n) \times \prod_{i=1(i\neq j)}^{n} u_i^{-\frac{\alpha_i+1}{p_i}}\, du_1\cdots du_{j-1}\, du_{j+1}\cdots du_n.$$

Then $\frac{1}{\lambda_1} W_1 = \frac{1}{\lambda_2} W_2 = \cdots = \frac{1}{\lambda n} W_n$, *and*

$$\begin{aligned}\omega_j(x_j) &= \int_{\mathbf{R}_+^{n-1}} K(x_1,\dots,x_n) \prod_{i=1(i\neq j)}^{n} x_i^{-\frac{\alpha_i+1}{p_i}}\, dx_1\cdots dx_{j-1}\, dx_{j+1}\cdots dx_n \\ &= x_j^{\lambda_j(\lambda+\sum_{i=1(i\neq j)}^{n} \frac{\alpha_i+1}{\lambda_i p_i}+\sum_{i=1(i\neq j)}^{n} \frac{1}{\lambda_i})} W_j,\end{aligned}$$

where $j = 1,\dots,n$.

Proof When $j \geq 2$, by $\sum_{i=1}^{n} \frac{\alpha_i+1}{\lambda_i p_i} = \lambda + \sum_{i=1}^{n} \frac{1}{\lambda_i}$,

$$\begin{aligned}W_j &= \int_{\mathbf{R}_+^{n-1}} u_1^{\lambda\lambda_1} K\left(1, \frac{u_2}{u_1^{\frac{\lambda_1}{\lambda_2}}},\dots,\frac{u_{j-1}}{u_1^{\frac{\lambda_1}{\lambda_{j-1}}}}, \frac{1}{u_1^{\frac{\lambda_1}{\lambda_j}}}, \frac{u_{j+1}}{u_1^{\frac{\lambda_1}{\lambda_{j+1}}}},\dots,\frac{u_n}{u_1^{\frac{\lambda_1}{\lambda_n}}}\right) \\ &\quad\times \prod_{i=1(i\neq j)}^{n} u_i^{-\frac{\alpha_i+1}{p_i}}\, du_1\cdots du_{j-1}\, du_{j+1}\cdots du_n \\ &= \frac{\lambda_j}{\lambda_1} \int_{\mathbf{R}_+^{n-1}} K(1,t_2,\dots,t_n) t_j^{\lambda_j(-\lambda-\sum_{i=1}^{n} \frac{1}{\lambda_i}+\sum_{i=1(i\neq j)}^{n} \frac{\alpha_i+1}{\lambda_i p_i})} \\ &\quad\times \prod_{i=2(i\neq j)}^{n} t_i^{-\frac{\alpha_i+1}{p_i}}\, dt_2\cdots dt_n \\ &= \frac{\lambda_j}{\lambda_1} \int_{\mathbf{R}_+^{n-1}} K(1,t_2,\dots,t_n) t_j^{\lambda_j(-\frac{\alpha_j+1}{\lambda_j p_j})} \prod_{i=2(i\neq j)}^{n} t_i^{-\frac{\alpha_i+1}{p_i}}\, dt_2\cdots dt_n\end{aligned}$$

$$= \frac{\lambda_j}{\lambda_1} \int_{\mathbf{R}_+^{n-1}} K(1, t_2, \ldots, t_n) \prod_{i=2}^{n} t_i^{-\frac{\alpha_i+1}{p_i}} \, dt_2 \cdots dt_n = \frac{\lambda_j}{\lambda_1} W_1.$$

Therefore $\frac{1}{\lambda_j} W_j = \frac{1}{\lambda_1} W_1$ $(j \geq 2)$. When $j = 1, \ldots, n$, we also get

$$\omega_j(x_j) = \int_{\mathbf{R}_+^{n-1}} x_j^{\lambda\lambda_j} K\left(\frac{x_1}{x_j^{\frac{\lambda_j}{\lambda 1}}}, \ldots, \frac{x_{j-1}}{x_j^{\frac{\lambda_j}{\lambda_{j-1}}}}, 1, \frac{x_{j+1}}{x_j^{\frac{\lambda_j}{\lambda_{j+1}}}}, \ldots, \frac{x_n}{x_j^{\frac{\lambda_j}{\lambda n}}} \right)$$

$$\times \prod_{i=1(i\neq j)}^{n} x_i^{-\frac{\alpha_i+1}{p_i}} \, dx_1 \cdots dx_{j-1} \, dx_{j+1} \cdots dx_n$$

$$= x_j^{\lambda\lambda_j - \lambda_j \sum_{i=1(i\neq j)}^{n} \frac{\alpha_i+1}{\lambda_i p_i} + \lambda_j \sum_{i=1(i\neq j)}^{n} \frac{1}{\lambda_i}}$$

$$\times \int_{\mathbf{R}_+^{n-1}} K(u_1, \ldots, u_{j-1}, 1, u_{j+1}, \ldots, u_n)$$

$$\times \prod_{i=1(i\neq j)}^{n} u_i^{-\frac{\alpha_i+1}{p_i}} \, du_1 \cdots du_{j-1} \, du_{j+1} \cdots du_n$$

$$= x_j^{\lambda_j(\lambda + \sum_{i=1(i\neq j)}^{n} \frac{\alpha_i+1}{\lambda_i p_i} + \sum_{i=1(i\neq j)}^{n} \frac{1}{\lambda_i})} W_j. \qquad \square$$

