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Estimates for iterated commutators of multilinear square fucntions with Dini-type kernels

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Abstract

Let $T_{\Pi\vec{b}}$ be the commutator generated by a multilinear square function and Lipschitz functions with kernel satisfying Dini-type condition. We show that $T_{\Pi\vec{b}}$ is bounded from product Lebesgue spaces into Lebesgue spaces, Lipschitz spaces, and Triebel–Lizorkin spaces.

Keywords: Multilinear square fucntions; Triebel–Lizorkin spaces; Lipschitz spaces; Iterated commutators

1 Introduction

Let $A(x)$ be an elliptic $n \times n$ matrix with complex-valued entries that are merely bounded and measurable, and let $T = \mathrm{div}(A(x)\nabla)$. The well-known problem of Kato is to show the boundedness of $T^{1/2}$ from the Sobolev space $H^1(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Fabes et al. [6] studied a family of multilinear square functions and applied it to the Kato problem. In fact, they obtained a collection of multilinear Littlewood–Paley estimates and then applied them to two problems in partial differential equations. The first problem is the estimation of the square root of an elliptic operator in divergence form, and the second is the estimation of solutions to the Cauchy problem for nondivergence-form parabolic equations. Such a square function has important applications in PDEs and other fields' we refer to [1–7, 9, 10, 13, 14, 17–19] and the references therein. We now give the definition of the multilinear square function of type $\omega(t)$.

Suppose that $\omega(t) : [0,\infty) \mapsto [0,\infty)$ is a nondecreasing function with $0 < \omega(1) < \infty$. For $a > 0$, we say that $\omega \in \mathrm{Dini}(a)$ if

$$|\omega|_{\mathrm{Dini}(a)} = \int_0^1 \omega^a(t)\frac{dt}{t} < \infty.$$

Let $K_t(x,y_1,\dots,y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$. We say that $K_t(x,y_1,\dots,y_m)$ is a kernel of type $\omega(t)$ if there is a positive constant A such that the following conditions hold.

Size condition:

$$\left(\int_0^\infty \bigl|K_t(x,y_1,\dots,y_m)\bigr|^2\frac{dt}{t}\right)^{\frac{1}{2}} \le \frac{A}{(\sum_{j=1}^m |x-y_j|)^{mn}}. \tag{1.1}$$

Smoothness condition:

$$\begin{aligned} &\left(\int_0^\infty \left|K_t(z,y_1,\dots,y_m)-K_t(x,y_1,\dots,y_m)\right|^2\frac{dt}{t}\right)^{\frac{1}{2}} \\ &\quad\le \frac{A}{(\sum_{j=1}^m |x-y_j|)^{mn}}\omega\left(\frac{|z-x|}{\sum_{j=1}^m |x-y_j|}\right) \end{aligned} \tag{1.2}$$

whenever $|z-x|\le \frac{1}{2}\max_{j=1}^m |x-y_j|$, and

$$\begin{aligned} &\left(\int_0^\infty \left|K_t(x,y_1,\dots,y_j,\dots,y_m)-K_t\big(x,y_1,\dots,y_j',\dots,y_m\big)\right|^2\frac{dt}{t}\right)^{\frac{1}{2}} \\ &\quad\le \frac{A}{(\sum_{j=1}^m |x-y_j|)^{mn}}\omega\left(\frac{|y_j-y_j'|}{\sum_{j=1}^m |x-y_j|}\right) \end{aligned} \tag{1.3}$$

whenever $|y_j-y_j'|\le \frac{1}{2}\max_{j=1}^m |x-y_j|$.

For any $x\notin \bigcap_{j=1}^m \operatorname{supp} f_j$ and $f_j\in C_c^\infty(\mathbb{R}^n)$, we say T is a multilinear square function of type $\omega(t)$ if

$$T(\vec{f})(x)=\left(\int_0^\infty \left|\int_{(\mathbb{R}^n)^m} K_t(x,y_1,\dots,y_m)\prod_{j=1}^m f_j(y_j)\,dy_1\cdots dy_m\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}. \tag{1.4}$$

In this paper, we always assume that T can be extended to bounded operators from $L^{q_1}\times\cdots\times L^{q_m}$ to L^q for some $1<q,q_1,\dots,q_m<\infty$ with $\frac{1}{q_1}+\cdots+\frac{1}{q_m}=\frac{1}{q}$.

Remark 1.1 When $\omega(x)=x^\gamma$ for some $\gamma>0$, the boundedness of a multilinear square function was studied by Xue et al. [18].

Definition 1.2 (Iterated commutators) Given a collection of locally integrable functions $\vec{b}=(b_1,\dots,b_m)$, the iterated commutator of a multilinear square function is defined by

$$\begin{aligned} &T_{\Pi\vec{b}}(\vec{f})(x) \\ &\quad=\left(\int_0^\infty \left|\int_{(\mathbb{R}^n)^m}\prod_{j=1}^m\big(b_j(x)-b_j(y_j)\big)K(x,y_1,\dots,y_m)K_t(x,y_1,\dots,y_m)\right.\right. \\ &\qquad\left.\left.\times\prod_{j=1}^m f_j(y_j)\,dy_1\cdots dy_m\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}. \end{aligned} \tag{1.5}$$

Definition 1.3 (Commutators in the jth entry) Given a collection of locally integrable functions $\vec{b}=(b_1,\dots,b_m)$, we define the commutator of a multilinear square function T as

$$[\vec{b},T](\vec{f})=T_{\vec{b}}(f_1,\dots,f_m)=\sum_{j=1}^m T_{\vec{b}}^j(\vec{f}),$$

where each term is the commutator of b_j, and T in the jth entry of T, that is,

$$T_{\vec{b}}^j(\vec{f})=b_jT(f_1,\dots,f_j,\dots,f_m)-T(f_1,\dots,b_jf_j,\dots,f_m).$$

For the commutators generated by the multilinear Calderón–Zygmud-type singular integrals and Lipschitz functions with the kernel of standard estimates, Wang and Xu [16] and Mo and Lu [11] obtained the boundedness from a product of Lebesgue spaces to the Lebesgue space, to the homogenous Triebel–Lizorkin space, and to Lipschitz spaces, respectively. Motivated by these results, we study the boundedness of commutators generated by the multilinear square functions and Lipschitz functions. The main results of this paper are as follows.

Theorem 1.1 *Let T be a multilinear square function of type $\omega(t)$ with $\omega \in \mathrm{Dini}(1)$. Suppose $b_j \in \dot{\wedge}_{\beta_j}$ with $0 < \beta_j < 1$ for $j = 1, \dots, m$ and $\beta = \beta_1 + \cdots + \beta_m$. If $1 < p_1, \dots, p_m < \infty$, $0 < q < \infty$, and $1/p_j > \beta_j/n$ with $1/q = 1/p_1 + \cdots + 1/p_m - \beta/n$, then $T_{\Pi\vec{b}}$ can be extended to a bounded operator from $L^{p_1} \times \cdots \times L^{p_m}$ into L^q.*

Theorem 1.2 *Let T be a multilinear square function of type $\omega(t)$ with $\omega \in \mathrm{Dini}(1)$. Suppose $b_j \in \dot{\wedge}_{\beta_j}$ with $0 < \beta_j < 1$ for $j = 1, \dots, m$ and $\beta = \beta_1 + \cdots + \beta_m$. Let $1 < p_1, \dots, p_m < \infty$, $0 < 1/p_j < \beta_j/n$, and $0 < \beta - n/p < 1$ with $1/p = 1/p_1 + \cdots + 1/p_m$. If ω satisfies*

$$\int_0^1 \frac{\omega(t)}{t^{1+\beta-n/p}}\, dt < \infty,$$

then $T_{\Pi\vec{b}}$ can be extended to a bounded operator from $L^{p_1} \times \cdots \times L^{p_m}$ into Lipschitz space $\dot{\wedge}_{\beta-n/p}$.

Theorem 1.3 *Let T be a multilinear square function of type $\omega(t)$ with $\omega \in \mathrm{Dini}(1)$. Suppose $b_j \in \dot{\wedge}_{\beta_j}$ with $0 < \beta_j < 1$ for $j = 1, \dots, m$ and $\beta = \beta_1 + \cdots + \beta_m$. If $1 < p_1, \dots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$ and ω satisfies*

$$\int_0^1 \frac{\omega(t)}{t^{1+\beta}}\, dt < \infty,$$

then $T_{\Pi\vec{b}}$ can be extended to a bounded operator from $L^{p_1} \times \cdots \times L^{p_m}$ into the Triebel–Lizorkin space $\dot{F}_p^{\beta,\infty}$.

The paper is organized as follows. Some definitions and preliminaries are given in Sect. 2. In Sect. 3, we focus ourselves on a key lemma, which will be used in the proof of Theorem 1.1. The proofs of Theorems 1.2 and 1.3 are given in Sect. 4.

2 Preliminaries

Definition 2.1 For $\delta > 0$, M_δ is the maximal function defined by

$$M_\delta f(x) = M\left(|f|^\delta\right)^{\frac{1}{\delta}}(x) = \left(\sup_{Q \ni x} \frac{1}{|Q|} \int_Q \left|f(y)\right|^\delta dy\right)^{\frac{1}{\delta}}.$$

In addition, $M^\sharp$ is the sharp maximal function of Feffeman and Stein,

$$M^\sharp f(x) = \sup_{Q \ni x} \inf_c \frac{1}{|Q|} \int_Q \left|f(y) - c\right| dy \approx \sup_{B \ni x} \frac{1}{|Q|} \int_Q \left|f(y) - f_Q\right| dy,$$

and

$$M_\delta^\sharp f(x) = M^\sharp\left(|f|^\delta\right)^{\frac{1}{\delta}}(x).$$

Given a locally integrable function f, for $0 \le \beta < n$, we define the fractional maximal function

$$M_{r,\beta}f(x) = \sup_{x\in B}\left(\frac{1}{|B|^{1-\beta r/n}}\int_B \left|f(y)\right|^r dy\right)^{\frac{1}{r}}, \quad r \ge 1.$$

If $\beta = 0$ and $r = 1$, then $M_{0,1}f = Mf$ denotes the usual Hardy–Littlewood maximal function. When $\beta = 0$, we denote $M_{r,\beta}$ simply by M_r.

