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The growth and approximation for an analytic function represented by Laplace–Stieltjes transforms with generalized order converging in the half plane

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Abstract

By utilizing the concept of generalized order, we investigate the growth of Laplace–Stieltjes transform converging in the half plane and obtain one equivalence theorem concerning the generalized order of Laplace–Stieltjes transforms. Besides, we also study the problem on the approximation of this Laplace–Stieltjes transform and give some results about the generalized order, the error, and the coefficients of Laplace–Stieltjes transforms. Our results are extension and improvement of the previous theorems given by Luo and Kong, Singhal, and Srivastava.

MSC: 44A10; 30D15

Keywords: Approximation; Laplace–Stieltjes transform; Half plane; Error

1 Introduction

Laplace–Stieltjes transforms

$$F(s)=\int_0^{+\infty} e^{sx}\,d\alpha(x),\qquad s=\sigma+it,\tag{1}$$

where $\alpha(x)$ is a bounded variation on any finite interval $[0,Y]$ $(0<Y<+\infty)$, and σ and t are real variables, as we know, if $\alpha(t)$ is absolutely continuous, then $F(s)$ becomes the classical Laplace integral of the form

$$F(s)=\int_0^{\infty} e^{st}g(t)\,dt.\tag{2}$$

If $\alpha(t)$ is a step-function and satisfies

$$\alpha(x)=\begin{cases} a_1+a_2+\cdots+a_n, & \lambda_n<x<\lambda_{n+1};\\ 0, & 0\le x<\lambda_1;\\ \frac{\alpha(x+)+\alpha(x-)}{2}, & x>0,\end{cases}$$

where the sequence $\{\lambda_n\}_0^\infty$ satisfies

$$0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \uparrow +\infty, \tag{3}$$

thus $F(s)$ becomes a Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \quad s = \sigma + it. \tag{4}$$

(σ, t are real variables), a_n are nonzero complex numbers. Obviously, if $\alpha(t)$ is an increasing continuous function which is not absolutely continuous, then the integral (1) defines a class of functions $F(s)$ which cannot be expressed either in the form (2) or (4).

Let a sequence $\{\lambda_n\}_{n=1}^\infty$ satisfy (3), and

$$\limsup_{n\to+\infty}(\lambda_{n+1} - \lambda_n) = h < +\infty, \qquad \limsup_{n\to+\infty}\frac{n}{\lambda_n} = D < \infty. \tag{5}$$

Set

$$A_n^* = \sup_{\lambda_n < x \le \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} e^{ity}\, d\alpha(y) \right|,$$

if

$$\limsup_{n\to+\infty}\frac{\log A_n^*}{\lambda_n} = 0, \tag{6}$$

it is easy to get $\sigma_u^F = 0$, that is, $F(s)$ is analytic in the left half plane; if

$$\limsup_{n\to+\infty}\frac{\log A_n^*}{\lambda_n} = -\infty, \tag{7}$$

it follows that $\sigma_u^F = +\infty$, that is, $F(s)$ is analytic in the whole plane. For convenience, we use $\overline{L}_\beta$ to be a class of all the functions $F(s)$ of the form (1) which are analytic in the half plane $\Re s < \beta$ ($-\infty < \beta < \infty$) and the sequence $\{\lambda_n\}$ satisfies (3) and (5); L_0 to be the class of all the functions $F(s)$ of the form (1) which are analytic in the half plane $\Re s < 0$ and the sequence $\{\lambda_n\}$ satisfies (3), (5), and (6); and L_∞ to be the class of all the functions $F(s)$ of the form (1) which are analytic in the whole plane $\Re s < +\infty$ and the sequence $\{\lambda_n\}$ satisfies (3), (5), and (7). Thus, if $-\infty < \beta < 0$ and $F(s) \in \overline{L}_\beta$, then $F(s) \in L_0$.

In 1963, Yu [26] first proved the Valiron–Knopp–Bohr formula of the associated abscissas of bounded convergence, absolute convergence, and uniform convergence of Laplace–Stieltjes transform. Moreover, Yu [26] also estimated the growth of the maximal molecule $M_u(\sigma, F)$, the maximal term $\mu(\sigma, F)$, by introducing the concepts of the order of $F(s)$, and investigated the singular direction–Borel line of entire functions represented by Laplace–Stieltjes transforms converging in the whole complex plane. After his wonderful works, considerable attention has been paid to the value distribution and the growth of analytic functions represented by Laplace–Stieltjes transforms converging in the whole plane or the half plane (see [1, 3, 4, 6–8, 11–15, 18–25, 27]).

Set

$$\mu(\sigma,F)=\max_{n\in\mathbb{N}}\left\{A_n^* e^{\lambda_n\sigma}\right\}\quad(\sigma<0),$$

$$M_u(\sigma,F)=\sup_{0<x<+\infty,-\infty<t<+\infty}\left|\int_0^x e^{(\sigma+it)y}\,d\alpha(y)\right|\quad(\sigma<0).$$

For $F(s)\in L_0$, in view of $M_u(\sigma,F)\to+\infty$ as $\sigma\to 0^-$, the concepts of order and type can be usually used in estimating the growth of $F(s)$ precisely.

Definition 1.1 If Laplace–Stieltjes transform (1) satisfies $\sigma_u^F=0$ and

$$\limsup_{\sigma\to 0^-}\frac{\log^+\log^+ M_u(\sigma,F)}{-\log(-\sigma)}=\rho,\quad 0\le\rho\le+\infty,$$

we call $F(s)$ of order ρ in the left half plane, where $\log^+ x=\max\{\log x,0\}$. Furthermore, if $\rho\in(0,+\infty)$, the type of $F(s)$ is defined by

$$\limsup_{\sigma\to 0^-}\frac{\log^+ M_u(\sigma,F)}{(-\frac{1}{\sigma})^\rho}=T,\quad 0\le T\le+\infty.$$

Remark 1.1 However, if $\rho=0$ and $\rho=+\infty$, we cannot estimate the growth of such functions precisely by using the concept of type.

In 2012 and 2014, Luo and Kong [9, 10] investigated the growth of Laplace–Stieltjes transform converging on the whole plane and obtained the following.

