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Characterization and stability of approximately dual g-frames in Hilbert spaces

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Abstract
This paper addresses approximately dual g-frames. First, we establish a connection between approximately dual g-frames and dual g-frames and obtain a characterization of approximately dual g-frames. Second, we give results on stability of approximately dual g-frames, which cover the results obtained by other authors.

MSC: 42C15; 42C40

Keywords: Frame; g-frame; Dual g-frame; Approximately dual g-frame

1 Introduction

The notion of frame dates back to Gabor [1] (1946) and Duffin and Schaeffer [2] (1952). Gabor [1] proposed the idea of decomposing a general signal in terms of elementary signals, and Duffin and Schaeffer [2] abstracted "these elementary signals" as the notion of frame. However, the frame theory had not attracted much attention until the celebrated work by Daubechies, Crossman, and Meyer [3] in 1986. So far, the theory of frame has seen great achievements in pure mathematics, science, and engineering ([4–13]). In 2006, Sun [14] introduced a generalized frame (simply g-frame), which covers all other generalizations of frames, for example, fusion frames [15], bounded quasiprojectors [16], and so on. Now, the research of g-frames has obtained many results [17–19]. This paper addresses approximately dual g-frames in Hilbert spaces.

Recall that a sequence $\{f_i\}_{i\in I}$ in a separable Hilbert space H is a frame if

$$A_1\|f\|^2 \le \sum_{i\in I}\bigl|\langle f,f_n\rangle\bigr|^2 \le B_1\|f\|^2$$

for all $f\in\mathcal{H}$ and some positive constants A_1, B_1. Given a frame $\{f_i\}_{i\in I}$, another frame $\{h_i\}_{i\in I}$ is said to be a dual frame of $\{f_i\}_{i\in I}$ if

$$f=\sum_{i\in I}\langle f,f_i\rangle h_i,\quad \forall f\in H,$$

or, equivalently,

$$f=\sum_{i\in I}\langle f,h_i\rangle f_i,\quad \forall f\in H.$$

To find the dual frames for a general frame is a fundamental problem in the frame theory. Usually, it is not easy due to involving complicated computation. In 2010, Christensen [20] introduced the notion of approximately dual frames. Bessel sequences $\{f_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ in a separable Hilbert space $\mathcal{H}$ are said to be approximately dual frames if

$$\left\| f - \sum_{i\in I} \langle f, h_i\rangle f_i \right\| \le \|f\|, \quad \forall f \in H,$$

or

$$\left\| f - \sum_{i\in I} \langle f, f_i\rangle h_i \right\| \le \|f\|, \quad \forall f \in H.$$

In 2014, Khosravi et al. [21] first introduced the notion of approximately dual g-frames, which generalize the usual approximately dual frames. They proved that a pair of operator sequences form approximately dual frames if and only if their induced sequences form a pair of approximately dual g-frames. They also obtained some important properties and applications of approximately dual frames. Later, many results on approximately dual g-frames were obtained (see [22, 23]).

Motivated by [21], in this paper, we focus on the characterization and stability of approximately dual g-frames and their connection with dual g-frames. Sect. 2 is an auxiliary one, where we recall some basic notions, properties, and some related results. In Sect. 3, we establish a characterization of approximately dual g-frames and discuss some properties of approximately dual (dual) g-frames. In Sect. 4, we give some stability results of approximately dual g-frames, which cover the results obtained by other authors.

2 Preliminaries

We begin with some basic notions and results of g-frames. See [14, 17, 18] for details.

Given separable Hilbert spaces H and V, let $\{V_j : j \in J\}$ be a sequence of closed subspaces of V with J being a subset of integers $\mathbb{Z}$. The identity operator on H is denoted by I_H. The set of all bounded linear operators from H into V_j is denoted by $L(H, V_j)$. Define

$$\bigoplus_{j\in J} V_j = \left\{ \{a_j\}_{j\in J} : a_j \in V_j, \left\| \{a_j\}_{j\in J} \right\|^2 = \sum_{j\in J} \|a_j\|^2 < \infty \right\}.$$

Then $\bigoplus_{j\in J} V_j$ is a Hilbert space under the inner product

$$\left\langle \{a_j\}_{j\in J}, \{b_j\}_{j\in J} \right\rangle = \sum_{j\in J} \langle a_j, b_j\rangle \quad \text{for } \{a_j\}_{j\in J}, \{b_j\}_{j\in J} \in \bigoplus_{j\in J} V_j.$$

Suppose $\{e_{j,k}\}_{k\in K_j}$ is an orthonormal basis (simply o.n.b.) for V_j, where $K_j \subset \mathbb{Z}$, $j \in J$. Define $\tilde{e}_{j,k} = \{\delta_{j,i} e_{i,k}\}_{i\in J}$, where δ is the Kronecker symbol. Then $\{\tilde{e}_{j,k}\}_{j\in J, k\in K_j}$ is an o.n.b. for $\bigoplus_{j\in J} V_j$ (see [17]).

Definition 2.1 ([14]) A sequence $\{\Lambda_j \in L(H, V_j)\}_{j\in J}$ is called a g-frame for H with respect to (w.r.t.) $\{V_j\}_{j\in J}$ if

$$A\|f\|^2 \le \sum_{j\in J} \|\Lambda_j f\|^2 \le B\|f\|^2 \tag{2.1}$$

for all $f \in H$ and some positive constants $A \leq B$. The numbers A, B are called the frame bounds. If only the right-hand inequality of (2.1) is satisfied, then $\{\Lambda_j\}_{j\in J}$ is called a g-Bessel sequence for H w.r.t. $\{V_j\}_{j\in J}$ with bound B. If $A = B = \lambda$, then $\{\Lambda_j\}_{j\in J}$ is called a λ-tight g-frame. In addition, if $\lambda = 1$, then $\{\Lambda_j\}_{j\in J}$ is called a Parsevel g-frame.