Chanillo [1] proved that if $0 < \beta < n, 0 < r < p < n/\beta$, and $1/q = 1/p - \beta/n$, then

$$\|M_{r,\beta}\|_q \le C\|f\|_p.$$

Definition 2.2 ([12]) For $\beta > 0$, the homogenous Lipschitz space $\dot\wedge_\beta(\mathbb{R}^n)$ is the space of functions f such that

$$\|f\|_{\dot\wedge_\beta(\mathbb{R}^n)} = \sup_{x,h\in\mathbb{R}^n, h\neq 0}\frac{|\Delta_h^{[\beta]+1}f(x)|}{|h|^\beta} < \infty,$$

where Δ_h^k denotes the kth difference operator.

To prove our theorem, we need the following lemmas.

Lemma 2.1 ([12]) *Let $b \in \dot\wedge_\beta$, $0 < \beta < 1$. For any cubes Q', Q in $\mathbb{R}^n$ such that $Q' \subset Q$, we have*

$$|b_{Q'} - b_Q| \le C\|b\|_{\dot\wedge_\beta}|Q|^{\beta/n}.$$

Lemma 2.2 ([12])

(1) *For $0 < \beta < 1$ and $1 \le q < \infty$, we have*

$$\|f\|_{\dot\wedge_\beta} \approx \sup_Q \frac{1}{|Q|^{1+n/\beta}}\int_Q |f - f_Q| \approx \sup_Q \frac{1}{|Q|^{n/\beta}}\left(\int_Q |f - f_Q|^q\right)^{\frac{1}{q}}.$$

(2) *For $0 < \beta < 1$ and $1 \le p < \infty$, we have*

$$\|f\|_{\dot F_p^{\beta,\infty}} \approx \left\|\sup_Q \frac{1}{|Q|^{1+n/\beta}}\int_Q |f - f_Q|\right\|_{L^p}.$$

Lemma 2.3 ([15]) *Let $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\vec\omega \in A_{\vec p}$. Let T be a multilinear square function of type $\omega(t)$ with $\omega \in$ Dini(1).*

(1) *If $1 < p_1, \ldots, p_m < \infty$, then*

$$\|T\vec f\|_{L^p(\nu_{\vec\omega})} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}.$$

(2) *If* $1 \le p_1, \dots, p_m < \infty$, *then*

$$\|T\vec{f}\|_{L^{p,\infty}(\nu_{\vec{\omega}})} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega_i)}.$$

3 Proof of Theorem 1.1

To prove Theorem 1.1, we need the following estimates for $T_{\Pi\vec{b}}$ and $T^j_{\vec{b}}$. We just consider the case $m = 2$ for simplicity; our method still holds for general m with little modifications.

Lemma 3.1 *Let* $0 < \delta < \epsilon < 1/2$, *and let* T *be a bilinear square function of type* $\omega(t)$ *with* $\omega \in \mathrm{Dini}(1)$.

(i) *If* $b_1 \in \dot{\wedge}_{\beta_1}$ *and* $b_2 \in \dot{\wedge}_{\beta_2}$ *with* $0 < \beta_1, \beta_2 < 1$ *such that* $\beta_1 + \beta_2 = \beta$, *then*

$$\begin{aligned} M^{\sharp}_{\delta} T_{\Pi\vec{b}}(f_1,f_2)(x) \le C \Biggl\{ &\prod_{i=1}^{2} \|b_i\|_{\dot{\wedge}_{\beta_i}} M_{\epsilon,\beta}\bigl(T(f_1,f_2)\bigr)(x) \\ &+ \|b_1\|_{\dot{\wedge}_{\beta_1}} M_{\epsilon,\beta_1}\bigl(T^2_{\vec{b}}(f_1,f_2)\bigr)(x) \\ &+ \|b_2\|_{\dot{\wedge}_{\beta_1}} M_{\epsilon,\beta_2}\bigl(T^1_{\vec{b}}(f_1,f_2)\bigr)(x) \\ &+ \prod_{i=1}^{2} \|b_i\|_{\dot{\wedge}_{\beta_i}} M_{1,\beta_1}(f_1)(x) M_{1,\beta_2}(f_2)(x) \Biggr\}. \end{aligned} \tag{3.1}$$

(ii) *If* $b_j \in \dot{\wedge}_{\beta}$, $j = 1, 2$, *and* $0 < \beta < 1$, *then*

$$M^{\sharp}_{\delta} T^j_{\vec{b}}(f_1,f_2)(x) \le C\|b_j\|_{\dot{\wedge}_{\beta}} \bigl\{ M_{\epsilon,\beta}\bigl(T(f_1,f_2)\bigr)(x) + M_{1,\beta}(f_j)(x) M(f_k)(x) \bigr\}, \tag{3.2}$$

where $k \ne j$, $k = 1, 2$.

Proof Fix a point x and a cube $Q(x_Q, l)$ containing x with side-length l, and set $Q^* = 8\sqrt{n}Q = Q(x_Q, 8\sqrt{n}l)$. We split f_j as $f_j = f_j^0 + f_j^\infty$, where $f_j^0 = f_j \chi_{Q^*}$ and $f_j^\infty = f_j \chi_{\mathbb{R}^n \setminus Q^*}$ for $j = 1, 2$. As is well known, to obtain (3.1), it suffices to show that

$$\begin{aligned} &\left(\frac{1}{|Q|} \int_Q \bigl| T_{\Pi\vec{b}}(f_1,f_2)(z) - c \bigr|^{\delta} \, dz \right)^{\frac{1}{\delta}} \\ &\quad \le C \Biggl\{ \prod_{i=1}^{2} \|b_i\|_{\dot{\wedge}_{\beta_i}} M_{\epsilon,\beta}\bigl(T(f_1,f_2)\bigr)(x) + \|b_1\|_{\dot{\wedge}_{\beta_1}} M_{\epsilon,\beta_1}\bigl(T^2_{\vec{b}}(f_1,f_2)\bigr)(x) \\ &\qquad + \|b_2\|_{\dot{\wedge}_{\beta_1}} M_{\epsilon,\beta_2}\bigl(T^1_{\vec{b}}(f_1,f_2)\bigr)(x) + \prod_{i=1}^{2} \|b_i\|_{\dot{\wedge}_{\beta_i}} M_{1,\beta_1}(f_1)(x) M_{1,\beta_2}(f_2)(x) \Biggr\} \end{aligned}$$

for some constant c to be determined.

Let $\lambda_1 = (b_1)_{Q^*}$ and $\lambda_2 = (b_2)_{Q^*}$. The sublinearity of $T_{\Pi\vec{b}}$ leads to

$$\begin{aligned} &\bigl| T_{\Pi\vec{b}}(f_1,f_2)(z) - c \bigr| \\ &\quad \le \bigl| \bigl(b_1(z) - \lambda_1\bigr)\bigl(b_2(z) - \lambda_2\bigr) T(f_1,f_2)(z) \bigr| + \bigl| \bigl(b_1(z) - \lambda_1\bigr) T^2_{\vec{b}}(f_1,f_2)(z) \bigr| \\ &\qquad + \bigl| \bigl(b_2(z) - \lambda_2\bigr) T^1_{\vec{b}}(f_1,f_2)(z) \bigr| + \bigl| T\bigl((b_1 - \lambda_1)f_1, (b_2 - \lambda_2)f_2\bigr)(z) - c \bigr|. \end{aligned}$$

Thus we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q\left|T_{\Pi\vec{b}}(f_1,f_2)(z)-c\right|^\delta dz\right)^{\frac{1}{\delta}}\\
&\quad\le\left(\frac{1}{|B|}\int_Q\left|\big(b_1(z)-\lambda_1\big)\big(b_2(z)-\lambda_2\big)T(f_1,f_2)(z)\right|^\delta dz\right)^{\frac{1}{\delta}}\\
&\qquad+\left(\frac{1}{|Q|}\int_Q\left|\big(b_1(z)-\lambda_1\big)T_{\vec{b}}^2(f_1,f_2)(z)\right|^\delta dz\right)^{\frac{1}{\delta}}\\
&\qquad+\left(\frac{1}{|Q|}\int_Q\left|\big(b_2(z)-\lambda_2\big)T_{\vec{b}}^1(f_1,f_2)(z)\right|^\delta dz\right)^{\frac{1}{\delta}}\\
&\qquad+\left(\frac{1}{|Q|}\int_Q\left|T\big((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2\big)(z)-c\right|^\delta dz\right)^{\frac{1}{\delta}}\\
&\quad\doteq T_1+T_2+T_3+T_4.
\end{aligned}$$

We now observe the elementary inequality

$$\left|b(z)-b_Q\right|\le C|Q|^{\beta/n}\|b\|_{\dot{\wedge}_\beta}$$

which follows from the fact $z\in Q$ and $b\in\dot{\wedge}_\beta$. From Hölder's inequality and the assumption $\beta_1+\beta_2=\beta$, for $0<\delta<\epsilon<1/2$, we have

$$\begin{aligned}
T_1&\le\prod_{i=1}^2\|b_i\|_{\dot{\wedge}_{\beta_i}}\left(\frac{1}{|Q|^{1-\frac{\delta\beta}{n}}}\int_Q\left|T(f_1,f_2)(z)\right|^\delta dz\right)^{\frac{1}{\delta}}\\
&\le\prod_{i=1}^2\|b_i\|_{\dot{\wedge}_{\beta_i}}\left(\frac{1}{|Q|^{1-\frac{\epsilon\beta}{n}}}\int_Q\left|T(f_1,f_2)(z)\right|^\epsilon dz\right)^{\frac{1}{\epsilon}}\\
&\le C\prod_{i=1}^2\|b_i\|_{\dot{\wedge}_{\beta_i}}M_{\epsilon,\beta}\big(T(f_1,f_2)\big)(x).
\end{aligned}$$

Similarly, we have

$$T_2\le C\|b_1\|_{\dot{\wedge}_{\beta_1}}M_{\epsilon,\beta_1}\big(T_{\vec{b}}^2(f_1,f_2)\big)(x)$$

and

$$T_3\le\|b_2\|_{\dot{\wedge}_{\beta_2}}M_{\epsilon,\beta_2}\big(T_{\vec{b}}^1(f_1,f_2)\big)(x).$$