Theorem 1.1 (see [10]) *If the L-S transform $F(s)\in L_\infty$ and is of order ρ $(0<\rho<\infty)$, then*

$$\rho=\limsup_{n\to+\infty}\frac{\lambda_n\log\lambda_n}{-\log A_n^*}.$$

Theorem 1.2 (see [9]) *If the L-S transform $F(s)\in L_\infty$, then for $p=1$, we have*

$$\limsup_{\sigma\to+\infty}\frac{h(\log M_u(\sigma,F))}{h(\sigma)}-1=\limsup_{n\to+\infty}\frac{h(\lambda_n)}{h(-\frac{1}{\lambda_n}\log A_n^*)},$$

and for $p=2,3,\dots$, we have

$$\limsup_{n\to+\infty}\frac{h(\lambda_n)}{h(-\frac{1}{\lambda_n}\log A_n^*)}\le\limsup_{\sigma\to+\infty}\frac{h(\log M_u(\sigma,F))}{h(\sigma)}$$
$$\le\limsup_{n\to+\infty}\frac{h(\lambda_n)}{h(-\frac{1}{\lambda_n}\log A_n^*)}+1,$$

where $h(x)$ satisfies the following conditions:

(i) *$h(x)$ is defined on $[a,+\infty)$ and is positive, strictly increasing, differentiable and tends to $+\infty$ as $x\to+\infty$;*

(ii) $\lim_{x\to+\infty}\frac{d(h(x))}{d(\log^{[p]}x)}=k\in(0,+\infty)$, $p\ge 1$, $p\in\mathbb{N}^+$, *where* $\log^{[0]}x=x$, $\log^{[1]}x=\log x$, *and* $\log^{[p]}x=\log(\log^{[p-1]}x)$.

In this paper, the first aim is to investigate the growth of analytic functions represented by Laplace–Stieltjes transforms with generalized order converging in the half plane, and we obtain some theorems about the generalized order A_n^* and λ_n, which are improvements of the previous results given by Luo and Kong [9, 10]. To state our results, we first introduce the following notations and definitions.

Let Γ be a class of continuous increasing functions $\mathcal{A}$ such that $\mathcal{A}(x) \geq 0$ for $x \geq x_0$, $\mathcal{A}(x) = \mathcal{A}(x_0)$ for $x \leq x_0$ and on $[x_0, +\infty)$ the function $\mathcal{A}$ increases to $+\infty$; and Γ^0 be a class such that $\Gamma^0 \subset \Gamma$ and $\mathcal{A}(x(1 + o(1)) = (1 + o(1))\mathcal{A}(x)$ as $x \to +\infty$, for $\mathcal{A} \in \Gamma^0$; further, $\mathcal{A} \in \Gamma^{0i}$ if $\mathcal{A} \in \Gamma$ and for any $\eta > 0$, $\mathcal{A}(\eta x) = (1 + o(1))\mathcal{A}(x)$ as $x \to +\infty$. Obviously, it follows $\Gamma^{0i} \subset \Gamma^0$ and $h(x) \in \Gamma$.

Definition 1.2 Let $F(s) \in L_0$ and $\mathcal{A} \in \Gamma$, $\mathcal{B} \in \Gamma$. If

$$\rho_{\mathcal{A}\mathcal{B}}(F) = \limsup_{\sigma \to 0^-} \frac{\mathcal{A}(\log M_u(\sigma, F))}{\mathcal{B}(-\frac{1}{\sigma})},$$

then $\rho_{\mathcal{A}\mathcal{B}}$ is called generalized order of $F(s)$.

Remark 1.2 Let $\mathcal{A}(x) = \log x$ and $\mathcal{B} = \log x$, then $\rho_{\mathcal{A}\mathcal{B}}(F) = \rho$.

Remark 1.3 Let $\mathcal{A}(x) = \log_p x$ and $\mathcal{B} = \log_q x$, then $\rho_{\mathcal{A}\mathcal{B}}(F) = \rho(p, q)(F)$, where $\rho(p, q)(F)$ is the (p, q)-order of $F(s)$ (see [2]).

Remark 1.4 Let $\mathcal{A}(x) = \log x$ and $\mathcal{B} = \log\log x$, then $\rho_{\mathcal{A}\mathcal{B}}(F) = \rho_l(F)$, where $\rho_l(F)$ is the logarithmic order of $F(s)$.

2 Results and discussion

For generalized order of Laplace–Stieltjes transform (1), we obtain the following.

Theorem 2.1 *Let $F(s) \in L_0$, $\mathcal{A} \in \Gamma^{0i}$ and $\mathcal{B} \in \Gamma^{0i}$ be continuously differentiable, and the function $\mathcal{B}$ increase more rapidly than $\mathcal{A}$ such that, for any constant $\eta \in (0, +\infty)$,*

$$\frac{x}{\mathcal{B}^{-1}(\eta\mathcal{A}(x))} \to +\infty \quad (x_0 \leq x \to +\infty), \tag{8}$$

and

$$\mathcal{A}\left(\frac{x}{\mathcal{B}^{-1}(\eta\mathcal{A}(x))}\right) = \bigl(1 + o(1)\bigr)\mathcal{A}(x) \quad (x \to +\infty). \tag{9}$$

If

$$\lim_{\sigma \to 0^-} \frac{\log M_u(\sigma, F)}{-\log(-\sigma)} = +\infty \tag{10}$$

and

$$\rho_{\mathcal{A}\mathcal{B}}(F) = \limsup_{\sigma \to 0^-} \frac{\mathcal{A}(\log M_u(\sigma, F))}{\mathcal{B}(-\frac{1}{\sigma})}, \quad 0 \leq \rho_{\mathcal{A}\mathcal{B}}(F) \leq +\infty,$$

then

$$\rho_{\mathcal{A}\mathcal{B}}(F) = \limsup_{n\to+\infty} \frac{\mathcal{A}(\lambda_n)}{\mathcal{B}(\frac{\lambda_n}{\log A_n^*})}.$$

Theorem 2.2 *Let $F(s) \in L_0$, $\mathcal{A} \in \Gamma^{0i}$, and $\mathcal{B} \in \Gamma^{0i}$ be continuously differentiable, and the function $\mathcal{A}$ increase more rapidly than $\mathcal{B}$ such that, for any constant $\eta \in (0, +\infty)$,*

$$\frac{x}{\mathcal{A}^{-1}(\eta\mathcal{B}(x))} \uparrow +\infty \quad (x_0 \le x \to +\infty), \tag{11}$$

and

$$\mathcal{B}\left(\frac{x}{\mathcal{A}^{-1}(\eta\mathcal{B}(x))}\right) = \big(1 + o(1)\big)\mathcal{B}(x) \quad (x \to +\infty). \tag{12}$$

If $F(s)$ satisfies (10) and

$$\rho_{\mathcal{A}\mathcal{B}}(F) = \limsup_{\sigma\to 0^-} \frac{\mathcal{A}(\log M_u(\sigma, F))}{\mathcal{B}(-\frac{1}{\sigma})}, \quad 0 \le \rho_{\mathcal{A}\mathcal{B}}(F) \le +\infty,$$

then

$$\rho_{\mathcal{A}\mathcal{B}}(F) = \limsup_{n\to+\infty} \frac{\mathcal{A}(\log A_n^*)}{\mathcal{B}(\lambda_n)}.$$