For a g-Bessel sequence $\{\Lambda_j\}_{j\in J}$ with bound B, the operator

$$T_\Lambda : \bigoplus_{j\in J} V_j \to H, \qquad T_\Lambda F = \sum_{j\in J} \Lambda_j^* f_j, \quad \forall F = \{f_j\}_{j\in J} \in \bigoplus_{j\in J} V_j,$$

is well-defined, and its adjoint is given by

$$T_\Lambda^* : H \to \bigoplus_{j\in J} V_j, \qquad T_\Lambda^* f = \{\Lambda_j f\}_{j\in J}, \quad \forall f \in H.$$

The operator T_Λ is called the synthesis operator, and T_Λ^* is called the analysis operator of $\{\Lambda_j\}_{j\in J}$. For g-frame $\{\Lambda_j\}_{j\in J}$ with bounds A and B, the operator

$$S_\Lambda : H \to H, \qquad S_\Lambda f = \sum_{j\in J} \Lambda_j^* \Lambda_j f, \quad \forall f \in H,$$

is called a g-frame operator of $\{\Lambda_j\}_{j\in J}$. It is bounded, invertible, self-adjoint, and positive, and $AI_H \leq S_\Lambda \leq BI_H$. Let $\tilde{\Lambda}_j = \Lambda_j S_\Lambda^{-1}$. Then $\{\tilde{\Lambda}_j\}_{j\in J}$ is also a g-frame for H w.r.t. $\{V_j\}_{j\in J}$ with the g-frame operator S_Λ^{-1} and frame bounds $\frac{1}{B}$ and $\frac{1}{A}$. $\{\tilde{\Lambda}_j\}_{j\in J}$ is called thebcanonical dual g-frame of $\{\Lambda_j\}_{j\in J}$ (see [14]).

Definition 2.2 ([14]) Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for H w.r.t. $\{V_j\}_{j\in J}$. A g-frame $\{\Gamma_j\}_{j\in J}$ is called an alternate dual g-frame for $\{\Lambda_j\}_{j\in J}$ if

$$f = \sum_{j\in J} \Gamma_j^* \Lambda_j f, \quad \forall f \in H.$$

Moreover, $\{\Lambda_j\}_{j\in J}$ is also an alternate dual g-frame for $\{\Gamma_j\}_{j\in J}$, that is,

$$f = \sum_{j\in J} \Lambda_j^* \Gamma_j f, \quad \forall f \in H.$$

Definition 2.3 ([20]) Let $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ be two Bessel sequences for H with their respective synthesis operators T_f and T_g. We say that$\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are approximately dual frames if $\|I_H - T_f T_g^*\| < 1$ or $\|I_H - T_g T_f^*\| < 1$.

It is clear that the operator $T_f T_g^*$ is invertible.

Definition 2.4 ([21]) Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be two g-Bessel sequences for H w.r.t. $\{V_j\}_{j\in J}$ with their respective synthesis operators T_Λ and T_Γ. Then $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are approximately dual g-frames if $\|I_H - T_\Lambda T_\Gamma^*\| < 1$ or $\|I_H - T_\Gamma T_\Lambda^*\| < 1$.

3 Dual and approximately dual g-frames

This section focuses on the connection between approximately dual g-frames and dual g-frames and on a characterization of approximately dual g-frames.

Lemma 3.1 ([19]) *Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be two g-Bessel sequences for H w.r.t. $\{V_j\}_{j\in J}$. Then the following are equivalent*:

(i) $f = \sum_{j\in J} \Gamma_j^* \Lambda_j f$, $\forall f \in H$.
(ii) $f = \sum_{j\in J} \Lambda_j^* \Gamma_j f$, $\forall f \in H$.
(iii) $\langle f, g\rangle = \sum_{j\in J} \langle \Lambda_j f, \Gamma_j g\rangle$, $\forall f, g \in H$.

In case the equivalent conditions are satisfied, $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are dual g-frames for H w.r.t. $\{V_j\}_{j\in J}$.

Lemma 3.2 ([14]) *Let $\Lambda_j \in L(H, V_j)$ for every $j \in J$, and let $\{e_{j,k}\}_{k\in K_j}$ be an o.n.b. for V_j. If $u_{j,k}$ is defined by $u_{j,k} = \Lambda_j^* e_{j,k}$, then $\{\Lambda_j\}_{j\in J}$ is a g-frame (g-Bessel sequence) for H if and only if $\{u_{j,k}\}_{j\in J,k\in K_j}$ is a frame (Bessel sequence) for H.*

The following two theorems give a method to construct new dual g-frames (approximately dual g-frames) from given dual g-fromes.

Theorem 3.1 *Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be dual g-frames for H w.r.t. $\{V_j\}_{j\in J}$, and let O_1 and O_2 be two bounded operators on H such that $O_2O_1^* = I_H$ ($\|I_H - O_2O_1^*\| < 1$). Then $\{\Lambda_j O_1\}_{j\in J}$ and $\{\Gamma_j O_2\}_{j\in J}$ are dual g-frames (approximately dual g-frames) for H w.r.t. $\{V_j\}_{j\in J}$.*