Lemma 2 ([25]) *Let $p_i > 0, a_i > 0, \alpha_i > 0$ $(i = 1, \ldots, n)$, $\psi(u)$ be measurable. Then*

$$\int \cdots \int_{x_i > 0, (\frac{x_1}{a_1})^{\alpha_1} + \cdots + (\frac{x_n}{a_n})^{\alpha_n} \leq 1} \psi\left(\left(\frac{x_1}{a_1}\right)^{\alpha_1} + \cdots + \left(\frac{x_n}{a_n}\right)^{\alpha_n} \right)$$

$$\times x_1^{p_1-1} \cdots x_n^{p_n-1} \, dx_1 \cdots dx_n$$

$$= \frac{a_1^{p_1} \cdots a_n^{p_n} \Gamma(\frac{p_1}{\alpha_1}) \cdots \Gamma(\frac{p_n}{\alpha n})}{\alpha_1 \cdots \alpha_n \Gamma(\frac{p_1}{\alpha_1} + \cdots + \frac{p_n}{\alpha n})} \int_0^1 \psi(t) t^{\frac{p_1}{\alpha_1} + \cdots + \frac{p_n}{\alpha n} - 1} \, dt,$$

where Γ represents the gamma function.

3 Main results and their proofs

Theorem 1 *Suppose that $n \geq 2, \sum_{i=1}^{n} \frac{1}{p_i} = 1$ $(p_i > 1), \lambda \in \mathbf{R}, \lambda_i > 0$ (or $\lambda_i < 0$), $\alpha_i \in \mathbf{R}$ $(i = 1, \ldots, n)$. If $K(x_1, \ldots, x_n)$ is a quasi-homogeneous positive function with parameters $(\lambda, \lambda_1, \ldots, \lambda_n)$, and*

$$W_1 = \int_{\mathbf{R}_+^{n-1}} K(1, u_2, \ldots, u_n) \prod_{i=2}^{n} u_i^{-\frac{\alpha_i+1}{p_i}} \, du_2 \cdots du_n$$

is convergent, then

(i) *the inequality*

$$\int_{\mathbf{R}_+^{n}} K(x_1, \ldots, x_n) \prod_{i=1}^{n} f_i(x_i) \, dx_1 \cdots dx_n \leq M \prod_{i=1}^{n} \|f_i\|_{p_i,\alpha_i} \qquad (1)$$

holds for some constant $M > 0$ if and only if $\sum_{i=1}^{n} \frac{\alpha_i+1}{\lambda_i p_i} = \lambda + \sum_{i=1}^{n} \frac{1}{\lambda_i}$, where $f_i(x_i) \in L_{\alpha_i}^{p_i}(0, +\infty)$ $(i = 1, \ldots, n)$;

(ii) *if* (1) *holds, then its best constant factor is* $\inf M = \frac{W_1}{|\lambda_1|}\prod_{i=1}^n |\lambda_i|^{1/p_i}$.

Proof (i) Sufficiency. Assume that $\sum_{i=1}^n \frac{\alpha_i+1}{\lambda_i p_i} = \lambda + \sum_{i=1}^n \frac{1}{\lambda_i}$. Since

$$\prod_{j=1}^n x_j^{\frac{\alpha_j+1}{p_j}}\left(\prod_{i=1}^n x_i^{-\frac{\alpha_i+1}{p_i}}\right)^{1/p_j} = \prod_{j=1}^n x_j^{\frac{\alpha_j+1}{p_j}} \prod_{i=1}^n x_i^{-\frac{\alpha_i+1}{p_i}\sum_{k=1}^n \frac{1}{p_k}} = 1,$$