Now we deal with T_4. Set $c=T((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)(x)$. We may bound T_4 as

$$T_4\le T_{41}+T_{42}+T_{43}+T_{44},$$

where

$$T_{41}=\left(\frac{1}{|Q|}\int_Q\left|T\big((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0\big)(z)\right|^\delta dx\right)^{\frac{1}{\delta}},$$

$$T_{42}=\left(\frac{1}{|Q|}\int_Q\left|T\left((b_1-\lambda_1)f_1^{\infty},(b_2-\lambda_2)f_2^{0}\right)(z)\right|^{\delta}dz\right)^{\frac{1}{\delta}},$$
$$T_{43}=\left(\frac{1}{|Q|}\int_Q\left|T\left((b_1-\lambda_1)f_1^{0},(b_2-\lambda_2)f_2^{\infty}\right)(z)\right|^{\delta}dz\right)^{\frac{1}{\delta}},$$

and

$$T_{44}=\left(\frac{1}{|Q|}\int_Q\left|T\left((b_1-\lambda_1)f_1^{\infty},(b_2-\lambda_2)f_2^{\infty}\right)(z)-T\left((b_1-\lambda_1)f_1^{\infty},(b_2-\lambda_2)f_2^{\infty}\right)(x)\right|^{\delta}dz\right)^{\frac{1}{\delta}}.$$

For T_{41}, by Kolmogorov's inequality and Lemma 2.3 we get

$$\begin{aligned}T_{41}&\le C\left\|T\left((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0\right)\right\|_{L^{1/2,\infty}(B,\frac{dx}{|Q|})}\\&\le\frac{C}{|Q|}\int_Q\left|(b_1-\lambda_1)f_1^0(z)\right|dz\frac{1}{|Q|}\int_Q\left|(b_2-\lambda_2)f_2^0(z)\right|dz\\&\le C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}M_{1,\beta_1}(f_1)(x)M_{1,\beta_2}(f_2)(x).\end{aligned}$$

For any $y\in\mathbb{R}^n\setminus Q^*$ and $b\in\dot\Lambda_\beta$, there exists Q' such that $Q^*\subset Q'$ and $|y-x_Q|\sim|Q'|^{1/n}$. Then, from Lemma 2.1 we have

$$\left|b(y)-b_{Q^*}\right|\le\left|b(y)-b_{Q'}\right|+|b_{Q'}-b_{Q^*}|\le C\|b\|_{\dot\Lambda_\beta}|y-x_Q|^{\beta}.\tag{3.3}$$

For any $y_2\in(Q^*)^c$ and $z\in Q$, we have $|z-y_2|\sim|y_2-x_Q|$. By Minkowski's inequality and the size condition (1.1) we get

$$\begin{aligned}T_{43}&\le\left(\frac{1}{|Q|}\int_Q\left|T\left((b_1-\lambda_1)f_1^{0},(b_2-\lambda_2)f_2^{\infty}\right)(z)\right|^{\delta}dz\right)^{\frac{1}{\delta}}\\&\le C\left(\frac{1}{|Q|}\int_Q\left|\int_{\mathbb{R}^{nm}}\left(\int_0^\infty\left|K_t(z,y_1,y_2)\right|^2\frac{dt}{t}\right)^{1/2}\right.\right.\\&\quad\left.\left.\times\left|(b_1(y_1)-\lambda_1)f_1^0(y_1)\right|\left|(b_2(y_2)-\lambda_2)f_2^{\infty}(y_2)\right|dy_1\,dy_2\right|^{\delta}dz\right)^{\frac{1}{\delta}}\\&\le C\left(\frac{1}{|Q|}\int_Q\left|\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*}\frac{|(b_1(y_1)-\lambda_1)f_1^0(y_1)||(b_2(y_2)-\lambda_2)f_2^{\infty}(y_2)|\,dy_1\,dy_2}{(|z-y_1|+|z-y_2|)^{2n}}\right|^{\delta}dz\right)^{\frac{1}{\delta}}\\&\le C\|b_1\|_{\dot\Lambda_{\beta_1}}M_{1,\beta_1}(f_1)(x)|Q|\sum_{k=1}^{\infty}\int_{2^{k+3}\sqrt{n}Q\setminus2^{k+2}\sqrt{n}Q}\frac{|f_2(y_2)(b_2(y_2)-\lambda_2)|\,dy_2}{|y_2-x_Q|^{2n}}\\&\le C\|b_1\|_{\dot\Lambda_{\beta_1}}M_{1,\beta_1}(f_1)(x)|Q|\sum_{k=1}^{\infty}\frac{1}{|2^{k+3}\sqrt{n}Q|^{1-\beta_2}}\int_{2^{k+3}\sqrt{n}Q}\left|f_2(y_2)\right|dy_2\\&\le C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}M_{1,\beta_1}(f_1)(x)M_{1,\beta_2}(f_2)(x)\sum_{k=1}^{\infty}2^{-k}\\&\le C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}M_{1,\beta_1}(f_1)(x)M_{1,\beta_2}(f_2)(x).\end{aligned}$$

By using the same technique we get $T_{42}\le C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}M_{1,\beta_1}(f_1)(x)M_{1,\beta_2}(f_2)(x)$.

To estimate T_{44}, we use Minkowski's inequality and (1.2) and (3.3). Since $(\mathbb{R}^n \setminus Q^*)^2 \subseteq \mathbb{R}^{2n} \setminus (Q^*)^2 \subseteq \bigcup_{k=1}^{\infty}(2^{k+3}\sqrt{n}Q)^2 \setminus (2^{k+2}\sqrt{n}Q)^2$, we deduce that

$$\begin{aligned}
T_{44} &\leq \left(\frac{1}{|Q|}\int_Q \left|\int_{\mathbb{R}^{nm}}\left(\int_0^\infty \left|K_t(z,\vec{y}) - K_t(x,\vec{y})\right|^2 \frac{dt}{t}\right)^{1/2}\right.\right. \\
&\quad \left.\left. \times \prod_{i=1}^{2}\left|(b_i(y_i) - \lambda_i)f_i^\infty(y_i)\right| d\vec{y}\right|^\delta dz\right)^{\frac{1}{\delta}} \\
&\leq C\left(\frac{1}{|Q|}\int_Q \left|\int_{(\mathbb{R}^n\setminus Q^*)^2} \frac{1}{(|x-y_1|+|x-y_2|)^{2n}}\omega\left(\frac{|z-x_Q|}{|x-y_1|+|x-y_2|}\right)\right.\right. \\
&\quad \left.\left. \times \prod_{i=1}^{2}\left|(b_i(y_i) - \lambda_i)f_i^\infty(y_i)\right| d\vec{y}\right|^\delta dz\right)^{\frac{1}{\delta}} \\
&\leq C\frac{1}{|Q|}\int_Q \sum_{k=1}^{\infty}\int_{(2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q)^2} \frac{1}{(|2^{k+3}\sqrt{n}Q|)^2}\omega(2^{-k}) \\
&\quad \times \prod_{i=1}^{2}\left|(b_i(y_i) - \lambda_i)f_i^\infty(y_i)\right| d\vec{y}\,dz \\
&\leq C\sum_{k=1}^{\infty}\frac{1}{(|2^{k+3}\sqrt{n}Q|)^{2-\beta_1-\beta_2}}\int_{(2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q)^2}\omega(2^{-k})\prod_{i=1}^{2}\left|f_i^\infty(y_i)\right| d\vec{y} \\
&\leq C\|b_1\|_{\dot{\wedge}_{\beta_1}}\|b_2\|_{\dot{\wedge}_{\beta_2}} M_{1,\beta_1}(f_1)(x)M_{1,\beta_2}(f_2)(x).
\end{aligned}$$

Combing all our estimates together, we obtain (3.1).

Now we are in the position to prove (3.2). It is sufficient to consider the operator with only one symbol. Fix $b \in \dot{\wedge}_\beta$ and consider the operator

$$T_b(\vec{f})(x) = \left|b(x)T(f_1,f_2)(x) - T(bf_1,f_2)(x)\right|.$$

We have to prove that

$$M_\delta^\sharp T_b(f_1,f_2)(x) \leq C\|b\|_{\dot{\wedge}_\beta}\left\{M_{\epsilon,\beta}\left(T(f_1,f_2)\right)(x) + M_{1,\beta}(f_1)(x)M(f_2)(x)\right\}.$$

Let $\lambda = b_{Q^*}$. We can control $T_b(\vec{f})(x)$ as

$$T_b(\vec{f})(x) \leq \left|\left(b(x) - \lambda\right)\right| T(f_1,\ldots,f_m)(x) + T\left((b-\lambda)f_1,\ldots,f_m\right)(x).$$

Then, for any constant c, we obtain that

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|\left|T_b(f_1,f_2)(z)\right|^\delta - |c|^\delta\right| dz\right)^{\frac{1}{\delta}} \\
&\quad \leq \left(\frac{1}{|Q|}\int_Q \left|T_b(f_1,f_2)(z) - c\right|^\delta dz\right)^{\frac{1}{\delta}} \\
&\quad \leq \left(\frac{1}{|Q|}\int_Q \left|\left(b(z)-\lambda\right)T(f_1,f_2)(z)\right|^\delta dz\right)^{\frac{1}{\delta}} + \left(\frac{1}{|Q|}\int_Q \left|T\left((b-\lambda)f_1,f_2\right)(z) - c\right|^\delta dz\right)^{\frac{1}{\delta}} \\
&\quad =: (P_1 + P_2).
\end{aligned}$$

By Hölder's inequality we get

$$\begin{aligned} P_1 &\leq C\|Q\|_{\dot{\Lambda}_\beta}\left(\frac{1}{|Q|^{1-\frac{\delta\beta}{n}}}\int_Q\left|T(f_1,f_2)(z)\right|^\delta dz\right)^{\frac{1}{\delta}} \\ &\leq C\|b\|_{\dot{\Lambda}_\beta}M_{\epsilon,\beta}\big(T(f_1,f_2)\big)(x). \end{aligned}$$