If Laplace–Stieltjes transform (1) satisfies $A_n^* = 0$ for $n \ge k + 1$ and $A_k^* \ne 0$, then $F(s)$ will be said to be an exponential polynomial of degree k usually denoted by p_k, *i.e.*, $p_k(s) = \int_0^{\lambda_k} \exp(sy)\,d\alpha(y)$. If we choose a suitable function $\alpha(y)$, the function $p_k(s)$ may be reduced to a polynomial in terms of $\exp(s\lambda_i)$, that is, $\sum_{i=1}^k b_i \exp(s\lambda_i)$. We denote Π_k to be the class of all exponential polynomials of degree almost k, that is,

$$\Pi_k = \left\{\sum_{i=1}^k b_i \exp(s\lambda_i) : (b_1, b_2, \ldots, b_k) \in \mathbb{C}^k\right\}.$$

For $F(s) \in \overline{L}_\beta$, $-\infty < \beta < 0$, we denote by $E_n(F, \beta)$ the error in approximating the function $F(s)$ by exponential polynomials of degree n in uniform norm as

$$E_n(F, \beta) = \inf_{p\in\Pi_n} \|F - p\|_\beta, \quad n = 1, 2, \ldots,$$

where

$$\|F - p\|_\beta = \max_{-\infty<t<+\infty} \big|F(\beta + it) - p(\beta + it)\big|.$$

In 2017, Singhal and Srivastava [17] studied the approximation of Laplace–Stieltjes transforms of finite order converging on the whole plane and obtained the following theorem.

Theorem 2.3 (see [17]) *If Laplace–Stieltjes transform $F(s) \in L_\infty$ and is of order ρ $(0 < \rho < \infty)$ and of type T, then for any real number $-\infty < \beta < +\infty$, we have*

$$\rho = \limsup_{n\to+\infty} \frac{\lambda_n \log \lambda_n}{-\log E_{n-1}(F,\beta)\exp(-\beta\lambda_n)} = \limsup_{n\to+\infty} \frac{\lambda_n \log \lambda_n}{-\log E_{n-1}(F,\beta)}$$

and

$$T = \limsup_{n\to+\infty} \frac{\lambda_n}{\rho e}\bigl(E_{n-1}(F,\beta)\exp(-\beta\lambda_n)\bigr)^{\frac{\rho}{\lambda_n}} = \limsup_{n\to+\infty} \frac{\lambda_n}{\rho\exp(\rho\beta+1)}\bigl(E_{n-1}(F,\beta)\bigr)^{\frac{\rho}{\lambda_n}}.$$

In the same year, the author and Kong [20] investigated the approximation of Laplace–Stieltjes transform $F(s) \in L_0$ with infinite order and obtained the following.

Theorem 2.4 (see [20, Theorem 2.5]) *If the Laplace–Stieltjes transform $F(s) \in L_0$ and is of infinite order, if $\lambda_n \sim \lambda_{n+1}$, then for any real number $-\infty < \beta < +\infty$, then for any fixed real number $-\infty < \alpha < 0$, we have*

$$\limsup_{\sigma\to 0^-} \frac{X(\log^+ M_u(\sigma,F))}{\log(-\frac{1}{\sigma})} = \rho^* \quad \Longleftrightarrow \quad \limsup_{n\to\infty} \frac{X(\lambda_n)}{\log^+ \frac{\lambda_n}{\log^+[E_{n-1}(F,\alpha)\exp\{-\alpha\lambda_n\}]}} = \rho^*, \tag{13}$$

where $0 < \rho^* < \infty$, $X(\cdot)$-order can be seen in [20].

The second purpose of this paper is to study the approximation of Laplace–Stieltjes transform $F(s) \in L_0$ with generalized order, and our results are listed as follows.

Theorem 2.5 *Let $F(s) \in L_0$, $\mathcal{A} \in \Gamma^{0i}$, and $\mathcal{B} \in \Gamma^{0i}$ be continuously differentiable satisfying* (8) *and* (9), *and let the function $\mathcal{B}$ increase more rapidly than $\mathcal{A}$. If $F(s)$ satisfies* (10) *and*

$$\rho_{\mathcal{A}\mathcal{B}}(F) = \limsup_{\sigma\to 0^-} \frac{\mathcal{A}(\log M_u(\sigma,F))}{\mathcal{B}(-\frac{1}{\sigma})}, \quad 0 \le \rho_{\mathcal{A}\mathcal{B}}(F) \le +\infty,$$

then for any real number $-\infty < \beta < 0$, we have

$$\rho_{\mathcal{A}\mathcal{B}}(F) = \limsup_{n\to+\infty} \frac{\mathcal{A}(\lambda_n)}{\mathcal{B}(\frac{\lambda_n}{\log[E_{n-1}(F,\beta)\exp\{-\beta\lambda_n\}]})}.$$

Theorem 2.6 *Under the assumptions of Theorem* 2.2, *then for any real number $-\infty < \beta < 0$, we have*

$$\rho_{\mathcal{A}\mathcal{B}}(F) = \limsup_{n\to+\infty} \frac{\mathcal{A}(\log E_{n-1}(F,\beta))}{\mathcal{B}(\lambda_n)}.$$

3 Conclusions

Regarding Theorems 2.1 and 2.2, the generalized order of Laplace–Stieltjes transforms are discussed by using the more abstract functions, and some related theorems among λ_n, A_n^* and the generalized order are obtained. Moreover, we also investigate some properties of approximation on analytic functions defined by Laplace–Stieltjes transforms of generalized order. For the topic of the growth and approximation of Laplace–Stieltjes transforms of generalized order, it seems that this topic has never been treated before. Our theorems are generalization and improvement of the previous results given by Luo and Kong [9, 10], Singhal and Srivastava [17].

4 Methods

To prove our results, we also need to give the following lemmas (see [16]).

