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Well-posedness for a class of generalized variational-hemivariational inequalities involving set-valued operators

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Abstract

The aim of present work is to study some kinds of well-posedness for a class of generalized variational-hemivariational inequality problems involving set-valued operators. Some systematic approaches are presented to establish some equivalence theorems between several classes of well-posedness for the inequality problems and some corresponding metric characterizations, which generalize many known results. Finally, the well-posedness for a class of generalized mixed equilibrium problems is also considered.

Keywords: Generalized variational-hemivariational inequality; Set-valued operator; α-well-posedness; Monotonicity

1 Introduction

Nowadays, well-posedness has been drawing great attention in the field of optimization problems and related problems such as variational inequalities, hemivariational inequalities, fixed point problems, equilibrium problems, and inclusion problems (see [1, 5, 9, 11, 17, 19, 21, 23, 33]). The classical concept of well-posedness for a global minimization problem was first introduced by Tikhonov [35], which required the existence and uniqueness of a solution to the global minimization problem and the convergence of every minimizing sequence toward the unique solution. Thereafter, the concept of well-posedness has been generalized to variational inequalities. The initial notion of well-posedness for variational inequality is due to Lucchetti and Patrone [28]. Fang [13, 14] generalized two kinds of well-posedness for a mixed variational inequality problem in a Banach space. For further results on the well-posedness of variational inequalities, we refer to [2, 4, 12–14, 16, 22, 27, 28] and the references therein.

As an important and useful generalization of variational inequality, hemivariational inequality, which was first studied by Panagiotopoulos [32], has a great development in recent years by several works [6, 29, 31]. Many authors are interested in generalizing the concept of well-posedness to hemivariational inequalities. In 1995, Goeleven and Mentagui [15] generalized the concept of the well-posedness to a hemivariational inequality and presented some basic results concerning the well-posed hemivariational inequality. Recently, using the concept of approximating sequence, Xiao et al. [37, 38] introduced a concept of well-posedness for a hemivariational inequality and a variational-hemivariational inequality. Ceng, Lur, and Wen [3] considered an extension of well-posedness for a minimization problem to a class of generalized variational-hemivariational inequalities with perturbations in reflexive Banach spaces. For more recent works on the well-posedness for variational-hemivariational inequalities, we refer to [3, 15, 18, 19, 26, 37, 38] and the references therein.

In the last years, many authors studied the existence results for some types of hemivariational inequalities involving set-valued operators [34, 36, 39]. In 2011, Zhang and He [39] studied a kind of hemivariational inequalities of Hartman–Stampacchia type by introducing the concept of stable quasimonotonicity. They supposed that the constraint set is a bounded (or unbounded), closed, and convex subset in a reflexive Banach space. The authors gave sufficient conditions for the existence and boundedness of solutions. In 2013, Tang and Huang [34] generalized the result of [39] by introducing the concept of stable ϕ-quasimonotonicity and obtained some existence theorems when the constrained set is nonempty, bounded (or unbounded), closed, and convex in a reflexive Banach space. Hereafter, Wangkeeree and Preechasilp [36] generalized the results of [34] and [39] by introducing the concept of stable f-quasimonotonicity. Very recently, Liu and Zeng obtained some existence results for a class of hemivariational inequalities involving the stable (g,f,α)-quasimonotonicity [25], a result on the well-posedness for mixed quasivariational hemivariational inequalities [26], and some existence results for a class of quasimixed equilibrium problems involving the (f,g,h)-quasimonotonicity [24].

Let K be a nonempty, closed, and convex subset of a real Banach space X with its dual X^*, and let $F : K \to P(X^*)$ be a set-valued operator, where $P(X^*)$ is the set of all nonempty subsets of X^*. Let $T : K \to X^*$ be a perturbation, and let $f \in X^*$ be a given element. Let $g : K \times K \to \overline{R} := R \cup \{\pm\infty\}$ be a function such that $\mathcal{D}(g) = \{u \in K : g(u,v) \neq -\infty, \forall v \in K\} \neq \emptyset$. Let $J : X \to R$ be a locally Lipschitz function, and let $J^\circ(u,v)$ denote the generalized directional derivative in the sense of Clarke of a locally Lipschitz functional $J : X \to R$ at u in the direction v. In this paper, we discuss the following generalized variational-hemivariational inequality (GVHVI):

Find $u \in K$ such that, for some $u^* \in F(u)$,

$$\langle u^* + Tu - f, v - u\rangle + g(u,v) + J^\circ(u; v-u) \geq 0, \quad \forall v \in K.$$

Now, let us consider some particular cases of GVHVI.