by Hölder's inequality and Lemma 1, we obtain

$$\begin{aligned}
&\int_{\mathbf{R}_+^n} K(x_1,\ldots,x_n)\prod_{i=1}^n f_i(x_i)\,dx_1\cdots dx_n \\
&= \int_{\mathbf{R}_+^n} K(x_1,\ldots,x_n)\left[\prod_{j=1}^n x_j^{\frac{\alpha_j+1}{p_j}}\left(\prod_{i=1}^n x_i^{-\frac{\alpha_i+1}{p_i}}\right)^{1/p_j} f_j(x_j)\right]dx_1\cdots dx_n \\
&\le \prod_{j=1}^n\left[\int_{\mathbf{R}_+^n} x_j^{\alpha_j+1}\left(\prod_{i=1}^n x_i^{-\frac{\alpha_i+1}{p_i}}\right) f_j^{p_j}(x_j)K(x_1,\ldots,x_n)\,dx_1\cdots dx_n\right]^{1/p_j} \\
&= \prod_{j=1}^n\left[\int_0^{+\infty} x_j^{\alpha_j+1-\frac{\alpha_j+1}{p_j}} f_j^{p_j}(x_j)\right. \\
&\quad \left.\times\left(\int_{\mathbf{R}_+^{n-1}} K(x_1,\ldots,x_n)\prod_{i=1(i\neq j)}^n x_i^{-\frac{\alpha_i+1}{p_i}}\,dx_1\cdots dx_{j-1}\,dx_{j+1}\cdots dx_n\right)dx_j\right]^{1/p_j} \\
&= \prod_{j=1}^n\left(\int_0^{+\infty} x_j^{\alpha_j+1-\frac{\alpha_j+1}{p_j}} f_j^{p_j}(x_j)\omega_j(x_j)\,dx_j\right)^{1/p_j} \\
&= \prod_{j=1}^n\left[\int_0^{+\infty} x_j^{\alpha_j+1-\frac{\alpha_j+1}{p_j}+\lambda_j(\lambda-\sum_{i=1(i\neq j)}^n \frac{\alpha_i+1}{\lambda_i p_i}+\sum_{i=1(i\neq j)}^n \frac{1}{\lambda_i})} f_j^{p_j}(x_j)W_j\,dx_j\right]^{1/p_j} \\
&= \prod_{j=1}^n W_j^{1/p_j}\prod_{j=1}^n\left[\int_0^{+\infty} x_j^{\lambda_j(\frac{\alpha_j}{\lambda_j}-\sum_{i=1}^n \frac{\alpha_i+1}{\lambda_i p_i}+\lambda+\sum_{i=1}^n \frac{1}{\lambda_i})} f_j^{p_j}(x_j)\,dx_j\right]^{1/p_j} \\
&= \prod_{i=1}^n W_i^{1/p_i}\prod_{i=1}^n\left(\int_0^{+\infty} x_i^{\alpha_i} f_i^{p_i}(x_i)\,dx_i\right)^{1/p_i} \\
&= \frac{W_1}{|\lambda_1|}\left(\prod_{i=1}^n |\lambda_i|^{1/p_i}\right)\prod_{i=1}^n \|f_i\|_{p_i,\alpha_i},
\end{aligned}$$

thus (1) holds when taking any constant $M \ge \frac{W_1}{|\lambda_1|}\prod_{i=1}^n |\lambda_i|^{1/p_i}$.

Necessity. Assume that (1) holds. Set $c = \sum_{i=1}^n \frac{\alpha_i+1}{\lambda_i p_i} - \lambda - \sum_{i=1}^n \frac{1}{\lambda_i}$. Next we will prove $c = 0$.

First consider the case of $\lambda_i > 0$ ($i = 1,\ldots,n$). If $c > 0$, for $0 < \varepsilon < c$, take

$$f_i(x_i) = \begin{cases} x_i^{(-\alpha_i-1+\lambda_i\varepsilon)/p_i}, & 0 < x_i \le 1, \\ 0, & x_i > 1, \end{cases}$$

where $i = 1, \dots, n$. Then

$$\prod_{i=1}^{n} \|f_i\|_{p_i,\alpha_i} = \prod_{i=1}^{n}\left(\int_0^1 x_i^{-1+\lambda_i\varepsilon}\,dx_i\right)^{1/p_i} = \frac{1}{\varepsilon}\prod_{i=1}^{n}\left(\frac{1}{\lambda_i}\right)^{1/p_i}, \tag{2}$$

$$\begin{aligned}
&\int_{\mathbf{R}_+^n} K(x_1,\dots,x_n)\prod_{i=1}^{n} f_i(x_i)\,dx_1\cdots dx_n \\
&\quad= \int_0^1 x_1^{(-\alpha_1-1+\lambda_1\varepsilon)/p_1} \\
&\qquad\times\left(\int_0^1\cdots\int_0^1 K(x_1,x_2,\dots,x_n)\prod_{i=2}^{n} x_i^{(-\alpha_i-1+\lambda_i\varepsilon)/p_i}\,dx_2\cdots dx_n\right)dx_1 \\
&\quad= \int_0^1 x_1^{(-\alpha_1-1+\lambda_1\varepsilon)/p_1+\lambda\lambda_1}\left(\int_0^1\cdots\int_0^1 K\big(1,x_1^{-\frac{\lambda_1}{\lambda_2}}x_2,\dots,x_1^{-\frac{\lambda_1}{\lambda_n}}x_n\big)\right. \\
&\qquad\left.\times\prod_{i=2}^{n} x_i^{(-\alpha_i-1+\lambda_i\varepsilon)/p_i}\,dx_2\cdots dx_n\right)dx_1 \\
&\quad= \int_0^1 x_1^{-\frac{\alpha_1+1-\lambda_1\varepsilon}{p_1}+\lambda\lambda_1+\lambda_1\sum_{i=2}^{n}\frac{1}{\lambda_i}-\lambda_1\sum_{i=2}^{n}\frac{\alpha_i+1-\lambda_i\varepsilon}{\lambda_i p_i}}\left(\int_0^{x_1^{-\lambda_1/\lambda_2}}\cdots\int_0^{x_n^{-\lambda_1/\lambda_n}}\right. \\
&\qquad\left.\times K(1,u_2,\dots,u_n)\prod_{i=2}^{n} u_i^{(-\alpha_i-1+\lambda_i\varepsilon)/p_i}\,du_2\cdots du_n\right)dx_1 \\
&\quad= \int_0^1 x_1^{\lambda_1(\lambda+\sum_{i=1}^{n}\frac{1}{\lambda_i}-\sum_{i=1}^{n}\frac{\alpha_i+1}{\lambda_i p_i}-\frac{1}{\lambda_1}+\varepsilon)}\left(\int_0^{x_1^{-\lambda_1/\lambda_2}}\cdots\int_0^{x_n^{-\lambda_1/\lambda_n}}\right. \\
&\qquad\left.\times K(1,u_2,\dots,u_n)\prod_{i=2}^{n} u_i^{(-\alpha_i-1+\lambda_i\varepsilon)/p_i}\,du_2\cdots du_n\right)dx_1 \\
&\quad\ge \int_0^1 x_1^{-1-\lambda_1 c+\lambda_1\varepsilon}\,dx_1\left(\int_0^1\cdots\int_0^1 K(1,u_2,\dots,u_n)\right. \\
&\qquad\left.\times\prod_{i=2}^{n} u_i^{(-\alpha_i-1+\lambda_i\varepsilon)/p_i}\,du_2\cdots du_n\right).
\end{aligned} \tag{3}$$