Let Ξ_0 denote the set of positive unbounded functions ϕ on $(-\infty,0)$ such that the derivative ϕ' is positive, continuous, and increasing to $+\infty$ on $(-\infty,0)$. Thus, if $\phi\in\Xi_0$, then $\phi(x)\to\zeta\geq 0$ and $\phi'(x)\to 0$ as $x\to-\infty$. Let φ be the inverse function of ϕ', then φ is continuous on $(0,+\infty)$ and increases to 0. Set $\phi\in\Xi_0$ and $\psi(x)=x-\frac{\phi(x)}{\phi'(x)}$. For $-\infty<x<x+\iota<0$, since ϕ' is increasing on $(-\infty,0)$, we have

$$\begin{aligned}\phi'(x)\phi(x+\iota)-\phi'(x+\iota)\phi(x)<\phi'(x)\bigl[\phi(x+\iota)-\phi(x)\bigr]&=\phi'(x)\int_x^{x+\iota}\phi'(t)\,dt\\&<(x+\iota-x)\phi'(x)\phi'(x+\iota),\end{aligned}$$

that is,

$$\psi(x)=x-\frac{\phi(x)}{\phi'(x)}<x+\iota-\frac{\phi(x+\iota)}{\phi'(x+\iota)}=\psi(x+\iota).$$

Thus, it means that ψ is an increasing function on $(-\infty,0)$.

Next, we will prove that $\psi(x)\to 0$ as $x\to 0$, that is, there is no constant $\eta<0$ such that $\psi(x)\leq\eta$ for all $x\in(-\infty,0)$. Assume that there exist two constants η, K_1 such that $\psi(x)\leq\eta$ for all $x\in(-\infty,0)$ and $\eta<K_1<0$. Since ψ is an increasing function and $\psi(x)<x<0$, then it follows $\frac{\phi'(x)}{\phi(x)}\leq\frac{1}{x-\eta}$ for $K_1\leq x<0$. Thus, it follows

$$\begin{aligned}\log\phi(x)&=\log\phi(K_1)+\int_{K_1}^{x}\frac{\phi'(t)}{\phi(t)}\,dt\leq\log\phi(K_1)+\int_{K_1}^{x}\frac{1}{t-\eta}\,dt\\&=\log\phi(K_1)+\log\frac{x-\eta}{K_1-\eta}.\end{aligned}$$

Hence $\phi(x)\leq\phi(K_1)\frac{x-\eta}{K_1-\eta}$. In view of $\phi'(x)\to+\infty$ $(x\to 0)$, we get a contradiction. Thus, it follows $\psi(x)\to 0$ as $x\to 0$.

Besides, let ψ^{-1} be the inverse function of ψ. Then ψ^{-1} is an increasing function on $(-\infty,0)$ and $\phi'(\psi^{-1}(\sigma))$ increases to $+\infty$ on $(-\infty,0)$.

Lemma 4.1 *Let* $\phi\in\Xi_0$, *then the conclusion that* $\log\mu(\sigma,F)\leq\phi(\sigma)$ *for any* $\sigma\in(-\infty,0)$ *holds if and only if* $\log A_n^*\leq-\lambda_n\psi(\varphi(\lambda_n))$ *for all* $n\geq 0$.

Proof Suppose that $\log\mu(\sigma,F)\leq\phi(\sigma)$ for any $\sigma\in(-\infty,0)$, then $\log A_n^*\leq\phi(\sigma)-\sigma\lambda_n$ for all $n>0$ and $\sigma\in(-\infty,0)$. Thus, take $\sigma=\varphi(\lambda_n)$, it follows for all $n\geq 0$ that

$$\log A_n^*\leq\phi\bigl(\varphi(\lambda_n)\bigr)-\lambda_n\varphi(\lambda_n)=-\lambda_n\biggl(\varphi(\lambda_n)-\frac{\phi(\varphi(\lambda_n))}{\phi'(\varphi(\lambda_n))}\biggr)=-\lambda_n\psi\bigl(\varphi(\lambda_n)\bigr).$$

On the contrary, assume that $\log A_n^*\leq-\lambda_n\psi(\varphi(\lambda_n))$ for all $n\geq 0$. Since, for any $\sigma<0$ and $x<0$,

$$(\sigma-x)\phi'(x)\leq\int_x^{\sigma}\phi'(t)\,dt=\phi(\sigma)-\phi(x),$$

then it follows

$$\begin{aligned}\log\mu(\sigma,F) &\le \max\bigl\{-\lambda_n\psi\bigl(\varphi(\lambda_n)\bigr)+\lambda_n\sigma : n\ge 0\bigr\} \le \max\bigl\{-t\psi\bigl(\varphi(t)\bigr)+t\sigma : t\ge 0\bigr\}\\ &= \max\bigl\{-\phi'(x)\psi(x)+\sigma+\phi'(x) : x>-\infty\bigr\}\\ &= \max\bigl\{(\sigma-x)\phi'(x)+\phi(x) : x>-\infty\bigr\}=\phi(\sigma).\end{aligned}$$

Therefore, this completes the proof of Lemma 4.1. □

Lemma 4.2 *If the L-S transform $F(s)\in L_\infty$, then for any σ $(-\infty<\sigma<0)$ and ε (>0), we have*

$$\frac{1}{p}\mu(\sigma,F)\le M_u(\sigma,F)\le C\mu\bigl((1-\varepsilon)\sigma,F\bigr)\frac{1}{-\sigma},$$

where $p>2$ and C $(\neq 0)$ are constants.

Proof We will adapt the method as in Yu [26] and Kong and Hong [5]. Set

$$I(x;\sigma+it)=\int_0^x \exp\bigl\{(\sigma+it)y\bigr\}\,d\alpha(y).$$

In view of (5), there exists a positive number ξ satisfying $0<\lambda_{n+1}-\lambda_n\le\xi$ $(n=1,2,3,\ldots)$. Thus, it yields $e^{-\xi\sigma}<\frac{p}{2}$ for σ sufficiently close to 0^-, where $p>2$ is a constant. For $x>\lambda_n$, it follows

$$\begin{aligned}\int_{\lambda_n}^x \exp\{ity\}\,d\alpha(y) &= \int_{\lambda_n}^x \exp\{-\sigma y\}d_yI(y;\sigma+it)\\ &= I(y;\sigma+it)\exp\{-\sigma y\}|_{\lambda_n}^x+\sigma\int_{\lambda_n}^x \exp\{-\sigma y\}I(y;\sigma+it)\,dy.\end{aligned}$$

Then, for any $\sigma<0$ and any $x\in(\lambda_n,\lambda_{n+1}]$, it yields

$$\begin{aligned}\left|\int_{\lambda_n}^x \exp\{ity\}\,d\alpha(y)\right| &\le M_u(\sigma,F)\bigl[\exp\{-x\sigma\}+\exp\{-\sigma\lambda_n\}+\bigl|\exp\{-x\sigma\}-\exp\{-\sigma\lambda_n\}\bigr|\bigr]\\ &\le 2M_u(\sigma,F)\exp\bigl\{-(\lambda_n+\xi)\sigma\bigr\}\le pM_u(\sigma,F)\exp\{-\lambda_n\sigma\},\end{aligned}$$

which implies

$$\frac{1}{p}\mu(\sigma,F)\le M_u(\sigma,F). \tag{14}$$

On the other hand, for any $x>0$, it follows that there exists a positive integer $n\in\mathbb{N}_+$ such that $\lambda_n<x\le\lambda_{n+1}$. Thus, it follows

$$\int_0^x \exp\bigl\{(\sigma+it)y\bigr\}\,d\alpha(y)=\sum_{k=1}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}}\exp\bigl\{(\sigma+it)y\bigr\}\,d\alpha(y)+\int_{\lambda_n}^x \exp\bigl\{(\sigma+it)y\bigr\}\,d\alpha(y).$$