(a) If $T \equiv 0$, $f \equiv 0$, and $g \equiv 0$, then GVHVI is reduced to the following form:

Find $u \in K$ and $u^* \in F(u)$ such that

$$\langle u^*, v - u\rangle + J^\circ(u; v-u) \geq 0, \quad \forall v \in K.$$

The existence of solutions to this inequality was recently studied by Zhang and He [39].

(b) If $T \equiv 0$ and $f \equiv 0$, and $g(u,v) = \phi(v) - \phi(u)$ for all $u, v \in K$, then GVHVI is reduced to the following form:

Find $u \in K$ and $u^* \in F(u)$ such that

$$\langle u^*, v - u\rangle + \phi(v) - \phi(u) + J^\circ(u; v-u) \geq 0, \quad \forall v \in K.$$

The existence of solutions to this inequality was studied by Tang and Huang [34].

(c) If $T \equiv 0$ and $f \equiv 0$, then GVHVI is reduced to the following form:
Find $u \in K$ and $u^* \in F(u)$ such that

$$\langle u^*, v - u \rangle + g(u, v) + J^\circ(u; v - u) \geq 0, \quad \forall v \in K.$$

The existence of solutions to this inequality was studied by Wangkeeree and Preechasilp [36].

Inspired by previous works, we study the well-posedness for GVHVI, which generalizes many known works. Under relatively weak conditions, we establish some equivalence results and some metric characterizations for the strong and weak α-well-posed GVHVI in the generalized sense. In particular, we present equivalence results on weak α-well-posedness for GVHVI, which were considered by few authors.

This paper is organized as follows. In Sect. 2, we recall some basic preliminaries of single-valued and set-valued mappings, metric concepts, Clarke's generalized directional derivative, and some classes of well-posedness for GVHVI. In Sect. 3, we show some equivalence results for the well-posedness for GVHVI and some corresponding metric characterizations. Theorems 3.3, 3.5, and 3.6 are the main results in this section. In the last section, we also present the well-posedness for a class of generalized mixed equilibrium problems.

2 Preliminaries

Let R, R_+, and N be the sets of real numbers, nonnegative real numbers, and natural numbers, respectively. Let X be a real Banach space with norm $\|\cdot\|_X$. Denote by X^* its dual space and by $\langle\cdot,\cdot\rangle_X$ the duality pairing between X^* and X. Let X_w be the Banach space X with weak topology.

Definition 2.1 Let K be a nonempty subset of X. A function $f : K \to R$ is said to be

(i) convex on K if for all finite subsets $\{u_1, \ldots, u_n\} \subset K$ and $\{\lambda_1, \ldots, \lambda_n\} \subset R_+$ such that $\sum_{i=1}^n \lambda_i = 1$ and $\sum_{i=1}^n \lambda_i u_i \in K$, we have

$$f\left(\sum_{i=1}^n \lambda_i u_i\right) \leq \sum_{i=1}^n \lambda_i f(u_i);$$

(ii) (weakly) upper semicontinuous (u.s.c. for short) at u if for any sequence $\{u_n\}_{n\geq 1} \subset K$ with $(u_n \rightharpoonup u)$ $u_n \to u$, we have

$$\limsup_{n\to\infty} f(u_n) \leq f(u).$$

(iii) (weakly) lower semicontinuous (l.s.c. for short) at u, if for any sequence $\{u_n\}_{n\geq 1} \subset K$ with $(u_n \rightharpoonup u)$ $u_n \to u$, we have

$$\liminf_{n\to\infty} f(u_n) \geq f(u).$$

The function f is said to be (weakly) u.s.c. (l.s.c.) on K if f is (weakly) u.s.c. (l.s.c.) at all $u \in K$.