Proof By a standard argument, $\{\Lambda_j\}_{j\in J}$ is a g-Bessel sequence with synthesis operator T_Λ. Since O_1 is a bounded operator on H, we see that $\{\Lambda_j O_1\}_{j\in J}$ is a g-Bessel sequence with synthesis operator $T_{O\Lambda} = O_1 T_\Lambda$. Similarly, $\{\Gamma_j O_2\}_{j\in J}$ is also a g-Bessel sequence with synthesis operator $T_{O\Gamma} = O_2 T_\Gamma$. By Lemma 3.1 we have

$$T_{O\Gamma} T_{O\Lambda}^* f = O_2 T_\Gamma T_\Lambda^* O_1^* f = O_2 O_1^* f = f$$
$$\left(\left\|I_H - T_{O\Gamma} T_{O\Lambda}^*\right\| = \left\|I_H - O_2 T_\Gamma T_\Lambda^* O_1^*\right\| = \left\|I_H - O_2 O_1^*\right\| < 1\right)$$

for all $f \in H$. □

Corollary 3.1 *Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be dual g-frames for H w.r.t. $\{V_j\}_{j\in J}$, and let T be a unitary operator on H. Then $\{\Lambda_j T\}_{j\in J}$ and $\{\Gamma_j T\}_{j\in J}$ are dual g-frames (approximately dual g-frames) for H w.r.t. $\{V_j\}_{j\in J}$.*

Theorem 3.2 *Assume that $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are dual g-frames for H w.r.t. $\{V_j\}_{j\in J}$, and let $\{\Lambda_j\}_{j\in J}$ and $\{\Delta_j\}_{j\in J}$ also be dual g-frames for H w.r.t. $\{V_j\}_{j\in J}$. Then for any $\alpha \in \mathbb{C}$, $\{\Lambda_j\}_{j\in J}$ and $\{\alpha\Gamma_j + (1-\alpha)\Delta_j\}_{j\in J}$ are dual g-frames for H w.r.t. $\{V_j\}_{j\in J}$.*

Proof By a standard argument, $\{\alpha\Gamma_j + (1-\alpha)\Delta_j\}_{j\in J}$ is a g-Bessel sequence for H w.r.t. $\{V_j\}_{j\in J}$. By Lemma 3.1 we have

$$\begin{aligned}\sum_{j\in J}\bigl\langle \Lambda_j f, \bigl(\alpha\Gamma_j + (1-\alpha)\Delta_j\bigr)g\bigr\rangle &= \sum_{j\in J}\langle \Lambda_j f, \alpha\Gamma_j g\rangle + \sum_{j\in J}\bigl\langle \Lambda_j f, (1-\alpha)\Delta_j g\bigr\rangle \\ &= \bar{\alpha}\sum_{j\in J}\langle \Lambda_j f, \Gamma_j g\rangle + (1-\bar{\alpha})\sum_{j\in J}\langle \Lambda_j f, \Delta_j g\rangle\end{aligned}$$

$$= \bar{\alpha}\langle f,g\rangle + (1-\bar{\alpha})\langle f,g\rangle$$
$$= \langle f,g\rangle$$

for all $f,g \in H$. □

Obviously, if $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are dual g-frames for H w.r.t. $\{V_j\}_{j\in J}$, then $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are approximately dual g-frames for H w.r.t. $\{V_j\}_{j\in J}$. However, the converse is not true in general. The following theorem gives a sufficient condition for approximately dual g-frames to be dual g-frames.

Theorem 3.3 *Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be approximately dual g-frames for H w.r.t. $\{V_j\}_{j\in J}$ with synthesis operators T_Λ and T_Γ, respectively. Then $T_\Lambda T_\Gamma^*$ is invertible; furthermore, the sequences $\{\Lambda_j\}_{j\in J}$ and $\{(T_\Lambda T_\Gamma^*)^{-1}\Gamma_j\}_{j\in J}$ are dual g-frames.*

Proof Since $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are approximately dual g-frames for H w.r.t. $\{V_j\}_{j\in J}$, we have $\|I_U - T_\Lambda T_\Gamma^*\| < 1$, and thus $T_\Lambda T_\Gamma^*$ is invertible on H. By Lemma 3.1 we have

$$\begin{aligned}\langle f,g\rangle &= \bigl\langle \bigl(T_\Lambda T_\Gamma^*\bigr)\bigl(T_\Lambda T_\Gamma^*\bigr)^{-1}f,g\bigr\rangle \\ &= \bigl\langle T_\Gamma^*\bigl(T_\Lambda T_\Gamma^*\bigr)^{-1}f, T_\Lambda^* g\bigr\rangle \\ &= \sum_{j\in J}\bigl\langle \Gamma_j\bigl(T_\Lambda T_\Gamma^*\bigr)^{-1}f, \Lambda_j g\bigr\rangle\end{aligned}$$

for all $f,g \in H$. □

For Theorem 3.3, a natural question is whether a g-frame always corresponds to an approximately dual g-frame. The following theorem gives an affirmative answer.

Theorem 3.4 *Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for H w.r.t. $\{V_j\}_{j\in J}$ with the synthesis operator T_Λ and frame bounds A and B. Then $\{B^{-1}\Lambda_j\}_{j\in J}$ is an approximately dual g-frame of $\{\Lambda_j\}_{j\in J}$.*

Proof Note that $\{\Lambda_j\}_{j\in J}$ is a g-frame for H w.r.t. $\{V_j\}_{j\in J}$ and T_Λ is its synthesis operator. So $\{B^{-1}\Lambda_j\}_{j\in J}$ is also g-frame with synthesis operator $B^{-1}T_\Lambda$, and

$$\begin{aligned}\bigl\|I_H - B^{-1}T_\Lambda T_\Lambda^*\bigr\| &= \sup_{\|f\|=1}\bigl|\bigl\langle \bigl(I_H - B^{-1}T_\Lambda T_\Lambda^*\bigr)f,f\bigr\rangle\bigr| \\ &\le \frac{B-A}{B} < 1.\end{aligned}$$

It follows that $\{B^{-1}\Lambda_j\}_{j\in J}$ is an approximately dual g-frame of $\{\Lambda_j\}_{j\in J}$. □

From Theorem 3.4 we know that every g-frame has at least an approximately dual g-frame. Next, we characterize all approximately dual g-frames for a given g-frame. For this purpose, we need to establish some lemmas.