Set $I_k(x;it)=\int_{\lambda_n}^x \exp\{ity\}\,d\alpha(y)$ $(\lambda_k<x\le\lambda_{k+1})$, then for any real number t and $\sigma<0$, it follows $|I_k(x;it)|\le A_k^*\le\mu(\sigma,F)e^{-\lambda_k\sigma}$. Thus, for any $\varepsilon\in(0,1)$ and $\sigma<0$, it yields $|I_k(x;it)|\le$

$\mu((1-\varepsilon)\sigma, F)e^{-\lambda_k(1-\varepsilon)\sigma}$ and

$$\int_0^x \exp\{(\sigma+it)y\}\,d\alpha(y) = \sum_{k=1}^{n-1}\left[\exp\{\lambda_{k+1}\sigma\}I_k(\lambda_{k+1};it) - \sigma\int_{\lambda_k}^{\lambda_{k+1}}\exp\{\sigma y\}I_k(y;it)\,dy\right] + \exp\{x\sigma\}I_n(x;it) - \sigma\int_{\lambda_n}^{x}\exp\{\sigma y\}I_n(y;it)\,dy.$$

Hence, we can deduce

$$\begin{aligned}
&\left|\int_0^x \exp\{(\sigma+it)y\}\,d\alpha(y)\right| \\
&\quad\le \sum_{k=1}^{n-1}\mu\big((1-\varepsilon)\sigma,F\big)\exp\{-\lambda_k(1-\varepsilon)\sigma\}\big(\exp\{\lambda_{k+1}\sigma\} + \left|\exp\{\lambda_{k+1}\sigma\} - \exp\{\lambda_k\sigma\}\right|\big) \\
&\qquad + \mu\big((1-\varepsilon)\sigma,F\big)\exp\{-\lambda_n(1-\varepsilon)\sigma\}\big(\exp\{x\sigma\} + \left|\exp\{x\sigma\} - \exp\{\lambda_n\sigma\}\right|\big) \\
&\quad= \mu\big((1-\varepsilon)\sigma,F\big)\sum_{k=1}^{+\infty}\exp\{\lambda_k\varepsilon\sigma\}.
\end{aligned}$$

In view of (5), for the above ε, there exists $N_1 \in N_+$ such that, for any $n > N_1$, we have $\lambda_n > \frac{n}{D+\varepsilon}$. Hence it follows for $\sigma \to 0^-$ that

$$\sum_{k=1}^{+\infty}\exp\{\lambda_k\varepsilon\sigma\} \le \sum_{k=1}^{N_1}\exp\{\lambda_k\varepsilon\sigma\} + \sum_{k=N_1+1}^{+\infty}\exp\left\{k\frac{\varepsilon\sigma}{D+\varepsilon}\right\} < C\frac{1}{-\sigma}, \tag{15}$$

where C is a constant on ε and (5). Therefore, this lemma is proved from (14) and (15). □

4.1 Proofs of Theorems 2.1 and 2.2

4.1.1 The proof of Theorem 2.1

Suppose that $\rho := \rho_{\mathcal{A}\mathcal{B}}(F) < +\infty$ and

$$\vartheta = \limsup_{n\to+\infty}\frac{\mathcal{A}(\lambda_n)}{\mathcal{B}(\frac{\lambda_n}{\log A_n^*})}.$$

In view of the definition of generalized order and Lemma 4.2, for any $\varepsilon > 0$, there exists a constant $\sigma_0 < 0$ such that, for all $0 > \sigma > \sigma_0$,

$$\log\mu(\sigma,F) \le \mathcal{A}^{-1}\left((\rho+\varepsilon)\mathcal{B}\left(-\frac{1}{\sigma}\right)\right) + \log p,$$

that is,

$$\log A_n^* \le \mathcal{A}^{-1}\left((\rho+\varepsilon)\mathcal{B}\left(-\frac{1}{\sigma}\right)\right) - \lambda_n\sigma + \log p, \quad n \ge 0. \tag{16}$$

Choosing

$$-\frac{1}{\sigma} = \mathcal{B}^{-1}\left(\frac{1}{\rho+\varepsilon}\mathcal{A}\left(\frac{\lambda_n}{\mathcal{B}^{-1}(\frac{\mathcal{A}(\lambda_n)}{\rho+\varepsilon})}\right)\right),$$

we conclude from (9) and (16) that

$$\log A_n^* \le \frac{\lambda_n}{\mathcal{B}^{-1}(\frac{\mathcal{A}(\lambda_n)}{\rho+\varepsilon})} + \frac{\lambda_n}{\mathcal{B}^{-1}(\frac{1}{\rho+\varepsilon}\mathcal{A}(\frac{\lambda_n}{\mathcal{B}^{-1}(\frac{\mathcal{A}(\lambda_n)}{\rho+\varepsilon})}))} + \log p$$
$$= \frac{(1+o(1))\lambda_n}{\mathcal{B}^{-1}((1+o(1))\frac{\mathcal{A}(\lambda_n)}{\rho+\varepsilon})}, \quad \text{as } n \to +\infty,$$

which implies

$$\mathcal{A}(\lambda_n) \le (\rho+\varepsilon)\mathcal{B}\left(\frac{(1+o(1))\lambda_n}{\log A_n^*}\right), \quad \text{as } n \to +\infty. \tag{17}$$

Since $\mathcal{A} \in \Gamma^{0i}$, $\mathcal{B} \in \Gamma^{0i}$ and let $\varepsilon \to 0^+$, we can conclude from (17) that $\vartheta \le \rho$.

Assume $\vartheta < \rho$, then we can choose a constant ρ_1 such that $\vartheta < \rho_1 < \rho$. Since $\mathcal{B}^{-1}(\frac{\mathcal{A}(x)}{\rho_1})$ is an increasing function, then there exists a positive integer n_0 such that, for $n \ge n_0$,

$$\log A_n^* \le \frac{\lambda_n}{\mathcal{B}^{-1}(\frac{\mathcal{A}(\lambda_n)}{\rho_1})} \le \int_{\lambda_{n_0}}^{\lambda_n} \frac{1}{\mathcal{B}^{-1}(\frac{\mathcal{A}(t)}{\rho_1})}\,dt + K_1, \tag{18}$$

where here and further K_j is a constant.