We bound the second part as

$$P_2 \leq P_{21}+P_{22}+P_{23}+P_{24},$$

where

$$\begin{aligned} P_{21} &= \left(\frac{1}{|Q|}\int_Q\left|T\big((b-\lambda)f_1^0,f_2^0\big)(z)\right|^\delta dx\right)^{\frac{1}{\delta}}, \\ P_{22} &= \left(\frac{1}{|Q|}\int_Q\left|T\big((b-\lambda)f_1^0,f_2^\infty\big)(z)\right|^\delta dz\right)^{\frac{1}{\delta}}, \\ P_{23} &= \left(\frac{1}{|Q|}\int_Q\left|T\big((b-\lambda)f_1^\infty,f_2^0\big)(z)\right|^\delta dz\right)^{\frac{1}{\delta}}, \end{aligned}$$

and

$$P_{24} = \left(\frac{1}{|Q|}\int_Q\left|T\big((b-\lambda)f_1^\infty,f_2^\infty\big)(z)-T\big((b-\lambda)f_1^\infty,f_2^\infty\big)(x)\right|^\delta dz\right)^{\frac{1}{\delta}}.$$

By Kolmogorov's inequality and Lemma 2.3 we get

$$\begin{aligned} P_{21} &\leq C\left\|T\big((b-\lambda)f_1^0,f_2^0\big)\right\|_{L^{1/2,\infty}(B,\frac{dx}{|Q|})} \\ &\leq \frac{C}{|Q|}\int_Q\left|(b-\lambda)f_1^0(z)\right|dz\frac{1}{|Q|}\int_Q\left|f_2^0(z)\right|dz \\ &\leq C\|b\|_{\dot{\Lambda}_\beta}\left|Q^*\right|^{\beta/n}\frac{1}{|Q|}\int_Q\left|f_1^0(z)\right|dz\frac{1}{|Q|}\int_Q\left|f_2^0(z)\right|dz \\ &\leq C\|b\|_{\dot{\Lambda}_\beta}M_{1,\beta}(f_1)(x)M(f_2)(x). \end{aligned}$$

By using the Minkowski inequality and (1.1) and (3.3) we obtain that

$$\begin{aligned} P_{22} &= \left(\frac{1}{|Q|}\int_Q\left|T\big((b-\lambda)f_1^0,f_2^\infty\big)(z)\right|^\delta dz\right)^{\frac{1}{\delta}} \\ &\leq C\frac{1}{|Q|}\int_Q\left|T\big((b-\lambda)f_1^0,f_2^\infty\big)(z)\right|dz \\ &\leq C\frac{1}{|Q|}\int_Q\int_{Q^*}\int_{(Q^*)^c}\left(\int_0^\infty\left|K_t(z,y_1,y_2)\right|^2\frac{dt}{t}\right)^{1/2} \\ &\quad\times\left|\big(b(y_1)-\lambda\big)f_1(y_1)\right|\left|f_2(y_2)\right|dy_2\,dy_1\,dz \\ &\leq C\frac{1}{|Q|}\int_Q\int_{Q^*}\int_{(Q^*)^c}\frac{1}{(|z-y_1|+|z-y_2|)^{2n}}\left|\big(b(y_1)-\lambda\big)f_1(y_1)\right|\left|f_2(y_2)\right|dy_2\,dy_1\,dz \end{aligned}$$

$$
\begin{aligned}
&\le C\int_{Q^*}\big|(b(y_1)-\lambda)f_1(y_1)\big|\,dy_1\int_{\mathbb{R}^n\setminus Q^*}\frac{|f_2(y_2)|\,dy_2}{|z-y_2|^{2n}}\\
&\le C\|b\|_{\dot\Lambda_\beta}\int_{Q^*}|y_1-x_Q|^{\beta}\big|f_1(y_1)\big|\,dy_1\sum_{k=1}^{\infty}\int_{2^{k+1}Q^*\setminus 2^kQ^*}\frac{|f_2(y_2)|\,dy_2}{|z-y_2|^{2n}}\\
&\le C\|b\|_{\dot\Lambda_\beta}\sum_{k=1}^{\infty}\big|2^kQ^*\big|^{-2}\big|Q^*\big|\int_{2^{k+1}Q^*\setminus 2^kQ^*}\big|f_2(y_2)\big|\,dy_2 M_{1,\beta}(f_1)(x)\\
&\le C\|b\|_{\dot\Lambda_\beta}\sum_{k=1}^{\infty}2^{-k}\frac{1}{|2^kQ^*|}\int_{2^{k+1}Q^*}\big|f_2(y_2)\big|\,dy_2 M_{1,\beta}(f_1)(x)\\
&\le C\|b\|_{\dot\Lambda_\beta}M_{1,\beta}(f_1)(x)M(f_2)(x).
\end{aligned}
$$

Similarly, we deduce that

$$
\begin{aligned}
P_{23}&=\left(\frac{1}{|Q|}\int_Q\big|T\big((b-\lambda)f_1^{\infty},f_2^0\big)(z)\big|^{\delta}\,dz\right)^{\frac{1}{\delta}}\\
&\le\frac{C}{|Q|}\int_Q\big|T\big((b-\lambda)f_1^{\infty},f_2^0\big)(z)\big|\,dz\\
&\le\frac{C}{|Q|}\int_Q\int_{Q^*}\int_{(Q^*)^c}\left(\int_0^{\infty}\big|K_t(z,y_1,y_2)\big|^2\frac{dt}{t}\right)^{1/2}\big|(b(y_1)-\lambda)f_1(y_1)\big|\big|f_2(y_2)\big|\,dy_2\,dy_1\,dz\\
&\le\frac{C}{|Q|}\int_Q\int_{Q^*}\int_{(Q^*)^c}\frac{1}{(|z-y_1|+|z-y_2|)^{2n}}\big|(b(y_1)-\lambda)f_1(y_1)\big|\big|f_2(y_2)\big|\,dy_2\,dy_1\,dz\\
&\le C\int_{(Q^*)^c}\frac{|(b(y_1)-\lambda)f_1(y_1)|\,dy_1}{|y_1-x_Q|^{2n}}\int_{Q^*}\big|f_2(y_2)\big|\,dy_2\\
&\le C\big|Q^*\big|\sum_{k=1}^{\infty}\int_{2^{k+1}Q^*\setminus 2^kQ^*}\frac{|(b(y_1)-\lambda)f_1(y_1)|\,dy_1}{|y_1-x_Q|^{2n}}M(f_2)(x)\\
&\le C\|b\|_{\dot\Lambda_\beta}\big|Q^*\big|\sum_{k=1}^{\infty}\int_{2^{k+1}Q^*\setminus 2^kQ^*}|y_1-x_Q|^{\beta-2n}\big|f_1(y_1)\big|\,dy_1M(f_2)(x)\\
&\le C\|b\|_{\dot\Lambda_\beta}\big|Q^*\big|\sum_{k=1}^{\infty}\big|2^kQ^*\big|^{\beta/n-2}\int_{2^{k+1}Q^*\setminus 2^kQ^*}\big|f_1(y_1)\big|\,dy_1M(f_2)(x)\\
&\le C\|b\|_{\dot\Lambda_\beta}\sum_{k=1}^{\infty}2^{-k}\frac{1}{|2^kQ^*|^{1-\beta/n}}\int_{2^{k+1}Q^*}\big|f_1(y_1)\big|\,dy_1M(f_2)(x)\\
&\le C\|b\|_{\dot\Lambda_\beta}M_{1,\beta}(f_1)(x)M(f_2)(x).
\end{aligned}
$$

Since $(\mathbb{R}^n\setminus Q^*)^2\subseteq\mathbb{R}^{2n}\setminus(Q^*)^2\subseteq\bigcup_{k=1}^{\infty}(2^{k+3}\sqrt{n}Q)^2\setminus(2^{k+2}\sqrt{n}Q)^2$, we can use Minkowski's inequality and (1.2) and (3.3) to get

$$
\begin{aligned}
P_{24}&\le\frac{C}{|Q|}\int_Q\big|T\big((b-\lambda)f_1^{\infty},f_2^{\infty}\big)(z)-T\big((b-\lambda)f_1^{\infty},f_2^{\infty}\big)(x)\big|\,dz\\
&\le\frac{C}{|Q|}\int_Q\int_{(\mathbb{R}^n\setminus Q^*)^2}\left(\int_0^{\infty}\big|K_t(z,\vec{y})-K_t(x,\vec{y})\big|^2\frac{dt}{t}\right)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left|\bigl(b(y_1)-\lambda\bigr)\prod_{i=1}^{2} f_i^{\infty}(y_i)\right| d\vec{y}\,dz \\
&\leq \frac{C}{|Q|}\int_Q\int_{(\mathbb{R}^n\setminus Q^*)^2}\frac{1}{(|x-y_1|+|x-y_2|)^{2n}}\omega\biggl(\frac{|z-x_Q|}{|x-y_1|+|x-y_2|}\biggr) \\
&\quad\times \left|\bigl(b(y_1)-\lambda\bigr)\prod_{i=1}^{2} f_i^{\infty}(y_i)\right| d\vec{y}\,dz \\
&\leq \frac{C}{|Q|}\int_Q\sum_{k=1}^{\infty}\int_{(2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q)^2}\frac{1}{(|2^{k+3}\sqrt{n}Q|)^2}\omega\bigl(2^{-k}\bigr) \\
&\quad\times \left|\bigl(b(y_1)-\lambda\bigr)\prod_{i=1}^{2} f_i^{\infty}(y_i)\right| d\vec{y}\,dz \\
&\leq C\frac{\|Q\|_{\dot{\Lambda}_\beta}}{|Q|}\int_Q\sum_{k=1}^{\infty}\frac{1}{(|2^{k+3}\sqrt{n}Q|)^2}\int_{(2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q)^2}\omega\bigl(2^{-k}\bigr)|y_1-x_Q|^{\beta} \\
&\quad\times \prod_{i=1}^{2}\bigl|f_i^{\infty}(y_i)\bigr|\,d\vec{y}\,dz \\
&\leq C\|b\|_{\dot{\Lambda}_\beta}\sum_{k=1}^{\infty}\frac{\omega(2^{-k})}{(|2^{k+3}\sqrt{n}Q|)^{1-\beta/n}}\int_{2^{k+3}\sqrt{n}Q}\bigl|f_1^{\infty}(y_1)\bigr|\,dy_1\frac{1}{|2^kQ^*|}\int_{2^{k+3}\sqrt{n}Q}\bigl|f_2^{\infty}(y_2)\bigr|\,dy_2 \\
&\leq C\|b\|_{\dot{\Lambda}_\beta}M_{1,\beta_1}(f_1)(x)M(f_2)(x).
\end{aligned}
$$

Thus we finish the proof of (3.2). Then Lemma 3.1 is proved. □

Proofs of Theorem 1.1 By using Lemma 3.1 and modifying the proof of Theorem 1.1 in [8] we can finish the proof of Theorem 1.1. We omit the proof. □

4 Proof of Theorems 1.2 and 1.3

For simplicity, we just consider the case $m = 2$; our method still holds for general m with little modifications.