Lemma 3.3 *Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for H w.r.t. $\{V_j\}_{j\in J}$, let T_Λ be its synthesis operator, and let $\{\tilde{e}_{j,k}\}_{j\in J,k\in K_j}$ be an o.n.b. for $\bigoplus_{j\in J} V_j$. Then $\{\Gamma_j\}_{j\in J}$ and $\{\Lambda_j\}_{j\in J}$ are approximately dual g-frames if and only if $\Gamma_j^* e_{j,k} = T\tilde{e}_{j,k}$ ($\forall j\in J, k\in K_j$), where $T : \bigoplus_{j\in J} V_j \to H$ is a linear bounded operator such that $\|I_H - TT_\Lambda^*\| < 1$.*

Proof Necessity. Suppose $\{\Gamma_j\}_{j\in J}$ is an approximately dual g-frame of $\{\Lambda_j\}_{j\in J}$. Then $\{\Gamma_j\}_{j\in J}$ is a g-frame, and $\|I_H - T_\Gamma T_\Lambda^*\| < 1$, where T_Γ is the synthesis operator of $\{\Gamma_j\}_{j\in J}$. Notice that

$$T_\Gamma \tilde{e}_{j,k} = T_\Gamma\big(\{\delta_{j,i}e_{i,k}\}_{i\in J}\big) = \sum_{i\in J}\Gamma_j^*\delta_{j,i}e_{i,k} = \Gamma_j^* e_{j,k}.$$

Denote $T = T_\Gamma$. Then $T : \bigoplus_{j\in J} V_j \to H$ is a linear bounded operator satisfying $\|I_H - TT_\Lambda^*\| < 1$ and $\Gamma_j^* e_{j,k} = T\tilde{e}_{j,k}$ for $j \in J$, $k \in K_j$.

Next, we prove the converse. Suppose $T : \bigoplus_{j\in J} V_j \to H$ is a linear bounded operator satisfying $\|I_H - TT_\Lambda^*\| < 1$ and $\Gamma_j^* e_{j,k} = T\tilde{e}_{j,k}$ for $j \in J$, $k \in K_j$. Then

$$\begin{aligned}
TT_\Lambda^* f &= T\big(\{\Lambda_j f\}_{j\in J}\big) \\
&= T\Big(\sum_{j\in J}\sum_{k\in K_j}\langle\Lambda_j f, e_{j,k}\rangle\tilde{e}_{j,k}\Big) \\
&= \sum_{j\in J}\sum_{k\in K_j}\langle\Lambda_j f, e_{j,k}\rangle T\tilde{e}_{j,k} \\
&= \sum_{j\in J}\sum_{k\in K_j}\langle\Lambda_j f, e_{j,k}\rangle\Gamma_j^* e_{j,k} \\
&= \sum_{j\in J}\Gamma_j^*\sum_{k\in K_j}\langle\Lambda_j f, e_{j,k}\rangle e_{j,k} \\
&= \sum_{j\in J}\Gamma_j^*\Lambda_j f
\end{aligned}$$

for $f \in H$. Since $\{\tilde{e}_{j,k}\}_{j\in J,k\in K_j}$ is an o.n.b. for $\bigoplus_{j\in J} V_j$, we have that $\{T\tilde{e}_{j,k}\}_{j\in J,k\in K_j}$ is a Bessel sequence for H. Let $u_{j,k} = T\tilde{e}_{j,k}$. Then $u_{j,k} = \Gamma_j^* e_{j,k}$. By Lemma 3.2 $\{\Gamma_j\}_{j\in J}$ is a g-Bessel sequence for H w.r.t. $\{V_j\}_{j\in J}$. Let T_Γ be the synthesis operator of $\{\Gamma_j\}_{j\in J}$. Then $T = T_\Gamma$ and $\|I_H - T_\Gamma T_\Lambda^*\| < 1$, and hence $\{\Gamma_j\}_{j\in J}$ and $\{\Lambda_j\}_{j\in J}$ are approximately dual g-frames. □

From Lemma 3.3 we know that T is very important. The following lemma gives an explicit expression of T in Lemma 3.3.

Lemma 3.4 *Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for H w.r.t. $\{V_j\}_{j\in J}$ with the synthesis operator T_Λ and the frame operator S_Λ. Then $\|I_H - TT_\Lambda^*\| < 1$ ($T : \bigoplus_{j\in J} V_j \to H$) if and only if $T = S_\Lambda^{-1}T_\Lambda + W(I - T_\Lambda^* QS_\Lambda^{-1}T_\Lambda)$, where I is the identity operator on $\bigoplus_{j\in J} V_j$, and $W : \bigoplus_{j\in J} V_j \to H$ and $Q : H \to H$ are linear bounded operators satisfying $\|WT_\Lambda^*(I_H - Q)\| < 1$.*

Proof First, we suppose that $\|I_H - TT_\Lambda^*\| < 1$ ($T \in L(\bigoplus_{j\in J} V_j, H)$). Then TT_Λ^* is invertible. Let $W = T$ and $Q = (TT_\Lambda^*)^{-1}$. Then

$$\begin{aligned}
S_\Lambda^{-1}T_\Lambda + W\big(I - T_\Lambda^* QS_\Lambda^{-1}T_\Lambda\big) &= S_\Lambda^{-1}T_\Lambda + T\big(I - T_\Lambda^*\big(TT_\Lambda^*\big)^{-1}S_\Lambda^{-1}T_\Lambda\big) \\
&= S_\Lambda^{-1}T_\Lambda + T - TT_\Lambda^*\big(TT_\Lambda^*\big)^{-1}S_\Lambda^{-1}T_\Lambda \\
&= S_\Lambda^{-1}T_\Lambda + T - S_\Lambda^{-1}T_\Lambda = T.
\end{aligned}$$

Conversely, assume that $T = S_\Lambda^{-1}T_\Lambda + W(I - T_\Lambda^* Q S_\Lambda^{-1} T_\Lambda)$. Then

$$\begin{aligned} TT_\Lambda^* &= \big(S_\Lambda^{-1}T_\Lambda + W\big(I - T_\Lambda^* Q S_\Lambda^{-1} T_\Lambda\big)\big)T_\Lambda^* \\ &= S_\Lambda^{-1}T_\Lambda T_\Lambda^* + WT_\Lambda^* - WT_\Lambda^* Q S_\Lambda^{-1} T_\Lambda T_\Lambda^* \\ &= I_U + WT_\Lambda^* - WT_\Lambda^* Q. \end{aligned}$$

Therefore

$$\|I_H - TT_\Lambda^*\| = \|WT_\Lambda^*(I_H - Q)\| < 1. \qquad \square$$

Now, we turn to characterizing all approximately dual g-frames for a given g-frame.