Definition 2.2 ([20]) Let K be a nonempty subset of X. An operator $\beta : K \to X$ is said to be affine if for any $u_i \in K$ ($i = 1, 2, \dots, n$) and $\lambda_i \in [0,1]$ with $\sum_{i=1}^{n} \lambda_i = 1$, we have

$$\beta\left(\sum_{i=1}^{n} \lambda_i v_i\right) = \sum_{i=1}^{n} \lambda_i \beta(u_i).$$

Definition 2.3 A set-valued operator $F : K \to P(X^*)$ is said to be

(i) lower semicontinuous (l.s.c.) at u_0 if for any $u_0^* \in F(u_0)$ and sequence $\{u_n\}_{n\geq 1} \subset K$ with $u_n \to u_0$, there exists a sequence $u_n^* \in F(u_n)$ that converges to u_0^*.

(ii) lower hemicontinuous (l.h.c.) if the restriction of F to every line segment of K is lower semicontinuous with respect to the weak topology in X^*.

Definition 2.4 A set-valued operator $F : K \to P(X^*)$ is said to be monotone if for all $u, v \in K$,

$$\langle v^* - u^*, v - u\rangle \geq 0, \quad \forall u^* \in F(u), \forall v^* \in F(v).$$

Definition 2.5 Let S be a nonempty subset of X. The measure μ of noncompactness for the set S is defined by

$$\mu(S) := \inf\left\{\epsilon > 0 : S = \bigcup_{i=1}^{n} S_i, \operatorname{diam}|S_i| < \epsilon, i = 1, 2, \dots, n\right\},$$

where $\operatorname{diam}|S_i|$ is the diameter of the set S_i.

Now, let us recall the definitions of the Clarke generalized directional derivative and generalized gradient for a locally Lipschitz function $\varphi : X \to R$ (see [6, 10]). The Clarke generalized directional derivative $\varphi^0(u; v)$ of φ at the point $u \in X$ in the direction $v \in X$ is defined as

$$\varphi^0(u; v) := \limsup_{\lambda \to 0^+, \zeta \to u} \frac{\varphi(\zeta + \lambda v) - \varphi(\zeta)}{\lambda}.$$

The Clarke subdifferential or generalized gradient of φ at $u \in X$, denoted by $\partial\varphi(u)$, is the subset of X^* given by

$$\partial\varphi(u) := \left\{u^* \in X^* : \varphi^0(u; v) \geq \langle u^*, v\rangle_X, \forall v \in X\right\}.$$

Lemma 2.6 ([6], Proposition 2.1.1) *Let $\varphi : X \to R$ be locally Lipschitz of rank $L_u > 0$ near u. Then*

(i) *$\varphi^0(u; v)$ is u.s.c. as a function of (u, v) and, as a function of v alone, is Lipschitz of rank L_u near u on X and satisfies*

$$\left|\varphi^0(u; v)\right| \leq L_u \|v\|_X;$$

(ii) *the gradient $\partial\varphi(u)$ is a nonempty, convex, and weakly* compact subset of X^* bounded by a Lipschitz constant L_u near x;*

(iii) *for every* $v \in X$, *we have*

$$\varphi^0(u;v) = \max\{\langle u^*, v\rangle | u^* \in \partial\varphi(u)\}.$$

We end this section with the notions of several classes of α-approximating sequences and α-well-posedness for GVHVI. Let $\alpha : X \to R_+$ be a functional.

Definition 2.7 A sequence $\{u_n\}$ in K is an α-approximating sequence for GVHVI if there exist $\{u_n^*\}$ in X^* with $u_n^* \in F(u_n)$ and a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that, for every $n \in N$,

$$\langle u_n^* + Tu_n - f, v - u_n\rangle + g(u_n, v) + J^\circ(u_n; v - u_n) \geq -\epsilon_n\alpha(v - u_n), \quad \forall v \in K.$$

In particular, if $\alpha(\cdot) = \|\cdot\|_X$, then $\{u_n\}$ is said to be an approximating sequence for GVHVI.

Definition 2.8 GVHVI is said to be strongly (respectively, weakly) α-well-posed if it has a unique solution u and every α-approximating sequence $\{u_n\}$ strongly (respectively, weakly) converges to u. In particular, if $\alpha(\cdot) = \|\cdot\|_X$, then GVHVI is said to be strongly (respectively, weakly) well-posed.