It follows from (1), (2), and (3) that

$$\begin{aligned}
&\int_0^1 x_1^{-1-\lambda_1 c+\lambda_1\varepsilon}\,dx_1\left(\int_0^1\cdots\int_0^1 K(1,u_2,\dots,u_n)\prod_{i=2}^{n} u_i^{(-\alpha_i-1+\lambda_i\varepsilon)/p_i}\,du_2\cdots du_n\right) \\
&\quad\le \frac{M}{\varepsilon}\prod_{i=1}^{n}\left(\frac{1}{\lambda_i}\right)^{1/p_i}.
\end{aligned} \tag{4}$$

Since $-1-\lambda_1 c+\lambda_1\varepsilon < -1$, $\int_0^1 x_1^{-1-\lambda_1 c+\lambda_1\varepsilon}\,dx_1$ diverges to $+\infty$. Whence it is a contradiction to (4). In other words, it is not valid for $c > 0$.

If $c<0$, for $0<\varepsilon<-c$, take

$$f_i(x_i)=\begin{cases} x_i^{(-\alpha_i-1-\lambda_i\varepsilon)/p_i}, & x_i\geq 1,\\ 0, & 0<x_i<1,\end{cases}$$

where $i=1,\ldots,n$. Similarly, we get

$$\begin{aligned}
&\int_1^{+\infty} x_1^{-1+\lambda_1 c-\lambda_1\varepsilon}\,dx_1\\
&\quad\times\left(\int_1^{+\infty}\cdots\int_1^{+\infty}K(1,u_2,\ldots,u_n)\prod_{i=2}^n u_i^{(-\alpha_i-1-\lambda_i\varepsilon)/p_i}\,du_2\cdots du_n\right)\\
&\quad\leq\frac{M}{\varepsilon}\prod_{i=1}^n\left(\frac{1}{\lambda_i}\right)^{1/p_i}.
\end{aligned}\tag{5}$$

Since $-1-\lambda_1 c+\lambda_1\varepsilon>-1$ and $\int_1^{+\infty}x_1^{-1-\lambda_1 c-\lambda_1\varepsilon}\,dx_1$ diverges to $+\infty$, which contradicts the above inequality, hence it does not hold for $c<0$.

To sum up, we have $c=0$ for $\lambda_i>0$ $(i=1,\ldots,n)$.

Now let us consider the case of $\lambda_i<0$ $(i=1,\ldots,n)$. If $c>0$, for $0<\varepsilon<c$, take

$$f_i(x_i)=\begin{cases} x_i^{(-\alpha_i-1+\lambda_i\varepsilon)/p_i}, & x_i\geq 1,\\ 0, & 0<x_i<1,\end{cases}$$

where $i=1,\ldots,n$. Consequently,

$$\prod_{i=1}^n\|f_i\|_{p_i,\alpha_i}=\prod_{i=1}^n\left(\int_1^{+\infty}x_i^{-1+\lambda_i\varepsilon}\,dx_i\right)^{1/p_i}=\frac{1}{\varepsilon}\prod_{i=1}^n\left(\frac{1}{-\lambda_i}\right)^{1/p_i},\tag{6}$$

$$\begin{aligned}
&\int_{\mathbf{R}_+^n}K(x_1,\ldots,x_n)\prod_{i=1}^n f_i(x_i)\,dx_1\cdots dx_n\\
&\quad=\int_1^{+\infty}x_1^{(-\alpha_1-1+\lambda_1\varepsilon)/p_1}\left(\int_1^{+\infty}\cdots\int_1^{+\infty}x_1^{\lambda\lambda_1}K\bigl(1,x_1^{-\frac{\lambda_1}{\lambda_2}}x_2,\ldots,x_1^{-\frac{\lambda_1}{\lambda_n}}x_n\bigr)\right.\\
&\quad\quad\left.\times\prod_{i=2}^n x_i^{(-\alpha_i-1+\lambda_i\varepsilon)/p_i}\,dx_2\cdots dx_n\right)dx_1\\
&\quad=\int_1^{+\infty}x_1^{-\frac{\alpha_1+1-\lambda_1\varepsilon}{p_1}+\lambda\lambda_1+\lambda_1\sum_{i=2}^n\frac{1}{\lambda_i}-\lambda_1\sum_{i=2}^n\frac{\alpha_i+1-\lambda_i\varepsilon}{\lambda_i p_i}}\left(\int_{x_1^{-\lambda_1/\lambda_2}}^{+\infty}\cdots\int_{x_n^{-\lambda_1/\lambda_n}}^{+\infty}\right.\\
&\quad\quad\left.\times K(1,u_2,\ldots,u_n)\prod_{i=2}^n u_i^{(-\alpha_i-1+\lambda_i\varepsilon)/p_i}\,du_2\cdots du_n\right)dx_1\\
&\quad\geq\int_1^{+\infty}x_1^{-1-\lambda_1 c+\lambda_1\varepsilon}\,dx_1\left(\int_1^{+\infty}\cdots\int_1^{+\infty}K(1,u_2,\ldots,u_n)\right.\\
&\quad\quad\left.\times\prod_{i=2}^n u_i^{(-\alpha_i-1+\lambda_i\varepsilon)/p_i}\,du_2\cdots du_n\right).
\end{aligned}\tag{7}$$