Proof of Theorem 1.2 The theorem will be proved if we show that

$$
\sup_Q \frac{1}{|Q|^{1+\beta/n-1/p}}\int_Q\bigl|T_{\Pi\vec{b}}(\vec{f})(z)-\bigl(T_{\Pi\vec{b}}(\vec{f})\bigr)_Q\bigr|\,dz \leq C\|b_1\|_{\dot{\Lambda}_{\beta_1}}\|b_2\|_{\dot{\Lambda}_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}. \tag{4.1}
$$

Let $c = c_1 + c_2 + c_3$, which will be determined later. Then we have

$$
\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n-1/p}}\int_Q\bigl|T_{\Pi\vec{b}}(\vec{f})(z)-\bigl(T_{\Pi\vec{b}}(\vec{f})\bigr)_Q\bigr|\,dz \\
&\quad\leq \frac{1}{|Q|^{1+\beta/n-1/p}}\int_Q\bigl|T_{\Pi\vec{b}}(f_1,f_2)(z)-c\bigr|\,dz \\
&\quad\leq \frac{C}{|Q|^{1+\beta/n-1/p}}\int_Q\bigl|T_{\Pi\vec{b}}\bigl(f_1^0,f_2^0\bigr)(z)\bigr|\,dz \\
&\qquad+\frac{C}{|Q|^{1+\beta/n-1/p}}\int_Q\bigl|T_{\Pi\vec{b}}\bigl(f_1^0,f_2^{\infty}\bigr)(z)-c_1\bigr|\,dz
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{C}{|Q|^{1+\beta/n-1/p}}\int_Q\bigl|T_{\Pi\vec{b}}\bigl(f_1^\infty,f_2^0\bigr)(z)-c_2\bigr|\,dz\\
&+\frac{C}{|Q|^{1+\beta/n-1/p}}\int_Q\bigl|T_{\Pi\vec{b}}\bigl(f_1^\infty,f_2^\infty\bigr)(z)-c_3\bigr|\,dz\\
&\doteq M_1+M_2+M_3+M_4.
\end{aligned}
$$

We can choose $1<q,q_j<\infty$, $q_j<n/\beta_j<p_j$, $j=1,2$, with $1/q=1/q_1+1/q_2-(\beta_1+\beta_2)/n$. By Hölder's inequality and Theorem 1.1 we obtain

$$
\begin{aligned}
M_1&\le\frac{C}{|Q|^{1+\beta/n-1/p}}\biggl(\int_Q\bigl|T_{\Pi\vec{b}}\bigl(f_1^0,f_2^0\bigr)(z)\bigr|^q\,dz\biggr)^{1/q}|Q|^{1-1/q}\\
&\le\frac{C}{|Q|^{1+\beta/n-1/p}}|Q|^{1-1/q}\bigl\|f_1^0\bigr\|_{L^{q_1}}\bigl\|f_2^0\bigr\|_{L^{q_2}}\\
&\le C\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.
\end{aligned}
$$

For the second term, we take $c_1=T((b_1-\lambda_1)f_1^0,f_2^\infty)(x_Q)$. Then

$$
\begin{aligned}
M_2&\le\frac{C}{|Q|^{1+\beta/n-1/p}}\int_Q\biggl(\int_0^\infty\biggl|\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*}\bigl(b_1(z)-\lambda_1\bigr)\bigl(b_2(z)-\lambda_2\bigr)\\
&\quad\times K_t(z,y_1,y_2)f_1(y_1)f_2(y_2)\,dy_1\,dy_2\biggr|^2\frac{dt}{t}\biggr)^{1/2}\,dz\\
&\quad+\frac{C}{|Q|^{1+\beta/n-1/p}}\int_Q\biggl(\int_0^\infty\biggl|\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*}\bigl(b_1(z)-\lambda_1\bigr)\bigl(b_2(y_2)-\lambda_2\bigr)\\
&\quad\times K_t(z,y_1,y_2)f_1(y_1)f_2(y_2)\,dy_1\,dy_2\biggr|^2\frac{dt}{t}\biggr)^{1/2}\,dz\\
&\quad+\frac{C}{|Q|^{1+\beta/n-1/p}}\int_Q\biggl(\int_0^\infty\biggl|\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*}\bigl(b_1(y_1)-\lambda_1\bigr)\bigl(b_2(z)-\lambda_2\bigr)\\
&\quad\times K_t(z,y_1,y_2)f_1(y_1)f_2(y_2)\,dy_1\,dy_2\biggr|^2\frac{dt}{t}\biggr)^{1/2}\,dz\\
&\quad+\frac{C}{|Q|^{1+\beta/n-1/p}}\int_Q\biggl(\int_0^\infty\biggl|\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*}\bigl(b_1(y_1)-\lambda_1\bigr)\bigl(b_2(y_2)-\lambda_2\bigr)\\
&\quad\times\bigl[K_t(z,y_1,y_2)-K_t(x_Q,y_1,y_2)\bigr]f_1(y_1)f_2(y_2)\,dy_1\,dy_2\biggr|^2\frac{dt}{t}\biggr)^{1/2}\,dz\\
&\doteq M_{21}+M_{22}+M_{23}+M_{24}.
\end{aligned}
$$

By Minkowski's inequality and the size condition (1.1) we have

$$
\begin{aligned}
M_{21}&\le\frac{C}{|Q|^{1+\beta/n-1/p}}\int_Q\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*}\bigl|\bigl(b_1(z)-\lambda_1\bigr)\bigl(b_2(z)-\lambda_2\bigr)\bigr|\\
&\quad\times\biggl(\int_0^\infty\bigl|K_t(z,y_1,y_2)\bigr|^2\frac{dt}{t}\biggr)^{1/2}\bigl|f_1(y_1)f_2(y_2)\bigr|\,dy_1\,dy_2\,dz\\
&\le\frac{C}{|Q|^{1+\beta/n-1/p}}\int_Q\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*}\bigl|\bigl(b_1(z)-\lambda_1\bigr)\bigl(b_2(z)-\lambda_2\bigr)\bigr|\\
&\quad\times\frac{1}{(|z-y_1|+|z-y_2|)^{2n}}\bigl|f_1(y_1)f_2(y_2)\bigr|\,dy_1\,dy_2\,dz
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{C}{|Q|^{1+\beta/n-1/p}} \int_Q \big|\big(b_1(z)-\lambda_1\big)\big(b_2(z)-\lambda_2\big)\big| \\
&\quad \times \int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*} \frac{1}{(|z-y_1|+|z-y_2|)^{2n}} \big|f_1(y_1)f_2(y_2)\big|\,dy_1\,dy_2\,dz \\
&\le \|b_1\|_{\dot{\wedge}_{\beta_1}}\|b_2\|_{\dot{\wedge}_{\beta_2}}|Q|^{1/p}\int_{Q^*}\big|f_1(y_1)\big|\,dy_1 \\
&\quad \times \sum_{k=1}^{\infty}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q} \frac{|f_2(y_2)|}{|y_2-x_Q|^{2n}}\,dy_2 \\
&\le \|b_1\|_{\dot{\wedge}_{\beta_1}}\|b_2\|_{\dot{\wedge}_{\beta_2}}|Q|^{1/p}\int_{Q^*}\big|f_1(y_1)\big|\,dy_1 \\
&\quad \times \sum_{k=1}^{\infty}\frac{1}{|2^{k+3}\sqrt{n}Q|^2}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q} \big|f_2(y_2)\big|\,dy_2 \\
&\le \|b_1\|_{\dot{\wedge}_{\beta_1}}\|b_2\|_{\dot{\wedge}_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\sum_{k=1}^{\infty}2^{kn(-1-1/p_2)} \\
&\le \|b_1\|_{\dot{\wedge}_{\beta_1}}\|b_2\|_{\dot{\wedge}_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.
\end{aligned}
$$

We now proceed as in the estimate of M_{21}:

$$
\begin{aligned}
M_{22} &\le \frac{C}{|Q|^{1+\beta/n-1/p}} \int_Q\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*} \big|\big(b_1(z)-\lambda_1\big)\big(b_2(y_2)-\lambda_2\big)\big| \\
&\quad \times \left(\int_0^{\infty}\big|K_t(z,y_1,y_2)\big|^2\frac{dt}{t}\right)^{1/2}\big|f_1(y_1)f_2(y_2)\big|\,dy_1\,dy_2\,dz \\
&\le \frac{C}{|Q|^{1+\beta/n-1/p}} \int_Q\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*} \big|\big(b_1(z)-\lambda_1\big)\big(b_2(y_2)-\lambda_2\big)\big| \\
&\quad \times \frac{1}{(|z-y_1|+|z-y_2|)^{2n}}\big|f_1(y_1)f_2(y_2)\big|\,dy_1\,dy_2\,dz \\
&\le \frac{C}{|Q|^{1+\beta/n-1/p}} \int_Q \big|\big(b_1(z)-\lambda_1\big)\big| \\
&\quad \times \int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*} \frac{|(b_2(z)-\lambda_2)|}{(|z-y_1|+|z-y_2|)^{2n}}\big|f_1(y_1)f_2(y_2)\big|\,dy_1\,dy_2\,dz \\
&\le \frac{\|b_1\|_{\dot{\wedge}_{\beta_1}}\|b_2\|_{\dot{\wedge}_{\beta_2}}}{|Q|^{\beta_2/n-1/p}}\int_{Q^*}\big|f_1(y_1)\big|\,dy_1\sum_{k=1}^{\infty}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\frac{|f_2(y_2)|}{|y_2-x_Q|^{2n-\beta_2}}\,dy_2 \\
&\le \|b_1\|_{\dot{\wedge}_{\beta_1}}\|b_2\|_{\dot{\wedge}_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\sum_{k=1}^{\infty}2^{kn(-1-1/p_2+\beta_2/n)} \\
&\le \|b_1\|_{\dot{\wedge}_{\beta_1}}\|b_2\|_{\dot{\wedge}_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}
\end{aligned}
$$

because of $-1-1/p_2+\beta_2/n<0$.