Since $\phi \in \Xi_0$, and let

$$\phi(\sigma) = \int_{-\frac{1}{\sigma_0}}^{-\frac{1}{\sigma}} \frac{\mathcal{A}^{-1}(\rho_1\mathcal{B}(t))}{t^2}\,dt + K_2 \quad \text{for } 0 > \sigma \ge \sigma_0.$$

Then it follows

$$\phi'(\sigma) = \mathcal{A}^{-1}\left(\rho_1\mathcal{B}\left(-\frac{1}{\sigma}\right)\right), \qquad \varphi(\lambda_n) = -\frac{1}{\mathcal{B}^{-1}(\frac{\mathcal{A}(\lambda_n)}{\rho_1})},$$
$$\left[\lambda_n\psi\left(\varphi(\lambda_n)\right)\right]' = \left[\lambda_n\varphi(\lambda_n) - \phi\left(\varphi(\lambda_n)\right)\right]' = \varphi(\lambda_n),$$

and

$$-\lambda_n\psi\left(\varphi(\lambda_n)\right) = -\int_{\lambda_{n_0}}^{\lambda_n} \varphi(t)\,dt + K_2 = \int_{\lambda_{n_0}}^{\lambda_n} \frac{1}{\mathcal{B}^{-1}(\frac{\mathcal{A}(t)}{\rho_1})}\,dt + K_2. \tag{19}$$

Thus, in view of (18) and (19), it follows

$$\log\mu(\sigma,F) \le \phi(\sigma) = \int_{-\frac{1}{\sigma_0}}^{-\frac{1}{\sigma}} \frac{\mathcal{A}^{-1}(\rho_1\mathcal{B}(t))}{t^2}\,dt + K_2$$
$$\le \mathcal{A}^{-1}\left(\rho_1\mathcal{B}\left(-\frac{1}{\sigma}\right)\right)\int_{-\frac{1}{\sigma_1}}^{-\frac{1}{\sigma}} \frac{dt}{t^2} + K_3$$
$$\le -\sigma_1\mathcal{A}^{-1}\left(\rho_1\mathcal{B}\left(-\frac{1}{\sigma}\right)\right) + K_3. \tag{20}$$

Since $\mathcal{A} \in \Gamma^{0i}$, in view of (10), (20) and by applying Lemma 4.2, we can deduce $\rho_{\mathcal{A}\mathcal{B}}(F) \le \rho_1$, which implies a contradiction with $\rho_{\mathcal{A}\mathcal{B}}(F) > \rho_1$. Hence $\vartheta = \rho_{\mathcal{A}\mathcal{B}}(F)$.

If $\rho_{\mathcal{A}\mathcal{B}}(F) = +\infty$, by using the same argument as above, it is easy to prove that the conclusion is true. Therefore, this completes the proof of Theorem 2.1.

4.1.2 The proof of Theorem 2.2

Suppose that $\rho := \rho_{\mathcal{A}\mathcal{B}}(F) < +\infty$ and

$$\vartheta_1 = \limsup_{n\to+\infty} \frac{\mathcal{A}(\log A_n^*)}{\mathcal{B}(\lambda_n)}.$$

In view of the definition of generalized order and Lemma 4.2, for any $\varepsilon > 0$, there exists a constant $\sigma_0 < 0$ such that, for all $0 > \sigma > \sigma_0$,

$$\log \mu(\sigma, F) \le \mathcal{A}^{-1}\left((\rho + \varepsilon)\mathcal{B}\left(-\frac{1}{\sigma}\right)\right) + \log p,$$

that is,

$$\log A_n^* \le \mathcal{A}^{-1}\left((\rho + \varepsilon)\mathcal{B}\left(-\frac{1}{\sigma}\right)\right) - \lambda_n\sigma + \log p, \quad n \ge 0. \tag{21}$$

Choosing $-\frac{1}{\sigma} = \lambda_n$, we conclude from (21) that

$$\begin{aligned}\log A_n^* &\le \mathcal{A}^{-1}\big((\rho + \varepsilon)\mathcal{B}(\lambda_n)\big) + 1 + \log p \\ &\le \big(1 + o(1)\big)\mathcal{A}^{-1}\big((\rho + \varepsilon)\mathcal{B}(\lambda_n)\big), \quad \text{as } n \to +\infty.\end{aligned} \tag{22}$$

Since $\mathcal{A} \in \Gamma^{0i}$, $\mathcal{B} \in \Gamma^{0i}$ and let $\varepsilon \to 0^+$, we can conclude from (22) that $\vartheta_1 \le \rho$.

Assume $\vartheta_1 < \rho$, then we can choose a constant ρ_2 such that $\vartheta_1 < \rho_2 < \rho$. It means that there exists a positive integer n_0 such that, for $n \ge n_0$,

$$\log A_n^* \le \mathcal{A}^{-1}\big(\rho_2\mathcal{B}(\lambda_n)\big),$$

that is,

$$\log \mu(\sigma, F) \le \max\big\{\mathcal{A}^{-1}\big(\rho_2\mathcal{B}(\lambda_n)\big) + \lambda_n\sigma : n \ge n_0\big\} + K_5. \tag{23}$$

In view of (10), the following equation

$$\mathcal{A}^{-1}\big(\rho_2\mathcal{B}(t)\big) + t\sigma = 0$$

has a unique solution $t_1 := t(\sigma)$ such that $t_1 \uparrow +\infty$ as $\sigma \to 0^-$, and for $t \ge t_1$ we can deduce that $\mathcal{A}^{-1}(\rho_2\mathcal{B}(t)) + t\sigma \le 0$. Hence, it follows

$$\begin{aligned}\log \mu(\sigma, F) &\le \max\big\{\mathcal{A}^{-1}\big(\rho_2\mathcal{B}(t)\big) + t\sigma : t_0 \le t \le t_1\big\} + K_6 \\ &\le \mathcal{A}^{-1}\big(\rho_2\mathcal{B}(t_1)\big) + K_6.\end{aligned} \tag{24}$$

In view of

$$-\frac{1}{\sigma} = \frac{t_1}{\mathcal{A}^{-1}(\rho_2\mathcal{B}(t_1))},$$

it follows from (12) that

$$\mathcal{B}\left(-\frac{1}{\sigma}\right) = \mathcal{B}\left(\frac{t_1}{\mathcal{A}^{-1}(\rho_2\mathcal{B}(t_1))}\right) = \bigl(1+o(1)\bigr)\mathcal{B}(t_1), \quad \sigma \to 0^-. \tag{25}$$

Hence, we can deduce from (24) and (25) that

$$\log \mu(\sigma, F) \le \mathcal{A}^{-1}\left(\rho_2\bigl(1+o(1)\bigr)\mathcal{B}\left(-\frac{1}{\sigma}\right)\right), \quad \text{as } \sigma \to 0^-,$$

which implies $\rho_{\mathcal{A}\mathcal{B}}(F) \le \rho_2 < \rho_{\mathcal{A}\mathcal{B}}(F)$ by combining Lemma 4.2 and (10), a contradiction. Therefore, $\vartheta_1 = \rho_{\mathcal{A}\mathcal{B}}(F)$.