It follows from (1), (6), and (7) that

$$\begin{aligned}
&\int_1^{+\infty} x_1^{-1-\lambda_1 c+\lambda_1\varepsilon}\,dx_1 \\
&\quad\times\left(\int_1^{+\infty}\cdots\int_1^{+\infty} K(1,u_2,\ldots,u_n)\prod_{i=2}^{n} u_i^{(-\alpha_i-1+\lambda_i\varepsilon)/p_i}\,du_2\cdots du_n\right) \\
&\quad\le \frac{M}{\varepsilon}\prod_{i=1}^{n}\left(\frac{1}{-\lambda_i}\right)^{1/p_i}.
\end{aligned} \tag{8}$$

Since $-1-\lambda_1 c+\lambda_1\varepsilon > -1$, $\int_1^{+\infty} x_1^{-1-\lambda_1 c+\lambda_1\varepsilon}\,dx_1$ diverges to $+\infty$. Thus it is a contradiction to the above inequality. That is, it does not hold for $c > 0$.

If $c < 0$, for $0 < \varepsilon < -c$, take

$$f_i(x_i) = \begin{cases} x_i^{(-\alpha_i-1-\lambda_i\varepsilon)/p_i}, & 0 < x_i \le 1, \\ 0, & x_i > 1, \end{cases}$$

where $i = 1,\ldots,n$. Similarly, one can get

$$\begin{aligned}
&\int_0^{1} x_1^{-1-\lambda_1 c-\lambda_1\varepsilon}\,dx_1\left(\int_0^{1}\cdots\int_0^{1} K(1,u_2,\ldots,u_n)\right. \\
&\quad\left.\times\prod_{i=2}^{n} u_i^{(-\alpha_i-1-\lambda_i\varepsilon)/p_i}\,du_2\cdots du_n\right) \\
&\quad\le \frac{M}{\varepsilon}\prod_{i=1}^{n}\left(\frac{1}{-\lambda_i}\right)^{1/p_i}.
\end{aligned} \tag{9}$$

Since $-1-\lambda_1 c-\lambda_1\varepsilon < -1$, $\int_0^{1} x_1^{-1-\lambda_1 c-\lambda_1\varepsilon}\,dx_1$ diverges to $+\infty$, which also contradicts the above inequality. It does not hold for $c < 0$.

To sum up, we also get $c = 0$ for $\lambda_i < 0$ ($i = 1,\ldots,n$).

(ii) Suppose that (1) holds. If the constant factor $\inf M \ne \frac{W_1}{|\lambda_1|}\prod_{i=1}^{n}|\lambda_i|^{1/p_i}$, then there exists a constant $M_0 < \frac{W_1}{|\lambda_1|}\prod_{i=1}^{n}|\lambda_i|^{1/p_i}$ such that

$$\int_{\mathbf{R}_+^n} K(x_1,\ldots,x_n)\prod_{i=1}^{n} f_i(x_i)\,dx_1\cdots dx_n \le M_0\prod_{i=1}^{n}\|f_i\|_{p_i,\alpha_i}.$$

For sufficiently small $\varepsilon > 0$ and $\delta > 0$, take

$$f_1(x_1) = \begin{cases} x_1^{(-\alpha_1-1-|\lambda_1|\varepsilon)/p_1}, & x_1 \ge 1, \\ 0, & 0 < x_1 < 1. \end{cases}$$

For $i = 2,3,\ldots,n$, take

$$f_i(x_i) = \begin{cases} x_i^{(-\alpha_i-1-|\lambda_i|\varepsilon)/p_i}, & x_i \ge \delta, \\ 0, & 0 < x_i < \delta. \end{cases}$$

Therefore,

$$\begin{aligned}
&\prod_{i=1}^{n}\|f_i\|_{p_i,\alpha_i}\\
&\quad=\left(\int_1^{+\infty}x_1^{-1-|\lambda_1|\varepsilon}\,dx_1\right)^{1/p_1}\prod_{i=2}^{n}\left(\int_\delta^{+\infty}x_i^{1-|\lambda_i|\varepsilon}\,dx_i\right)^{1/p_i}\\
&\quad=\left(\frac{1}{|\lambda_1|\varepsilon}\right)^{1/p_1}\prod_{i=2}^{n}\left(\frac{1}{|\lambda_i|\varepsilon}\cdot\frac{1}{\delta^{|\lambda_i|\varepsilon}}\right)^{1/p_i}\\
&\quad=\frac{1}{\varepsilon}\prod_{i=1}^{n}\left(\frac{1}{|\lambda_i|}\right)^{1/p_i}\prod_{i=2}^{n}\left(\frac{1}{\delta^{|\lambda_i|\varepsilon}}\right)^{1/p_i},
\end{aligned}\tag{10}$$