Similarly,

$$
M_{23} \le C\|b_1\|_{\dot{\wedge}_{\beta_1}}\|b_2\|_{\dot{\wedge}_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.
$$

By Minkowski's inequality and (1.2) we have

$$\begin{aligned} M_{24} &\le \frac{C}{|Q|^{1+\beta/n-1/p}} \int_Q \int_{Q^*} \int_{\mathbb{R}^n\setminus Q^*} \big|\big(b_1(z)-\lambda_1\big)\big(b_2(y_2)-\lambda_2\big)\big| \\ &\quad \times \left(\int_0^\infty \big|K_t(z,y_1,y_2)-K_t(x_Q,y_1,y_2)\big|^2 \frac{dt}{t}\right)^{1/2} \big|f_1(y_1)f_2(y_2)\big|\,dy_1\,dy_2\,dz \\ &\le C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}} \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q \int_{Q^*} \int_{\mathbb{R}^n\setminus Q^*} \big|\big(b_1(z)-\lambda_1\big)\big(b_2(y_2)-\lambda_2\big)\big| \\ &\quad \times \frac{\omega(\frac{|z-x_Q|}{|z-y_1|+|z-y_2|})}{(|z-y_1|+|z-y_2|)^{2n}} \big|f_1(y_1)f_2(y_2)\big|\,dy_1\,dy_2\,dz \\ &\le C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}} \frac{1}{|Q|^{1+\beta_2/n-1/p}} \int_Q \int_{Q^*} \int_{\mathbb{R}^n\setminus Q^*} \big|\big(b_1(z)-\lambda_1\big)\big(b_2(y_2)-\lambda_2\big)\big| \\ &\quad \times \frac{\omega(\frac{|z-x_Q|}{|x_Q-y_2|})}{(|z-y_1|+|z-y_2|)^{2n-\beta_2}} \big|f_1(y_1)f_2(y_2)\big|\,dy_1\,dy_2\,dz \\ &\le C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}} \frac{C}{|Q|^{\beta_2/n-1/p}} \int_Q \big|\big(b_1(z)-\lambda_1\big)\big| \\ &\quad \times \int_{Q^*} \int_{\mathbb{R}^n\setminus Q^*} \frac{|(b_2(z)-\lambda_2)|}{(|z-y_1|+|z-y_2|)^{2n}} \big|f_1(y_1)f_2(y_2)\big|\,dy_1\,dy_2\,dz \\ &\le C\frac{\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|^{\beta_2/n-1/p}} \int_{Q^*} \big|f_1(y_1)\big|\,dy_1 \\ &\quad \times \sum_{k=1}^\infty \frac{\omega(2^{-k})}{|2^{k+3}\sqrt{n}Q|^{2-\beta_2/n}} \int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q} \big|f_2(y_2)\big|\,dy_2 \\ &\le C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}} \sum_{k=1}^\infty \omega\big(2^{-k}\big)2^{kn(1-\beta_2/n+1/p_2)} \\ &\le C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}, \end{aligned}$$

where we have used the fact $1-\beta_2/n+1/p_2>0$.

Thus

$$M_2 \le C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.$$

Similarly,

$$M_3 \le C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.$$

We deal with M_4 as follows:

$$\begin{aligned} M_4 &\le \frac{C}{|Q|^{1+\beta/n-1/p}} \int_Q \left(\int_0^\infty \left|\int_{(\mathbb{R}^n\setminus Q^*)^2} \big(b_1(z)-\lambda_1\big)\big(b_2(z)-\lambda_2\big)\right.\right. \\ &\quad \left.\left. \times K_t(z,y_1,y_2)f_1(y_1)f_2(y_2)\,dy_1\,dy_2\right|^2 \frac{dt}{t}\right)^{1/2} dz \\ &\quad + \frac{C}{|Q|^{1+\beta/n-1/p}} \int_Q \left(\int_0^\infty \left|\int_{(\mathbb{R}^n\setminus Q^*)^2} \big(b_1(z)-\lambda_1\big)\big(b_2(y_2)-\lambda_2\big)\right.\right. \end{aligned}$$

$$
\begin{aligned}
&\times \big[K_t(z,y_1,y_2)-K_t(x_Q,y_1,y_2)\big]f_1(y_1)f_2(y_2)\,dy_1\,dy_2\bigg|^2\frac{dt}{t}\bigg)^{1/2}dz \\
&+\frac{C}{|Q|^{1+\beta/n-1/p}}\int_Q\bigg(\int_0^\infty\bigg|\int_{(\mathbb{R}^n\setminus Q^*)^2}\big(b_1(y_1)-\lambda_1\big)\big(b_2(z)-\lambda_2\big) \\
&\times \big[K_t(z,y_1,y_2)-K_t(x_Q,y_1,y_2)\big]f_1(y_1)f_2(y_2)\,dy_1\,dy_2\bigg|^2\frac{dt}{t}\bigg)^{1/2}dz \\
&+\frac{C}{|Q|^{1+\beta/n-1/p}}\int_Q\bigg(\int_0^\infty\bigg|\int_{(\mathbb{R}^n\setminus Q^*)^2}\big(b_1(y_1)-\lambda_1\big)\big(b_2(y_2)-\lambda_2\big) \\
&\times \big[K_t(z,y_1,y_2)-K_t(x_Q,y_1,y_2)\big]f_1(y_1)f_2(y_2)\,dy_1\,dy_2\bigg|^2\frac{dt}{t}\bigg)^{1/2}dz \\
&\doteq M_{41}+M_{42}+M_{43}+M_{44}.
\end{aligned}
$$

By Minkowski's inequality and the size condition (1.1) we have

$$
\begin{aligned}
M_{41} &\le \frac{C}{|Q|^{1+\beta/n-1/p}}\int_Q\int_{(\mathbb{R}^n\setminus Q^*)^2}\big|\big(b_1(z)-\lambda_1\big)\big(b_2(z)-\lambda_2\big)\big| \\
&\quad\times\bigg(\int_0^\infty\big|K_t(z,y_1,y_2)\big|^2\frac{dt}{t}\bigg)^{1/2}\big|f_1(y_1)f_2(y_2)\big|\,dy_1\,dy_2\,dz \\
&\le \frac{C}{|Q|^{1+\beta/n-1/p}}\int_Q\int_{(\mathbb{R}^n\setminus Q^*)^2}\big|\big(b_1(z)-\lambda_1\big)\big(b_2(z)-\lambda_2\big)\big| \\
&\quad\times\frac{1}{(|z-y_1|+|z-y_2|)^{2n}}\big|f_1(y_1)f_2(y_2)\big|\,dy_1\,dy_2\,dz \\
&\le \|b_1\|_{\dot\wedge_{\beta_1}}\|b_2\|_{\dot\wedge_{\beta_2}}|Q|^{1/p}\sum_{k=1}^\infty\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\frac{|f_1(y_1)|}{|y_1-x_Q|^n}\,dy_1 \\
&\quad\times\sum_{k=1}^\infty\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\frac{|f_2(y_2)|}{|y_2-x_Q|^n}\,dy_2 \\
&\le \|b_1\|_{\dot\wedge_{\beta_1}}\|b_2\|_{\dot\wedge_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}|Q|^{1/p}\sum_{k=1}^\infty\big|2^{k+3}\sqrt{n}Q\big|^{-1/p_1}\sum_{k=1}^\infty\big|2^{k+3}\sqrt{n}Q\big|^{-1/p_2} \\
&\le \|b_1\|_{\dot\wedge_{\beta_1}}\|b_2\|_{\dot\wedge_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.
\end{aligned}
$$

By Minkowski's inequality and the smooth condition (1.2) we have

$$
\begin{aligned}
M_{42} &\le \frac{C}{|Q|^{1+\beta/n-1/p}}\int_Q\int_{(\mathbb{R}^n\setminus Q^*)^2}\big|\big(b_1(z)-\lambda_1\big)\big(b_2(y_2)-\lambda_2\big)\big| \\
&\quad\times\bigg(\int_0^\infty\big|K_t(z,y_1,y_2)-K_t(x_Q,y_1,y_2)\big|^2\frac{dt}{t}\bigg)^{1/2}\big|f_1(y_1)f_2(y_2)\big|\,dy_1\,dy_2\,dz \\
&\le \frac{\|b_1\|_{\dot\wedge_{\beta_1}}\|b_2\|_{\dot\wedge_{\beta_2}}}{|Q|^{\beta_2/n-1/p}}\sum_{k=1}^\infty\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\frac{|f_1(y_1)|}{|y_1-x_Q|^n}\,dy_1 \\
&\quad\times\sum_{i=1}^\infty\int_{2^{i+3}\sqrt{n}Q\setminus 2^{i+2}\sqrt{n}Q}\frac{|f_2(y_2)|\omega(2^{-i})}{|y_2-x_Q|^{n-\beta_2}}\,dy_2 \\
&\le \|b_1\|_{\dot\wedge_{\beta_1}}\|b_2\|_{\dot\wedge_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\sum_{k=1}^\infty\big|2^{k+3}\sqrt{n}Q\big|^{-1/p_1}|Q|^{1/p}
\end{aligned}
$$

$$
\begin{aligned}
&\times \sum_{i=1}^{\infty} \omega(2^{-i}) \left|2^{i+3}\sqrt{n}Q\right|^{\beta_2/n-1/p_2} |Q|^{-\beta_2/n+1/p_2} \\
&\le \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \sum_{i=1}^{\infty} \omega(2^{-i}) 2^{in(\beta/n-1/p)} \\
&\le \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.
\end{aligned}
$$