Theorem 3.5 *Let $\{\Gamma_j \in L(H, V_j)\}$ be a sequence, and let $\{\Lambda_j\}_{j\in J}$ be a g-frame for H w.r.t. $\{V_j\}_{j\in J}$ with the synthesis operator T_Λ and the frame operator S_Λ. Then $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are approximately dual g-frames if and only if*

$$\Gamma_j^* e_{j,k} = S_\Lambda^{-1}\Lambda_j^* e_{j,k} + W\tilde{e}_{j,k} - \sum_{j'\in J}\sum_{k'\in K_j}\big\langle Q S_\Lambda^{-1}\Lambda_j^* e_{j,k}, \Lambda_{j'}^* e_{j',k'}\big\rangle W\tilde{e}_{j',k'}, \quad \forall j\in J, k\in K_j, \tag{3.1}$$

where $W : \bigoplus_{j\in J} V_j \to H$ and $Q : H \to H$ are linear bounded operators satisfying $\|WT_\Lambda^*(I_H - Q)\| < 1$.

Proof First, we assume that $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are approximately dual g-frames. By Lemmas 3.3 and 3.4 we have

$$\Gamma_j^* e_{j,k} = \big(S_\Lambda^{-1}T_\Lambda + W\big(I - T_\Lambda^* Q S_\Lambda^{-1} T_\Lambda\big)\big)\tilde{e}_{j,k}, \tag{3.2}$$

where I is the identity operator on $\bigoplus_{j\in J} V_j$, and $W : \bigoplus_{j\in J} V_j \to H$ and $Q : H \to H$ are linear bounded operators satisfying $\|WT_\Lambda^*(I_U - Q)\| < 1$. Set $z_{j,k} = W\tilde{e}_{j,k}$. We know that $\{z_{j,k}\}_{j\in J,k\in K_j}$ is a Bessel sequence for H. Using the notations $u_{j,k} := \Lambda_j^* e_{j,k}$ and $v_{j,k} := \Gamma_j^* e_{j,k}$, we have

$$\big\{\big\langle Q S_\Lambda^{-1} u_{j,k}, u_{j',k'}\big\rangle\big\}_{j'\in J, k'\in K_j} \in l^2$$

for any $j \in J$ and $k \in K_j$. So $\sum_{j'\in J}\sum_{k'\in K_j}\langle Q S_\Lambda^{-1} u_{j,k}, u_{j',k'}\rangle z_{j',k'}$ converges unconditionally. By (3.2) we have

$$\begin{aligned} v_{j,k} &= S_\Lambda^{-1}T_\Lambda\tilde{e}_{j,k} + W\tilde{e}_{j,k} - WT_\Lambda^* Q S_\Lambda^{-1} T_\Lambda \tilde{e}_{j,k} \\ &= S_\Lambda^{-1}u_{j,k} + z_{j,k} - WT_\Lambda^* Q S_\Lambda^{-1} u_{j,k} \\ &= S_\Lambda^{-1}u_{j,k} + z_{j,k} - W\Bigg(\sum_{j'\in J}\sum_{k'\in K_j}\big\langle \Lambda_{j'} Q S_\Lambda^{-1} u_{j,k}, e_{j',k'}\big\rangle \tilde{e}_{j',k'}\Bigg) \\ &= S_\Lambda^{-1}u_{j,k} + z_{j,k} - \sum_{j'\in J}\sum_{k'\in K_j}\big\langle Q S_\Lambda^{-1} u_{j,k}, \Lambda_{j'}^* e_{j',k'}\big\rangle W\tilde{e}_{j',k'} \\ &= S_\Lambda^{-1}u_{j,k} + z_{j,k} - \sum_{j'\in J}\sum_{k'\in K_j}\big\langle Q S_\Lambda^{-1} u_{j,k}, u_{j',k'}\big\rangle z_{j',k'}, \end{aligned}$$

that is,

$$\Gamma_j^* e_{j,k} = S_\Lambda^{-1}\Lambda_j^* e_{j,k} + W\tilde{e}_{j,k} - \sum_{j'\in J}\sum_{k'\in K_j}\langle QS_\Lambda^{-1}\Lambda_j^* e_{j,k}, \Lambda_{j'}^* e_{j',k'}\rangle W\tilde{e}_{j',k'}$$

for all $j \in J$, $k \in K_j$.