$$\begin{aligned}
&\int_{\mathbf{R}_+^n}K(x_1,\ldots,x_n)\prod_{i=1}^{n}f_i(x_i)\,dx_1\cdots dx_n\\
&\quad=\int_0^1 x_1^{-\frac{\alpha_1+1+|\lambda_1|\varepsilon}{p_1}}\Bigg(\int_\delta^{+\infty}\cdots\int_\delta^{+\infty}K(x_1,\ldots,x_n)\\
&\qquad\times\prod_{i=2}^{n}x_i^{-\frac{\alpha_i+1+|\lambda_i|\varepsilon}{p_i}}\,dx_2\cdots dx_n\Bigg)\,dx_1\\
&\quad=\int_0^1 x_1^{\lambda\lambda_1-\frac{\alpha_1+1+|\lambda_1|\varepsilon}{p_1}}\Bigg(\int_\delta^{+\infty}\cdots\int_\delta^{+\infty}K\big(1,x_1^{-\lambda_1/\lambda_2}x_2,\ldots,x_1^{-\lambda_1/\lambda_n}x_n\big)\\
&\qquad\times\prod_{i=2}^{n}x_i^{-\frac{\alpha_i+1+|\lambda_i|\varepsilon}{p_i}}\,dx_2\cdots dx_n\Bigg)\,dx_1\\
&\quad=\int_1^{+\infty}x_1^{\lambda\lambda_1-\frac{\alpha_1+1+|\lambda_1|\varepsilon}{p_1}-\lambda_1\sum_{i=2}^{n}\frac{\alpha_i+1+|\lambda_i|\varepsilon}{\lambda_i p_i}+\sum_{i=2}^{n}\frac{\lambda_1}{\lambda_i}}\Bigg(\int_{\delta x_1^{-\lambda_1/\lambda_2}}^{+\infty}\cdots\int_{\delta x_n^{-\lambda_1/\lambda_n}}^{+\infty}\\
&\qquad\times K(1,u_2,\ldots,u_n)\prod_{i=2}^{n}u_i^{-\frac{\alpha_i+1+|\lambda_i|\varepsilon}{p_i}}\,du_2\cdots du_n\Bigg)\,dx_1\\
&\quad\ge\int_1^{+\infty}x_1^{\lambda_1(\lambda-\sum_{i=1}^{n}\frac{\alpha_i+1}{\lambda_i p_i}+\sum_{i=1}^{n}\frac{1}{\lambda_i}-\frac{1}{\lambda_1}-\sum_{i=1}^{n}\frac{|\lambda_i|\varepsilon}{\lambda_i p_i})}\,dx_1\Bigg(\int_\delta^{+\infty}\cdots\int_\delta^{+\infty}\\
&\qquad\times K(1,u_2,\ldots,u_n)\prod_{i=2}^{n}u_i^{-\frac{\alpha_i+1+|\lambda_i|\varepsilon}{p_i}}\,du_2\cdots du_n\Bigg)\\
&\quad=\int_1^{+\infty}x_1^{-1-|\lambda_1|\varepsilon}\,dx_1\left(\int_\delta^{+\infty}\cdots\int_\delta^{+\infty}K(1,u_2,\ldots,u_n)\times\prod_{i=2}^{n}u_i^{-\frac{\alpha_i+1+|\lambda_i|\varepsilon}{p_i}}\,du_2\cdots du_n\right)\\
&\quad=\frac{1}{|\lambda_1|\varepsilon}\int_\delta^{+\infty}\cdots\int_\delta^{+\infty}K(1,u_2,\ldots,u_n)\prod_{i=2}^{n}u_i^{-\frac{\alpha_i+1+|\lambda_i|\varepsilon}{p_i}}\,du_2\cdots du_n.
\end{aligned}\tag{11}$$

It follows from (1), (10), and (11) that

$$\frac{1}{|\lambda_1|}\int_\delta^{+\infty}\cdots\int_\delta^{+\infty}K(1,u_2,\ldots,u_n)\prod_{i=2}^{n}u_i^{-\frac{\alpha_i+1+|\lambda_i|\varepsilon}{p_i}}\,du_2\cdots du_n$$

$$\le M_0 \prod_{i=1}^{n} \left(\frac{1}{|\lambda_i|}\right)^{1/p_i} \prod_{i=2}^{n} \left(\frac{1}{\delta^{|\lambda_i|\varepsilon}}\right)^{1/p_i}.$$

Consequently,

$$\frac{1}{|\lambda_1|} \prod_{i=1}^{n} |\lambda_i|^{1/p_i} \int_{\delta}^{+\infty} \cdots \int_{\delta}^{+\infty} K(1, u_2, \ldots, u_n) \prod_{i=2}^{n} u_i^{-(\alpha_i+1)/p_i} \, du_2 \cdots du_n \le M_0$$

as $\varepsilon \to 0^+$. And then let $\delta \to 0^+$, we eventually get

$$\begin{aligned} &\frac{W_1}{|\lambda_1|} \prod_{i=1}^{n} |\lambda_i|^{1/p_i} \\ &= \frac{1}{|\lambda_1|} \prod_{i=1}^{n} |\lambda_i|^{1/p_i} \int_{\mathbf{R}_+^{n-1}} K(1, u_2, \ldots, u_n) \prod_{i=2}^{n} u_i^{-(\alpha_i+1)/p_i} \, du_2 \cdots du_n \\ &\le M_0, \end{aligned}$$

this is a contradiction. Hence $\inf M = \frac{W_1}{|\lambda_1|} \prod_{i=1}^{n} |\lambda_i|^{1/p_i}$, i.e., the constant factor $\frac{W_1}{|\lambda_1|} \prod_{i=1}^{n} |\lambda_i|^{1/p_i}$ is the best. □