Similarly,

$$
M_{43} \le C\|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.
$$

Now we estimate M_{44}:

$$
\begin{aligned}
M_{44} &\le \frac{C}{|Q|^{1+\beta/n-1/p}} \int_Q \int_{(\mathbb{R}^n \setminus Q^*)^2} \left|\big(b_1(y_1)-\lambda_1\big)\big(b_2(y_2)-\lambda_2\big)\right| \\
&\quad \times \left(\int_0^{\infty} \left|K_t(z,y_1,y_2) - K_t(x_Q,y_1,y_2)\right|^2 \frac{dt}{t} \right)^{1/2} \left|f_1(y_1)f_2(y_2)\right| \, dy_1\, dy_2\, dz \\
&\le \frac{\|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}}}{|Q|^{\beta/n-1/p}} \sum_{k=1}^{\infty} \int_{(2^{k+3}\sqrt{n}Q)^2 \setminus (2^{k+2}\sqrt{n}Q)^2} \frac{|f_1(y_1)|}{|y_2-x_Q|^{2n-\beta_1-\beta_2}} \\
&\quad \times \omega\left(\frac{|z-x_Q|}{|y_2-x_Q|} \right) dy_1\, dy_2 \\
&\le \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \sum_{k=1}^{\infty} \omega(2^{-k}) 2^{kn(\beta/n-1/p)} \\
&\le \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.
\end{aligned}
$$

Combing the estimates for M_1, M_2, M_3, M_4, we get (4.1). Thus the proof of Theorem 1.2 is completed. □

Proof of Theorem 1.3 Let $c = c_1 + c_2 + c_3$, which will be determined later. Then we have

$$
\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}} \int_Q \left| T_{\Pi\vec{b}}(\vec{f})(z) - \big(T_{\Pi\vec{b}}(\vec{f})\big)_Q \right| dz \\
&\quad \le \frac{1}{|Q|^{1+\beta/n}} \int_Q \left| T_{\Pi\vec{b}}(f_1,f_2)(z) - c \right| dz \\
&\quad \le \frac{C}{|Q|^{1+\beta/n}} \int_Q \left| \big(b_1(z)-\lambda_1\big)\big(b_2(z)-\lambda_2\big) T(f_1,f_2)(z) \right| dz \\
&\qquad + \frac{C}{|Q|^{1+\beta/n}} \int_Q \left| \big(b_2(z)-\lambda_2\big) T_{\vec{b}}^1(f_1,f_2)(z) - c_1 \right| dz \\
&\qquad + \frac{C}{|Q|^{1+\beta/n}} \int_Q \left| \big(b_1(z)-\lambda_1\big) T_{\vec{b}}^2(f_1,f_2)(z) - c_2 \right| dz \\
&\qquad + \frac{C}{|Q|^{1+\beta/n}} \int_Q \left| T\big((b_1-\lambda_1)f_1, (b_2-\lambda_2)f_2\big)(z) - c_3 \right| dz \\
&\quad \doteq N_1 + N_2 + N_3 + N_4.
\end{aligned}
$$

In what follows, we estimate each term separately. For $1 < r < p$, by the Hölder inequality we have

$$N_1 \leq C\|b_1\|_{\dot{\wedge}_{\beta_1}}\|b_2\|_{\dot{\wedge}_{\beta_2}} M_r\big(T(f_1,f_2)\big)(x).$$

Observe that

$$\begin{aligned}
&[b_1,T](f_1,f_2)(z)\\
&\quad < \big|\big(b_1(z)-\lambda_1\big)\big|T(f_1,f_2)(z) + T\big(f_1^0,f_2^0\big)(z)\\
&\qquad + \left(\int_0^\infty\left|\int_{(\mathbb{R}^n)^m}\big(b_1(y_1)-\lambda_1\big)K_t(x,y_1,y_2)f_1^\infty(y_1)f_2^0(y_2)\,dy_1\,dy_2\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}\\
&\qquad + \left(\int_0^\infty\left|\int_{(\mathbb{R}^n)^m}\big(b_1(y_1)-\lambda_1\big)K_t(x,y_1,y_2)f_1^0(y_1)f_2^\infty(y_2)\,dy_1\,dy_2\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}\\
&\qquad + \left(\int_0^\infty\left|\int_{(\mathbb{R}^n)^m}\big(b_1(y_1)-\lambda_1\big)K_t(x,y_1,y_2)f_1^\infty(y_1)f_2^\infty(y_2)\,dy_1\,dy_2\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}.
\end{aligned}$$

Let

$$\begin{aligned}
c_1' &= \|b_2\|_{\dot{\wedge}_{\beta_2}}|Q|^{\beta_2/n}\\
&\quad \times \left(\int_0^\infty\left|\int_{(\mathbb{R}^n)^m}\big(b_1(y_1)-\lambda_1\big)K_t(x,y_1,y_2)f_1^\infty(y_1)f_2^0(y_2)\,dy_1\,dy_2\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}\\
&\quad + \|b_2\|_{\dot{\wedge}_{\beta_2}}|Q|^{\beta_2/n}\\
&\quad \times \left(\int_0^\infty\left|\int_{(\mathbb{R}^n)^m}\big(b_1(y_1)-\lambda_1\big)K_t(x,y_1,y_2)f_1^0(y_1)f_2^\infty(y_2)\,dy_1\,dy_2\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}\\
&\quad + \|b_2\|_{\dot{\wedge}_{\beta_2}}|Q|^{\beta_2/n}\\
&\quad \times \left(\int_0^\infty\left|\int_{(\mathbb{R}^n)^m}\big(b_1(y_1)-\lambda_1\big)K_t(x,y_1,y_2)f_1^\infty(y_1)f_2^\infty(y_2)\,dy_1\,dy_2\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}.
\end{aligned}$$

Then

$$\begin{aligned}
N_2 &\leq \frac{C}{|Q|^{1+\beta/n}}\int_Q\Big|\|b_2\|_{\dot{\wedge}_{\beta_2}}|Q|^{\beta_2/n}[b_1,T](f_1,f_2)(z)-c_1'\Big|\,dz\\
&\leq \frac{C\|b_2\|_{\dot{\wedge}_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q\big|\big(b_1(z)-\lambda_1\big)\big|T(f_1,f_2)(z)\,dz\\
&\quad + \frac{C\|b_2\|_{\dot{\wedge}_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q T\big(f_1^0,f_2^0\big)(z)\,dz\\
&\quad + \frac{C\|b_2\|_{\dot{\wedge}_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q\bigg(\int_0^\infty\bigg|\int_{(\mathbb{R}^n)^m}\big(b_1(y_1)-\lambda_1\big)\\
&\quad \times \big[K_t(z,y_1,y_2)-K_t(x_Q,y_1,y_2)\big]f_1^0(y_1)f_2^\infty(y_2)\,dy_1\,dy_2\bigg|^2\frac{dt}{t}\bigg)^{\frac{1}{2}}\,dz\\
&\quad + \frac{C\|b_2\|_{\dot{\wedge}_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q\bigg(\int_0^\infty\bigg|\int_{(\mathbb{R}^n)^m}\big(b_1(y_1)-\lambda_1\big)
\end{aligned}$$

$$\times \big[K_t(z,y_1,y_2) - K_t(x_Q,y_1,y_2)\big]f_1^\infty(y_1)f_2^0(y_2)\,dy_1\,dy_2\bigg|^2 \frac{dt}{t}\bigg)^{\frac{1}{2}} dz$$

$$+ \frac{C\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|^{1+\beta_1/n}} \int_Q \bigg(\int_0^\infty \bigg|\int_{(\mathbb{R}^n)^m} \big(b_1(y_1) - \lambda_1\big)$$

$$\times \big[K_t(z,y_1,y_2) - K_t(x_Q,y_1,y_2)\big]f_1^\infty(y_1)f_2^\infty(y_2)\,dy_1\,dy_2\bigg|^2 \frac{dt}{t}\bigg)^{\frac{1}{2}} dz$$

$$\doteq N_{21} + N_{22} + N_{23} + N_{24} + N_{25}.$$

By the Hölder inequality we have

$$N_{21} \le C\|b_2\|_{\dot\Lambda_{\beta_2}} \bigg(\frac{1}{|Q|^{r'\beta_1/n+1}} \int_Q \big|b_1(z) - \lambda_1\big|^{r'}\,dz\bigg)^{1/r'} \bigg(\frac{1}{|Q|}\int_Q \big|T(f_1,f_2)(z)\big|^r\,dz\bigg)^{1/r}$$
$$\le \|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}} M_r\big(T(f_1,f_2)\big)(x).$$

Take $1 < q_1 < p_1$, $1 < q_2 < p_2$, and $1 < q < \infty$ such that $1/q = 1/q_1 + 1/q_2$. Then by the Hölder inequality and Lemma 2.3 we have

$$N_{22} \le \frac{C\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|^{\beta_1/n+1/q}} \bigg(\int_Q |T\big((b_1-\lambda_1)f_1^0,f_2^0\big)(z)|^q\,dz\bigg)^{1/q}$$
$$\le \frac{C\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|^{\beta_1/n+1/q}} \big\|(b_1-\lambda_1)f_1^0\big\|_{L^{q_1}} \big\|f_2^0\big\|_{L^{q_2}}$$
$$\le \frac{C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|^{1/q}} \big\|f_1^0\big\|_{L^{q_1}} \big\|f_2^0\big\|_{L^{q_2}}$$
$$\le \|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}} M_{q_1}(f_1)(x)M_{q_2}(f_2)(x).$$