If $\rho_{\mathcal{A}\mathcal{B}}(F) = +\infty$, by using the same argument as above, it is easy to prove that the conclusion is true. Therefore, this completes the proof of Theorem 2.2.

4.2 Proofs of Theorems 2.5 and 2.6

4.2.1 The proof of Theorem 2.5

Suppose that $\rho := \rho_{\mathcal{A}\mathcal{B}}(F) < +\infty$ and

$$\vartheta_3 = \limsup_{n\to+\infty} \frac{\mathcal{A}(\lambda_n)}{\mathcal{B}(\frac{\lambda_n}{\log[E_{n-1}(F,\beta)\exp\{-\beta\lambda_n\}]})}.$$

In view of the definition of generalized order and Lemma 4.2, for any $\varepsilon > 0$, there exists a constant $\sigma_0 < 0$ such that, for all $0 > \sigma > \sigma_0$,

$$\log M_u(\sigma, F) \le \mathcal{A}^{-1}\left((\rho+\varepsilon)\mathcal{B}\left(-\frac{1}{\sigma}\right)\right). \tag{26}$$

Since $F(s) \in L_0$, and for any constant β ($-\infty < \beta < 0$), then $F(s) \in \overline{L}_\beta$. Hence, for $\beta < \sigma < 0$ and $p_k \in \Pi_k$, it follows

$$\begin{aligned} E_k(F,\beta) &\le \|F - p_k\|_\beta \le \bigl|F(\beta+it) - p_k(\beta+it)\bigr| \\ &\le \left|\int_0^{+\infty} \exp\{sy\}\,d\alpha(y) - \int_0^{\lambda_k} \exp\{sy\}\,d\alpha(y)\right| = \left|\int_{\lambda_k}^{\infty} \exp\{sy\}\,d\alpha(y)\right|. \end{aligned} \tag{27}$$

Let

$$I_{j+k}(b;it) = \int_{\lambda_{j+k}}^{b} \exp\{ity\}\,d\alpha(y) \quad (\lambda_{j+k} < b \le \lambda_{j+k+1}),$$

then $|I_{j+k}(b;it)| \le A^*_{j+k}$. In view of

$$\left|\int_{\lambda_k}^{\infty} \exp\{(\beta+it)y\}\,d\alpha(y)\right| = \lim_{b\to+\infty}\left|\int_{\lambda_k}^{b} \exp\{(\beta+it)y\}\,d\alpha(y)\right|,$$

where $-\infty < \beta < 0$, hence

$$\begin{aligned}
&\left|\int_{\lambda_k}^{b} \exp\{(\beta + it)y\}\, d\alpha(y)\right| \\
&\quad = \left|\sum_{j=k}^{n+k-1} \int_{\lambda_j}^{\lambda_{j+1}} \exp\{\beta y\} d_y I_j(y;it) + \int_{\lambda_{n+k}}^{b} \exp\{\beta y\} d_y I_{n+k}(y;it)\right| \\
&\quad = \left|\left[\sum_{j=k}^{n+k-1} e^{\lambda_{j+1}\beta} I_j(\lambda_{j+1};it) - \beta \int_{\lambda_j}^{\lambda_{j+1}} e^{\beta y} I_j(y;it)\, dy\right]\right. \\
&\qquad \left. + e^{\beta b} I_{n+k}(b;it) - \beta \int_{\lambda_{n+k}}^{b} e^{\beta y} I_j(y;it)\, dy\right| \\
&\quad \le \sum_{j=k}^{n+k-1} \left[A_j^* e^{\lambda_{j+1}\beta} + A_j^*\left(e^{\lambda_{j+1}\beta} - e^{\lambda_j\beta}\right)\right] + 2e^{\beta\lambda_{n+k+1}} A_{n+k}^* - e^{\beta\lambda_{n+k}} A_{n+k}^* \\
&\quad \le 2\sum_{j=k}^{n+k} A_n^* e^{\lambda_{n+1}\beta}.
\end{aligned}$$

Therefore, we conclude

$$\left|\int_{\lambda_k}^{\infty} \exp\{(\beta + it)y\}\, d\alpha(y)\right| \le 2\sum_{n=k}^{+\infty} A_n^* \exp\{\beta\lambda_{n+1}\}, \quad \text{as } n \to +\infty. \tag{28}$$

In view of Lemma 4.2, it follows $A_n^* \le pM_u(\sigma, F)e^{-\sigma\lambda_n}$. So, for any σ ($\beta < \sigma < 0$), it yields from (27) and (28) that

$$E_n(F,\beta) \le 2\sum_{k=n+1}^{\infty} A_{k-1}^* \exp\{\beta\lambda_k\} \le 2pM_u(\sigma, F)\sum_{k=n+1}^{\infty} \exp\{(\beta - \sigma)\lambda_k\}. \tag{29}$$

In view of (5), we can choose h' ($0 < h' < h$) such that $(\lambda_{n+1} - \lambda_n) \ge h'$ for $n \ge 0$. Then, for $\sigma \ge \frac{\beta}{2}$, it follows from (29) that

$$\begin{aligned}
E_n(F,\beta) &\le M_u(\sigma, F)\exp\{\lambda_{n+1}(\beta - \sigma)\}\sum_{k=n+1}^{\infty} \exp\{(\lambda_k - \lambda_{n+1})(\beta - \sigma)\} \\
&\le M_u(\sigma, F)\exp\{\lambda_{n+1}(\beta - \sigma)\}\exp\left\{-\frac{\beta}{2}h'(n+1)\right\}\sum_{k=n+1}^{\infty} \exp\left\{\frac{\beta}{2}h'k\right\} \\
&= M_u(\sigma, F)\exp\{\lambda_{n+1}(\beta - \sigma)\}\left(1 - \exp\left\{\frac{\beta}{2}h'\right\}\right)^{-1},
\end{aligned}$$

that is,

$$E_{n-1}(F,\beta) \le KM_u(\sigma, F)\exp\{\lambda_n(\beta - \sigma)\}, \tag{30}$$

where K is a constant. Hence, it follows from (26) and (30) that

$$\log\left[E_{n-1}(F,\beta)\exp\{-\beta\lambda_n\}\right] \le \mathcal{A}^{-1}\left((\rho + \varepsilon)\mathcal{B}\left(-\frac{1}{\sigma}\right)\right) - \lambda_n\sigma + \log K, \quad n \ge 0. \tag{31}$$