4 Applications

Theorem 2 *Suppose that* $n \ge 2$, $\sum_{i=1}^{n} \frac{1}{p_i} = 1$ $(p_i > 1)$, $\lambda_i > 0$ (*or* $\lambda_i < 0$), $f_i(x_i) \in L_{p_i-1}^{p_i}(0, +\infty)$, $i = 1, \ldots, n$. *Then*

$$\int_{\mathbf{R}_+^n} \frac{\min\{x_1^{\lambda_1}, \ldots, x_n^{\lambda_n}\}}{\max\{x_1^{\lambda_1}, \ldots, x_n^{\lambda_n}\}} \prod_{i=1}^{n} f_i(x_i) \, dx_1 \cdots dx_n \le \left(n! \prod_{i=1}^{n} |\lambda_i|^{\frac{1}{p_i}-1} \right) \prod_{i=1}^{n} \|f_i\|_{p_i, p_i-1},$$

where the constant factor is the best.

Proof Set $\alpha_i = p_i - 1$, $\lambda = 0$, then $\sum_{i=1}^{n} \frac{\alpha_i+1}{\lambda_i p_i} = \lambda + \sum_{i=1}^{n} \frac{1}{\lambda_i}$. Take

$$K(x_1, \ldots, x_n) = \frac{\min\{x_1^{\lambda_1}, \ldots, x_n^{\lambda_n}\}}{\max\{x_1^{\lambda_1}, \ldots, x_n^{\lambda_n}\}},$$

then $K(x_1, \ldots, x_n)$ is a quasi-homogeneous positive function with parameters $(\lambda, \lambda_1, \ldots, \lambda_n)$, and

$$\begin{aligned} W_1 &= \int_{\mathbf{R}_+^{n-1}} K(1, u_2, \ldots, u_n) \prod_{i=2}^{n} u_i^{-(\alpha_i+1)/p_i} \, du_2 \cdots du_n \\ &= \int_{\mathbf{R}_+^{n-1}} \frac{\min\{1, u_2^{\lambda_2}, \ldots, u_n^{\lambda_n}\}}{\max\{1, u_2^{\lambda_2}, \ldots, u_n^{\lambda_n}\}} \prod_{i=2}^{n} u_i^{-1} \, du_2 \cdots du_n \\ &= \prod_{i=2}^{n} \frac{1}{|\lambda_i|} \int_{\mathbf{R}_+^{n-1}} \frac{\min\{1, t_2, \ldots, t_n\}}{\max\{1, t_2, \ldots, t_n\}} \prod_{i=2}^{n} t_i^{-1} \, dt_2 \cdots dt_n. \end{aligned}$$

In view of [1], we get

$$\int_{\mathbf{R}_+^{n-1}} \frac{\min\{1, t_2, \ldots, t_n\}}{\max\{1, t_2, \ldots, t_n\}} \prod_{i=2}^{n} t_i^{-1} \, dt_2 \cdots dt_n = n!,$$

it follows that

$$\frac{W_1}{|\lambda_1|}\prod_{i=1}^{n}|\lambda_i|^{\frac{1}{p_i}} = n!\frac{1}{|\lambda_1|}\prod_{i=2}^{n}|\lambda_i|^{-1}\prod_{i=1}^{n}|\lambda_i|^{\frac{1}{p_i}} = n!\prod_{i=1}^{n}|\lambda_i|^{\frac{1}{p_i}-1}.$$

According to Theorem 1, we know that Theorem 2 holds. □

Theorem 3 *Suppose that* $n \geq 2, \sum_{i=1}^{n}\frac{1}{p_i} = 1\ (p_i > 1), a > 0, \lambda_i > 0, \alpha_i \in \mathbf{R}, p_i > 1 + \alpha_i$. *Then*

(i) *the inequality*

$$\int_{\mathbf{R}_+^n}\frac{1}{(x_1^{\lambda_1}+x_2^{\lambda_2}+\cdots+x_n^{\lambda_n})^a}\prod_{i=1}^{n}x_i^{-\frac{\alpha_i+1}{p_i}}\,dx_1\cdots dx_n \leq M\prod_{i=1}^{n}\|f_i\|_{p_i,\alpha_i} \tag{12}$$

holds for some constant $M > 0$ *if and only if* $\sum_{i=1}^{n}\frac{\alpha_i+1}{\lambda_i p_i} = -a + \sum_{i=1}^{n}\frac{1}{\lambda_i}$.