For $y_2 \in (Q^*)^c$, $|y_2 - x_Q| \sim |y_2 - z|$, and $|z - x_Q| \le \frac{|y_2-z|}{2} \le \frac{1}{2}\max\{|z-y_1|,|z-y_2|\}$, by Minkowski's inequality and the smooth condition (1.2) we get

$$N_{23} \le \frac{C\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|^{1+\beta_1/n}} \int_Q \int_{(\mathbb{R}^n)^2} \big|\big(b_1(y_1) - \lambda_1\big)\big|$$
$$\times \bigg(\int_0^\infty \big|K_t(z,y_1,y_2) - K_t(x_Q,y_1,y_2)\big|^2 \frac{dt}{t}\bigg)^{\frac{1}{2}} \big|f_1^0(y_1)f_2^\infty(y_2)\big|\,dy_1\,dy_2\,dz$$
$$\le \frac{C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|^{1+\beta_1/n}} \int_Q \int_{(\mathbb{R}^n)^2} \frac{|y_1 - x_Q|^{\beta_1}}{(|z-y_1| + |z-y_2|)^{2n}}$$
$$\times\, \omega\bigg(\frac{|z-x_Q|}{|z-y_1| + |z-y_2|}\bigg) \big|f_1^0(y_1)f_2^\infty(y_2)\big|\,dy_1\,dy_2\,dz$$
$$\le \frac{C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|^{1+\beta_1/n}} \int_Q \int_{(\mathbb{R}^n)^2} \frac{|f_1^0(y_1)f_2^\infty(y_2)|}{(|z-y_1| + |z-y_2|)^{2n-\beta_1}}$$
$$\times\, \omega\bigg(\frac{|z-x_Q|}{|z-y_1| + |z-y_2|}\bigg)\,dy_1\,dy_2\,dz$$
$$\le \frac{C}{|Q|}\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}} \int_Q \int_{Q^*} |f_1(y_1)| \int_{(Q^*)^c} \frac{|f_2(y_2)|}{|z-y_2|^{2n}}\,\omega\bigg(\frac{|z-x_Q|}{|z-y_2|}\bigg)\,dy_2\,dy_1\,dz$$

$$\begin{aligned}
&\leq \frac{C}{|Q|}\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}\int_Q\int_{Q^*}|f_1(y_1)| \\
&\quad\times\sum_{k=1}^{\infty}\int_{2^{k+3}\sqrt{n}Q\backslash 2^{k+2}\sqrt{n}Q}|f_2(y_2)|\big|2^k\sqrt{n}Q\big|^{-2}\omega\big(2^{-k}\big)\,dy_2\,dy_1\,dz \\
&\leq C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}\frac{1}{|Q|}\int_{Q^*}|f_1(y_1)|\,dy_1 \\
&\quad\times\sum_{k=1}^{\infty}|Q|\big|2^{k+3}\sqrt{n}Q\big|^{-1}\omega(2^{-k})\frac{1}{|2^{k+3}\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q\backslash 2^{k+2}\sqrt{n}Q}|f_2(y_2)|\,dy_2 \\
&\leq C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}M(f_1)(x)\sum_{k=1}^{\infty}2^{-k}\omega\big(2^{-k}\big)\frac{1}{|2^{k+3}\sqrt{n}Q|} \\
&\quad\times\int_{2^{k+3}\sqrt{n}Q\backslash 2^{k+2}\sqrt{n}Q}|f_2(y_2)|\,dy_2 \\
&\leq C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}M(f_1)(x)M(f_2)(x).
\end{aligned}$$

Similarly,

$$N_{24}\leq C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}M(f_1)(x)M(f_2)(x).$$

For $y_1,y_2\in(Q^*)^c$, $|y_1-x_Q|\sim|y_1-z|$, and $|y_2-x_Q|\sim|y_2-z|$, by Minkowski's inequality and the smooth condition (1.2) we get

$$\begin{aligned}
N_{25}&\leq\frac{C\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q\int_{(\mathbb{R}^n)^2}\big|\big(b_1(y_1)-\lambda_1\big)\big| \\
&\quad\times\left(\int_0^\infty\big|K_t(z,y_1,y_2)-K_t(x_Q,y_1,y_2)\big|^2\frac{dt}{t}\right)^{\frac12}\big|f_1^\infty(y_1)f_2^\infty(y_2)\big|\,dy_1\,dy_2\,dz \\
&\leq\frac{C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q\int_{(\mathbb{R}^n)^2}\frac{|y_1-x_Q|^{\beta_1}}{(|z-y_1|+|z-y_2|)^{2n}} \\
&\quad\times\omega\left(\frac{|z-x_Q|}{|z-y_1|+|z-y_2|}\right)\big|f_1^\infty(y_1)f_2^\infty(y_2)\big|\,dy_1\,dy_2\,dz \\
&\leq\frac{C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q\int_{(\mathbb{R}^n)^2}\frac{|y_1-x_Q|^{\beta_1}|f_1^0(y_1)f_2^\infty(y_2)|}{(|z-y_1|+|z-y_2|)^{2n}} \\
&\quad\times\omega\left(\frac{|z-x_Q|}{|z-y_1|+|z-y_2|}\right)dy_1\,dy_2\,dz \\
&\leq\frac{C}{|Q|^{1+\beta_1/n}}\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}\int_Q\int_{((Q^*)^c)^2}\frac{|f_1(y_1)||f_2(y_2)|}{|y_1-x_Q|^{2n-\beta_1}}\omega\left(\frac{|z-x_Q|}{|z-y_1|}\right)dy_1\,dy_2\,dz \\
&\leq\frac{C}{|Q|^{1+\beta_1/n}}\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}} \\
&\quad\times\int_Q\sum_{k=1}^{\infty}\int_{2^{k+3}\sqrt{n}Q\backslash 2^{k+2}\sqrt{n}Q}\frac{|f_1(y_1)||f_2(y_2)|}{|y_1-x_Q|^{2n-\beta_1}}\omega\left(\frac{|z-x_Q|}{|z-y_1|}\right)dy_1\,dy_2\,dz \\
&\leq C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}\sum_{k=1}^{\infty}\frac{2^{k\beta_1}\omega(2^{-k})}{|2^{k+3}\sqrt{n}Q|^2}
\end{aligned}$$

$$\begin{aligned}&\times \int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q} |f_1(y_1)|\,dy_1 \int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q} |f_2(y_2)|\,dy_2 \\ &\leq C\|b_1\|_{\dot{\Lambda}_{\beta_1}}\|b_2\|_{\dot{\Lambda}_{\beta_2}} M(f_1)(x)M(f_2)(x).\end{aligned}$$

Combining the estimates for $N_{21}, N_{22}, N_{23}, N_{24}, N_{25}$, we get

$$N_2 \leq C\|b_1\|_{\dot{\Lambda}_{\beta_1}}\|b_2\|_{\dot{\Lambda}_{\beta_2}} \bigl\{M_r\bigl(T(f_1,f_2)\bigr)(x) + M_{q_1}(f_1)(x)M_{q_2}(f_2)(x) + M(f_1)(x)M(f_2)(x)\bigr\}.$$

Similarly, we have

$$N_3 \leq C\|b_1\|_{\dot{\Lambda}_{\beta_1}}\|b_2\|_{\dot{\Lambda}_{\beta_2}} \bigl\{M_r\bigl(T(f_1,f_2)\bigr)(x) + M_{q_1}(f_1)(x)M_{q_2}(f_2)(x) + M(f_1)(x)M(f_2)(x)\bigr\}$$

and

$$N_4 \leq C\|b_1\|_{\dot{\Lambda}_{\beta_1}}\|b_2\|_{\dot{\Lambda}_{\beta_2}} \bigl\{M_r\bigl(T(f_1,f_2)\bigr)(x) + M_{q_1}(f_1)(x)M_{q_2}(f_2)(x) + M(f_1)(x)M(f_2)(x)\bigr\}.$$

Thus we deduce that

$$\begin{aligned}&\frac{1}{|Q|^{1+\beta/n}} \int_Q \bigl|T_{\Pi\vec{b}}(\vec{f})(z) - \bigl(T_{\Pi\vec{b}}(\vec{f})\bigr)_Q\bigr|\,dz \\ &\quad\leq C\|b_1\|_{\dot{\Lambda}_{\beta_1}}\|b_2\|_{\dot{\Lambda}_{\beta_2}} \bigl\{M_r\bigl(T(f_1,f_2)\bigr)(x) + M_{q_1}(f_1)(x)M_{q_2}(f_2)(x) + M(f_1)(x)M(f_2)(x)\bigr\}.\end{aligned}$$

By the Hölder inequality, Lemma 2.3, and the normal inequalities for the maximal operators, we arrive at

$$\begin{aligned}&\bigl\|T_{\Pi\vec{b}}(\vec{f})\bigr\|_{\dot{F}_p^{\beta,\infty}} \\ &\quad\approx \biggl\|\sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q \bigl|T_{\Pi\vec{b}}(\vec{f})(z) - \bigl(T_{\Pi\vec{b}}(\vec{f})\bigr)_Q\bigr|\,dz\biggr\|_{L^p} \\ &\quad\leq C\|b_1\|_{\dot{\Lambda}_{\beta_1}}\|b_2\|_{\dot{\Lambda}_{\beta_2}} \bigl\{\bigl\|M_r\bigl(T(f_1,f_2)\bigr)\bigr\|_{L^p} + \bigl\|M_{q_1}(f_1)M_{q_2}(f_2)\bigr\|_{L^p} + \bigl\|M(f_1)M(f_2)\bigr\|_{L^p}\bigr\} \\ &\quad\leq C\|b_1\|_{\dot{\Lambda}_{\beta_1}}\|b_2\|_{\dot{\Lambda}_{\beta_2}} \bigl\{\bigl\|T(f_1,f_2)\bigr\|_{L^p} + \bigl\|M_{q_1}(f_1)\bigr\|_{L^{p_1}}\bigl\|M_{q_2}(f_2)\bigr\|_{L^{p_2}} \\ &\qquad + \bigl\|M(f_1)\bigr\|_{L^{p_1}}\bigl\|M(f_2)\bigr\|_{L^{p_2}}\bigr\} \\ &\quad\leq \|b_1\|_{\dot{\Lambda}_{\beta_1}}\|b_2\|_{\dot{\Lambda}_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}},\end{aligned}$$

where we have used that facts $1 < r < p$, $1 < q_1 < p_1$, and $1 < q_2 < p_2$. This finishes the proof of Theorem 1.3. □

5 Conclusions

In this paper,we studied the boundedness properties of the commutator generated by a multilinear square function and Lipschitz functions with kernel satisfying Dini-type condition. We showed that such commutators are bounded from product Lebesgue spaces into the Lebesgue spaces, Lipschitz spaces, and Triebel–Lizorkin spaces.

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