Let

$$-\frac{1}{\sigma} = \mathcal{B}^{-1}\left(\frac{1}{\rho+\varepsilon}\mathcal{A}\left(\frac{\lambda_n}{\mathcal{B}^{-1}(\frac{\mathcal{A}(\lambda_n)}{\rho+\varepsilon})}\right)\right),$$

we conclude from (9) and (31) that

$$\begin{aligned}\log\bigl[E_{n-1}(F,\beta)\exp\{-\beta\lambda_n\}\bigr] &\le \frac{\lambda_n}{\mathcal{B}^{-1}(\frac{\mathcal{A}(\lambda_n)}{\rho+\varepsilon})} + \frac{\lambda_n}{\mathcal{B}^{-1}(\frac{1}{\rho+\varepsilon}\mathcal{A}(\frac{\lambda_n}{\mathcal{B}^{-1}(\frac{\mathcal{A}(\lambda_n)}{\rho+\varepsilon})}))} + \log K \\ &= \frac{(1+o(1))\lambda_n}{\mathcal{B}^{-1}((1+o(1))\frac{\mathcal{A}(\lambda_n)}{\rho+\varepsilon})}, \quad \text{as } n \to +\infty,\end{aligned}$$

which implies

$$\mathcal{A}(\lambda_n) \le (\rho+\varepsilon)\mathcal{B}\left(\frac{(1+o(1))\lambda_n}{\log[E_{n-1}(F,\beta)\exp\{-\beta\lambda_n\}]}\right), \quad \text{as } n \to +\infty. \tag{32}$$

Since $\mathcal{A} \in \Gamma^{0i}$, $\mathcal{B} \in \Gamma^{0i}$ and let $\varepsilon \to 0^+$, we can conclude from (32) that $\vartheta_3 \le \rho$.

Assume $\vartheta_3 < \rho$, then we can choose a constant ρ_3 such that $\vartheta_3 < \rho_3 < \rho$. Since $\mathcal{B}^{-1}(\frac{\mathcal{A}(x)}{\rho_3})$ is an increasing function, then there exists a positive integer n_0 such that, for $n \ge n_0$,

$$\log\bigl[E_{n-1}(F,\beta)\exp\{-\beta\lambda_n\}\bigr] \le \frac{\lambda_n}{\mathcal{B}^{-1}(\frac{\mathcal{A}(\lambda_n)}{\rho_3})} \le \int_{\lambda_{n_0}}^{\lambda_n} \frac{1}{\mathcal{B}^{-1}(\frac{\mathcal{A}(t)}{\rho_3})}\,dt + K_1. \tag{33}$$

For any $\beta < 0$, then there exists $p_1 \in \Pi_{n-1}$ such that

$$\|F - p_1\| \le 2E_{n-1}(F,\beta). \tag{34}$$

And since

$$\begin{aligned}A_n^* \exp\{\beta\lambda_n\} &= \sup_{\lambda_n<x\le\lambda_{n+1},-\infty<t<+\infty}\left|\int_{\lambda_n}^x \exp\{ity\}\,d\alpha(y)\right|\exp\{\beta\lambda_n\} \\ &\le \sup_{\lambda_n<x\le\lambda_{n+1},-\infty<t<+\infty}\left|\int_{\lambda_n}^x \exp\bigl\{(\beta+it)y\bigr\}\,d\alpha(y)\right| \\ &\le \sup_{-\infty<t<+\infty}\left|\int_{\lambda_n}^\infty \exp\bigl\{(\beta+it)y\bigr\}\,d\alpha(y)\right|,\end{aligned}$$

thus, for any $p \in \Pi_{n-1}$, it follows

$$A_n^* \exp\{\beta\lambda_n\} \le \bigl|F(\beta+it) - p(\beta+it)\bigr| \le \|F-p\|_\beta. \tag{35}$$

Hence, for any $\beta < 0$ and $F(s) \in L_0$, it follows from (34) and (35) that

$$A_n^* \le 2E_{n-1}(F,\beta)\exp\{-\beta\lambda_n\}. \tag{36}$$

Hence, (18) follows from (33) and (36).

Since $\phi \in \Xi_0$, and let

$$\phi(\sigma) = \int_{-\frac{1}{\sigma_0}}^{-\frac{1}{\sigma}} \frac{\mathcal{A}^{-1}(\rho_1 \mathcal{B}(t))}{t^2}\,dt + K_2 \quad \text{for } 0 > \sigma \ge \sigma_0,$$

and

$$\varphi(\lambda_n) = -\frac{1}{\mathcal{B}^{-1}(\frac{\mathcal{A}(\lambda_n)}{\rho_1})}.$$

By using the same argument as in the proof of Theorem 2.1, we conclude

$$\begin{aligned}\log \mu(\sigma, F) \le \phi(\sigma) &= \int_{-\frac{1}{\sigma_0}}^{-\frac{1}{\sigma}} \frac{\mathcal{A}^{-1}(\rho_1 \mathcal{B}(t))}{t^2}\,dt + K_2 \\ &\le -\sigma_1 \mathcal{A}^{-1}\left(\rho_1 \mathcal{B}\left(-\frac{1}{\sigma}\right)\right) + K_3. \end{aligned} \tag{37}$$

Since $\mathcal{A} \in \Gamma^{0i}$, in view of (10), (37) and by applying Lemma 4.2, we can deduce $\rho_{\mathcal{A}\mathcal{B}}(F) \le \rho_3$, which implies a contradiction with $\rho_{\mathcal{A}\mathcal{B}}(F) > \rho_1$. Hence $\vartheta_3 = \rho_{\mathcal{A}\mathcal{B}}(F)$.

If $\rho_{\mathcal{A}\mathcal{B}}(F) = +\infty$, by using the same argument as above, it is easy to prove that the conclusion is true. Therefore, this completes the proof of Theorem 2.5.

4.2.2 The proof of Theorem 2.6

By combining the arguments as in the proofs of Theorems 2.2 and 2.5, we can easily prove the conclusion of Theorem 2.6.

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Competing interests

The authors declare that none of the authors have any competing interests in the manuscript.

Authors' contributions

HYX and HW completed the main part of this article, HYX and HW corrected the main theorems. All authors read and approved the final manuscript.

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