Now we prove the converse. Assume that (3.1) holds. For any $f \in H$, using the notations $u_{j,k} := \Lambda_j^* e_{j,k}$, $v_{j,k} := \Gamma_j^* e_{j,k}$, and $z_{j,k} := W\tilde{e}_{j,k}$, by a standard argument we get that $\sum_{j\in J}\sum_{k\in K_j}\langle f, u_{j,k}\rangle S_\Lambda^{-1} u_{j,k}$ converges unconditionally to f. Therefore

$$\begin{aligned}
\sum_{j\in\mathcal{J}}\Gamma_j^*\Lambda_j f &= \sum_{j\in\mathcal{J}}\Gamma_j^*\sum_{k\in\mathcal{K}_j}\langle\Lambda_j f, e_{j,k}\rangle e_{j,k} \\
&= \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}_j}\langle f, \Lambda_j^* e_{j,k}\rangle\Gamma_j^* e_{j,k} \\
&= \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}_j}\langle f, u_{j,k}\rangle v_{j,k} \\
&= \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}_j}\langle f, u_{j,k}\rangle\left(S_\Lambda^{-1}u_{j,k} + z_{j,k} - \sum_{j'\in\mathcal{J}}\sum_{k'\in\mathcal{K}_j}\langle QS_\Lambda^{-1}u_{j,k}, u_{j',k'}\rangle z_{j',k'}\right) \\
&= \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}_j}\langle f, u_{j,k}\rangle S_\Lambda^{-1}u_{j,k} + \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}_j}\langle f, u_{j,k}\rangle z_{j,k} \\
&\quad - \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}_j}\langle f, u_{j,k}\rangle\sum_{j'\in\mathcal{J}}\sum_{k'\in\mathcal{K}_j}\langle QS_\Lambda^{-1}u_{j,k}, u_{j',k'}\rangle z_{j',k'} \\
&= f + \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}_j}\langle f, u_{j,k}\rangle z_{j,k} - \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}_j}\left\langle Q\sum_{j'\in\mathcal{J}}\sum_{k'\in\mathcal{K}_j}\langle f, u_{j,k}\rangle S_\Lambda^{-1}u_{j,k}, u_{j',k'}\right\rangle z_{j',k'} \\
&= f + \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}_j}\langle f, u_{j,k}\rangle z_{j,k} - \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}_j}\langle Qf, u_{j',k'}\rangle z_{j',k'} \\
&= f + \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}_j}\langle f - Qf, u_{j,k}\rangle z_{j,k}
\end{aligned}$$

for all $f \in H$. Next, we prove that $\{\Gamma_j\}_{j\in J}$ is a g-Bessel sequence for H w.r.t. $\{V_j\}_{j\in J}$. Indeed,

$$\begin{aligned}
\sum_{j\in J}\|\Gamma_j f\|^2 &= \sum_{j\in\mathcal{J}}\sum_{k\in K_j}\left|\langle\Gamma_j f, e_{j,k}\rangle\right|^2 \\
&= \sum_{j\in J}\sum_{k\in K_j}\left|\langle f, v_{j,k}\rangle\right|^2 \\
&= \sum_{j\in J}\sum_{k\in K_j}\left|\left\langle f, S_\Lambda^{-1}u_{j,k} + z_{j,k} - \sum_{j'\in J}\sum_{k'\in K_j}\langle QS_\Lambda^{-1}u_{j,k}, u_{j',k'}\rangle z_{j',k'}\right\rangle\right|^2 \\
&\le C_1\left(\sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}_j}\left|\langle f, S_\Lambda^{-1}u_{j,k}\rangle\right|^2 + \sum_{j\in J}\sum_{k\in K_j}\left|\langle f, z_{j,k}\rangle\right|^2\right. \\
&\quad \left. + \sum_{j\in J}\sum_{k\in K_j}\left|\left\langle Q^*\sum_{j'\in J}\sum_{k'\in K_j}\langle f, z_{j',k'}\rangle u_{j',k'}, S_\Lambda^{-1}u_{j,k}\right\rangle\right|^2\right)
\end{aligned}$$

$$\begin{aligned}
&\le C_2\left(\|f\|^2 + \left\|Q^*\sum_{j'\in J}\sum_{k'\in K_j}\langle f, z_{j',k'}\rangle u_{j',k'}\right\|^2\right)\\
&\le C_3\left(\|f\|^2 + \sum_{j'\in J}\sum_{k'\in K_j}\left|\langle f, z_{j'k'}\rangle\right|^2\right)\\
&\le C_4\|f\|^2
\end{aligned}$$

for all $f \in H$, where C_1, C_2, C_3, and C_4 are different positive constants. Let T_Γ be the synthesis operator of $\{\Gamma_j\}_{j\in J}$. Then

$$\begin{aligned}
\left\|\left(I_H - T_\Gamma T_\Lambda^*\right)f\right\| &= \left\|\sum_{j\in J}\sum_{k\in K_j}\langle f - Qf, u_{j,k}\rangle z_{j,k}\right\|\\
&= \left\|\sum_{j\in J}\sum_{k\in K_j}\langle f - Qf, u_{j,k}\rangle W\tilde{e}_{j,k}\right\|\\
&= \left\|W\sum_{j\in J}\sum_{k\in K_j}\langle f - Qf, u_{j,k}\rangle\tilde{e}_{j,k}\right\|\\
&= \left\|W\sum_{j\in J}\sum_{k\in K_j}\langle\Lambda_j(f - Qf), e_{j,k}\rangle\tilde{e}_{j,k}\right\|\\
&= \left\|WT_\Lambda^*(f - Qf)\right\|\\
&\le \left\|WT_\Lambda^*(I_H - Q)\right\|\|f\|
\end{aligned}$$

for all $f \in H$. Therefore $\|I_H - T_\Gamma T_\Lambda^*\| < 1$, and thus $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are approximately dual g-frames. □

4 Perturbations of approximately dual g-frames

The stability of frames is of great importance in frame theory, and it is studied widely by a lot of authors ([4, 18]). In this section, we show that, under some conditions, approximately dual g-frames and g-frames are stable under some perturbations. We first introduce some lemmas.