(ii) *if* (12) *holds, then its best constant factor is*

$$\inf M = \frac{1}{\Gamma(a)}\prod_{i=1}^{n}\lambda_i^{\frac{1}{p_i}-1}\prod_{i=1}^{n}\Gamma\left(\frac{1}{\lambda_i}\left(1-\frac{\alpha_i+1}{p_i}\right)\right).$$

Proof Set $K(x_1,\ldots,x_n) = 1/(x_1^{\lambda_1}+x_2^{\lambda_2}+\cdots+x_n^{\lambda_n})^a$, then $K(x_1,\ldots,x_n)$ is a quasi-homogeneous positive funct+ion with parameters $(-a,\lambda_1,\ldots,\lambda_n)$. By Lemma 2,

$$\begin{aligned}
W_1 &= \int_{\mathbf{R}_+^{n-1}}\frac{1}{(1+x_2^{\lambda_2}+\cdots+x_n^{\lambda_n})^a}\prod_{i=2}^{n}x_i^{-\frac{\alpha_i+1}{p_i}}\,dx_2\cdots dx_n \\
&= \lim_{r\to+\infty}\int_{x_i>0,x_2^{\lambda_2}+\cdots+x_n^{\lambda_n}\leq r}\frac{1}{\{1+r[(\frac{x_1}{r^{1/\lambda_1}})^{\lambda_1}+\cdots+(\frac{x_n}{r^{1/\lambda_n}})^{\lambda_n}]\}^n} \\
&\quad\times\prod_{i=2}^{n}x_i^{(1-\frac{\alpha_i+1}{p_i})-1}\,dx_2\cdots dx_n \\
&= \lim_{r\to+\infty}\frac{r^{\sum_{i=2}^{n}\frac{1}{\lambda_i}-\sum_{i=2}^{n}\frac{1}{\lambda_i}(1-\frac{\alpha_i+1}{p_i})}\prod_{i=2}^{n}\Gamma(\frac{1}{\lambda_i}(1-\frac{\alpha_i+1}{p_i}))}{\prod_{i=2}^{n}\lambda_i\Gamma(\sum_{i=2}^{n}\frac{1}{\lambda_i}(1-\frac{\alpha_i+1}{p_i}))} \\
&\quad\times\int_0^1\frac{1}{(1+rt)^a}t^{\sum_{i=2}^{n}\frac{1}{\lambda_i}(1-\frac{\alpha_i+1}{p_i})-1}\,dt \\
&= \lim_{r\to+\infty}\frac{\prod_{i=2}^{n}\Gamma(\frac{1}{\lambda_i}(1-\frac{\alpha_i+1}{p_i}))}{\prod_{i=2}^{n}\lambda_i\Gamma(\sum_{i=2}^{n}\frac{1}{\lambda_i}(1-\frac{\alpha_i+1}{p_i}))}\int_0^r\frac{1}{(1+u)^a}u^{\sum_{i=2}^{n}\frac{1}{\lambda_i}(1-\frac{\alpha_i+1}{p_i})-1}\,du \\
&= \frac{\prod_{i=2}^{n}\Gamma(\frac{1}{\lambda_i}(1-\frac{\alpha_i+1}{p_i}))}{\prod_{i=2}^{n}\lambda_i\Gamma(\sum_{i=2}^{n}\frac{1}{\lambda_i}(1-\frac{\alpha_i+1}{p_i}))}\int_0^{+\infty}\frac{1}{(1+u)^a}u^{\sum_{i=2}^{n}\frac{1}{\lambda_i}(1-\frac{\alpha_i+1}{p_i})-1}\,du \\
&= \frac{\prod_{i=2}^{n}\Gamma(\frac{1}{\lambda_i}(1-\frac{\alpha_i+1}{p_i}))}{\prod_{i=2}^{n}\lambda_i\Gamma(\sum_{i=2}^{n}\frac{1}{\lambda_i}(1-\frac{\alpha_i+1}{p_i}))} \\
&\quad\times\frac{\Gamma(\sum_{i=2}^{n}\frac{1}{\lambda_i}(1-\frac{\alpha_i+1}{p_i}))\Gamma(a-\sum_{i=2}^{n}\frac{1}{\lambda_i}(1-\frac{\alpha_i+1}{p_i}))}{\Gamma(a)} \\
&= \frac{1}{\Gamma(a)}\prod_{i=2}^{n}\frac{1}{\lambda_i}\prod_{i=2}^{n}\Gamma\left(\frac{1}{\lambda_i}\left(1-\frac{\alpha_i+1}{p_i}\right)\right)\Gamma\left(\frac{1}{\lambda_1}\left(1-\frac{\alpha_1+1}{p_1}\right)\right)
\end{aligned}$$

$$= \frac{1}{\Gamma(a)} \prod_{i=2}^{n} \frac{1}{\lambda_i} \prod_{i=1}^{n} \Gamma\left(\frac{1}{\lambda_i}\left(1 - \frac{\alpha_i + 1}{p_i}\right)\right).$$

Based on this, we can obtain

$$\frac{W_1}{\lambda_1} \prod_{i=1}^{n} \lambda_i^{\frac{1}{p_i}} = \frac{1}{\Gamma(a)} \prod_{i=1}^{n} \lambda_i^{\frac{1}{p_i}-1} \prod_{i=1}^{n} \Gamma\left(\frac{1}{\lambda_i}\left(1 - \frac{\alpha_i + 1}{p_i}\right)\right).$$

According to Theorem 1, we know that Theorem 3 holds. □

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The authors declare that they have no competing interests.

Authors' contributions
JC carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. BH, YH, and BY participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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