Lemma 4.1 ([17]) *Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for H w.r.t. $\{V_j\}_{j\in J}$ with bounds A and B, $\lambda_1, \lambda_2 \in (-1, 1)$, $\mu \ge 0$, and $\max\{\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2\} < 1$. If $\{\Gamma_j \in L(H, V_j)\}_{j\in J}$ satisfies*

$$\left\|\sum_{j\in J_1}(\Lambda_j - \Gamma_j)^* g_j\right\| \le \lambda_1\left\|\sum_{j\in J_1}\Lambda_j^* g_j\right\| + \lambda_2\left\|\sum_{j\in J_1}\Gamma_j^* g_j\right\| + \mu\left(\sum_{j\in J_1}\|g_j\|^2\right)^{\frac{1}{2}}$$

for an arbitrary finite subset $J_1 \subset J$ and $g_j \in V_j$, then $\{\Gamma_j\}_{j\in J}$ is a g-frame for H w.r.t. $\{V_j\}_{j\in J}$ with bounds

$$\frac{((1-\lambda_1)\sqrt{A} - \mu)^2}{(1+\lambda_2)^2}, \qquad \frac{((1+\lambda_1)\sqrt{B} + \mu)^2}{(1-\lambda_2)^2}.$$

Lemma 4.2 ([14]) *Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for H w.r.t. $\{V_j\}_{j\in J}$. Then for $g_j \in V_j$ satisfying $f = \sum_{j\in J} \Lambda_j^* g_j$, we have*

$$\sum_{j\in J} \|g_j\|^2 \geq \sum_{j\in J} \|\tilde{\Lambda}_j f\|^2.$$

Lemma 4.3 ([14]) *$\{\Lambda_j\}_{j\in J}$ is a g-Bessel sequence with an upper bound B if and only if*

$$\left\| \sum_{j\in J_1} \Lambda_j^* g_j \right\|^2 \leq B \sum_{j\in J_1} \|g_j\|^2, \quad g_j \in V_j,$$

where J_1 is an arbitrary finite subset of J.

Theorem 4.1 *Let $\Lambda_j \in L(H, V_j)$, let $\{\Gamma_j\}_{j\in J}$ be a g-frame for H w.r.t. $\{V_j\}_{j\in J}$ with bounds A and B and the synthesis operator T_Λ, and let $\{\Delta_j\}_{j\in J}$ be alternate dual for $\{\Gamma_j\}_{j\in J}$ with the upper bound C and the synthesis operator T_Δ. Assume that there are constants $\lambda_1, \mu \geq 0$, and $0 \leq \lambda_2 < 1$ satisfying*

$$\left\| \sum_{j\in J_1} (\Gamma_j - \Lambda_j)^* g_j \right\| \leq \lambda_1 \left\| \sum_{j\in J_1} \Gamma_j^* g_j \right\| + \lambda_2 \left\| \sum_{j\in J_1} \Lambda_j^* g_j \right\| + \mu \left(\sum_{j\in J_1} \|g_j\|^2 \right)^{\frac{1}{2}}, \tag{4.1}$$

where J_1 is an arbitrary finite subset of J, and $g_j \in V_j$. If

$$\lambda_1 + \lambda_2 \sqrt{BC} \left(1 + \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{B}}}{1 - \lambda_2} \right) + \mu \sqrt{C} < 1,$$

then $\{\Lambda_j\}_{j\in J}$ and $\{\Delta_j\}_{j\in J}$ are approximately dual g-frames.

Proof By Lemma 4.2 we have $C \geq \frac{1}{A}$ and $BC \geq \frac{B}{A} \geq 1$. Note that

$$\lambda_1 + \lambda_2 \sqrt{BC} \left(1 + \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{B}}}{1 - \lambda_2} \right) + \mu \sqrt{C} < 1.$$

It follows that $\lambda_1 + \frac{\mu}{\sqrt{A}} < 1$. By Lemma 4.1 $\{\Lambda_j\}_{j\in J}$ is a g-frame for H w.r.t. $\{V_j\}_{j\in J}$ with bounds

$$A \left(1 - \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{A}}}{1 + \lambda_2} \right)^2, \qquad B \left(1 + \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{B}}}{1 - \lambda_2} \right)^2.$$

Denote by T_Λ the synthesis operator of $\{\Lambda_j\}_{j\in J}$. From (4.1) we have

$$\|T_\Gamma c - T_\Lambda c\| \leq \lambda_1 \|T_\Gamma c\| + \lambda_2 \|T_\Lambda c\| + \mu \|c\|_{\bigoplus_{j\in J} V_j} \tag{4.2}$$

for any $c = \{c_j\}_{j\in J} \in \bigoplus_{j\in J} V_j$. Take $c = T_\Delta^* f$ in (4.2). Then

$$\begin{aligned} \left\| \left(I_H - T_\Lambda T_\Delta^* \right) f \right\| &\leq \lambda_1 \|f\| + \lambda_2 \left\| T_\Lambda T_\Delta^* f \right\| + \mu \left\| T_\Delta^* f \right\|_{\bigoplus_{j\in J} V_j} \\ &\leq \lambda_1 \|f\| + \lambda_2 \sqrt{C} \|T_\Lambda\| \|f\| + \mu \sqrt{C} \|f\| \end{aligned}$$

$$\leq \lambda_1\|f\| + \lambda_2\sqrt{BC}\left(1 + \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{B}}}{1-\lambda_2}\right)\|f\| + \mu\sqrt{C}\|f\|$$
$$= \left(\lambda_1 + \lambda_2\sqrt{BC}\left(1 + \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{B}}}{1-\lambda_2}\right) + \mu\sqrt{C}\right)\|f\|$$

for any $f \in H$. So

$$\left\|I_H - T_\Lambda T_\Delta^*\right\| \leq \lambda_1 + \lambda_2\sqrt{BC}\left(1 + \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{B}}}{1-\lambda_2}\right) + \mu\sqrt{C} < 1.$$

Thus $\{\Lambda_j\}_{j\in J}$ and $\{\Delta_j\}_{j\in J}$ are approximately dual g-frames if $\lambda_1 + \lambda_2\sqrt{BC}(1 + \frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{B}}}{1-\lambda_2}) + \mu\sqrt{C} < 1$. □

From Theorem 4.1 we can obtain immediately the following corollary.

Corollary 4.1 *Let $\Lambda_j \in L(H, V_j)$, let $\{\Gamma_j\}_{j\in J}$ be a g-frame for H w.r.t. $\{V_j\}_{j\in J}$ with bounds A and B and the synthesis operator T_Γ, and let $\{\Delta_j\}_{j\in J}$ be the canonical dual for $\{\Gamma_j\}_{j\in J}$ with the synthesis operator T_Δ. Suppose that there are constants $\lambda_1, \mu \geq 0$, and $0 \leq \lambda_2 < 1$ such that*

$$\left\|\sum_{j\in J_1}(\Gamma_j - \Lambda_j)^* g_j\right\| \leq \lambda_1\left\|\sum_{j\in J_1}\Gamma_j^* g_j\right\| + \lambda_2\left\|\sum_{j\in J_1}\Lambda_j^* g_j\right\| + \mu\left(\sum_{j\in J_1}\|g_j\|^2\right)^{\frac{1}{2}}, \tag{4.3}$$

where J_1 is an arbitrary finite subset of J, and $g_j \in V_j$. If $\lambda_1 + \lambda_2\sqrt{\frac{B}{A}}(1 + \frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{B}}}{1-\lambda_2}) + \frac{\mu}{\sqrt{A}} < 1$, then $\{\Lambda_j\}_{j\in J}$ and $\{\Delta_j\}_{j\in J}$ are approximately dual g-frames.

Note that $\{\Gamma_j\}_{j\in J}$ is a Parseval g-frame for H w.r.t. $\{V_j\}_{j\in J}$. Then $\{\Gamma_j\}_{j\in J}$ is the canonical dual for itself. We have the following:

Corollary 4.2 *Let $\Lambda_j \in L(H, V_j)$, and let $\{\Gamma_j\}_{j\in J}$ be a Parseval g-frame for H w.r.t. $\{V_j\}_{j\in J}$. Assume that there are constants $\lambda, \mu \geq 0$ such that*

$$\left\|\sum_{j\in J_1}(\Gamma_j - \Lambda_j)^* g_j\right\| \leq \lambda\left\|\sum_{j\in J_1}\Gamma_j^* g_j\right\| + \mu\left(\sum_{j\in J_1}\|g_j\|^2\right)^{\frac{1}{2}} \tag{4.4}$$

for an arbitrary finite subset $J_1 \subset J$ and $g_j \in V_j$. If $\lambda + \mu < 1$, then $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are approximately dual g-frames.

Corollary 4.3 *Let $\{\Lambda_j \in L(H, V_j)\}_{j\in J}$ be a sequence, and let $\{\Gamma_j\}_{j\in J}$ be a g-frame for H w.r.t. $\{V_j\}_{j\in J}$. Also, let $\{\Delta_j\}_{j\in J}$ be an alternate dual for $\{\Gamma_j\}_{j\in J}$ with the upper bound C. If there exists a constant R such that $CR < 1$ and*

$$\sum_{j\in J}\left\|(\Gamma_j - \Lambda_j)f\right\|^2 \leq R\|f\|^2, \quad \forall f \in H, \tag{4.5}$$

then $\{\Lambda_j\}_{j\in J}$ and $\{\Delta_j\}_{j\in J}$ are approximately dual g-frames.

Proof Take $\lambda_1 = \lambda_2 = 0$ and $\mu = \sqrt{R}$ in Theorem 4.1. From Lemma 4.3 we know that (4.1) is equivalent to (4.5). Since $CR < 1$, we have that $\{\Lambda_j\}_{j\in J}$ and $\{\Delta_j\}_{j\in J}$ are approximately dual g-frames. □

Remark 4.1 Corollary 4.1 and Corollary 4.3 are Proposition 3.10(i) and Theorem 3.1(i) in [21], respectively. They are particular cases of our Theorem 4.1.

Theorem 4.2 *Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for H w.r.t. $\{V_j\}_{j\in J}$ with synthesis operator T_Λ and bounds A and B. Assume that $\Gamma_j \in L(H, V_j)$ for all $j \in J$ and there exist constants $\lambda_1, \lambda_2, \mu \geq 0$ such that*

$$\left\|\sum_{j\in J_1}(\Lambda_j - \Gamma_j)^* g_j\right\| \leq \lambda_1 \left\|\sum_{j\in J_1}\Lambda_j^* g_j\right\| + \lambda_2 \left\|\sum_{j\in J_1}\Gamma_j^* g_j\right\| + \mu\left(\sum_{j\in J_1}\|g_j\|^2\right)^{\frac{1}{2}} \tag{4.6}$$

for an arbitrary finite subset $J_1 \subset J$ and $g_j \in V_j$. If $\lambda_1 + \frac{\mu}{\sqrt{A}} < 1$ and $\lambda_2 + (\lambda_1\sqrt{\frac{B}{A}} + \frac{\mu}{\sqrt{A}}) \times \frac{1+\lambda_2}{1-(\lambda_1+\frac{\mu}{\sqrt{A}})} < 1$, then $\{\Gamma_j\}_{j\in J}$ is a g-frame for H w.r.t. $\{V_j\}_{j\in J}$, and $\{\tilde{\Gamma}_j\}_{j\in J}$ and $\{\Lambda_j\}_{j\in J}$ are approximately dual g-frames.

Proof We can prove the theorem by an argument similar to that of Theorem 4.1. □

5 Conclusions

For a given frame, it is usually not easy to find a dual frame. The notion of approximately dual frames was introduced by Christensen in 2010. It is a generalization of dual frames. In this paper, on one hand, we obtain the link between approximately dual g-frames and dual g-frames and characterize approximately dual g-frames. On the other hand, we give stability results of approximately dual g-frames, which cover the results obtained by other authors.

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