Definition 2.9 GVHVI is said to be strongly (respectively, weakly) α-well-posed in the generalized sense if the solution set Γ of GVHVI is nonempty and every α-approximating sequence $\{u_n\}$ has a subsequence that strongly (respectively, weakly) converges to some point of Γ. In particular, if $\alpha(\cdot) = \|\cdot\|_X$, then GVHVI is said to be strongly (respectively, weakly) well-posed in the generalized sense.

Remark 2.10 Strong α-well-posedness (in the generalized sense) implies weak α-well-posedness (in the generalized sense), but the converse is not true in general.

3 The characterizations of well-posedness for GVHVI

In this section, we establish metric characterizations and derive some conditions under which GVHVI is strongly (weakly) α-well-posed.

For any $\epsilon > 0$, we define the following two sets:

$$\Omega_\alpha(\epsilon) = \{u \in K : \exists u^* \in F(u) \text{ such that } \langle u^* + Tu - f, v - u\rangle + g(u, v) + J^\circ(u; v - u) \geq -\epsilon\alpha(v - u), \forall v \in K\}$$

and

$$\Phi_\alpha(\epsilon) = \{u \in K : \langle v^* + Tu - f, v - u\rangle + g(u, v) + J^\circ(u; v - u) \geq -\epsilon\alpha(v - u), \forall v \in K, \forall v^* \in F(v)\}.$$

Denote by Γ the set of solutions to GVHVI. It is clear that $\Gamma = \Omega_0(\epsilon)$.

Lemma 3.1 *Assume that*:

(i) K *is a nonempty closed subset of a real Banach space* X;

(ii) $T : K \to X_w^*$ *is continuous*;
(iii) $g : K \times K \to R$ *is u.s.c. with respect to the first variable*;
(iv) $\alpha : X \to R_+$ *is such that* $\liminf_{n\to\infty} \alpha(v_n) \le \alpha(v)$ *whenever* $v_n \to v$.
Then, for every $\epsilon > 0$, *the set* $\Phi_\alpha(\epsilon)$ *is closed in* X.

Proof Let $\{u_n\} \subset \Phi_\alpha(\epsilon)$ be s sequence such that $u_n \to u$ in X. Then $u \in K$, and, for all $v \in K$ and $v^* \in F(v)$,

$$\langle v^* + Tu_n - f, v - u_n\rangle + g(u_n, v) + J^\circ(u_n; v - u_n) \ge -\epsilon\alpha(v - u_n).$$

By the assumptions and the properties of J° we have

$$\begin{aligned} &\langle v^* + Tu - f, v - u\rangle + g(u, v) + J^\circ(u; v - u) \\ &\quad\ge \limsup_{n\to\infty}\bigl[\langle v^* + Tu_n - f, v - u_n\rangle + g(u_n, v) + J^\circ(u_n; v - u_n)\bigr] \\ &\quad\ge \limsup_{n\to\infty} -\epsilon\alpha(v - u_n) \\ &\quad= -\epsilon \liminf_{n\to\infty} \alpha(v - u_n) \\ &\quad\ge -\epsilon\alpha(v - u), \end{aligned}$$

and hence

$$\langle v^* + Tu - f, v - u\rangle + g(u, v) + J^\circ(u; v - u) - \epsilon\alpha(v - u), \quad \forall v \in K, \forall v^* \in F(v),$$

which shows that $u \in \Phi_\alpha(\epsilon)$. □

Lemma 3.2 *Assume that*:
(i) K *is a nonempty convex subset of a real Banach space* X;
(ii) $F : K \to P(X^*)$ *is l.h.c. and monotone*;
(iii) $g : K \times K \to R$ *is convex with respect to the second variable*;
(iv) $\alpha : X \to R_+$ *is convex with* $\alpha(tv) = t\alpha(v)$ *for all* $t \ge 0$ *and* $v \in X$.
Then $\Omega_\alpha(\epsilon) = \Phi_\alpha(\epsilon)$ *for all* $\epsilon > 0$.

Proof We first show that $\Omega_\alpha(\epsilon) \subset \Phi_\alpha(\epsilon)$. Indeed, take arbitrary $u \in \Omega_\alpha(\epsilon)$. Then there exists $u^* \in F(u)$ such that

$$\langle u^* + Tu - f, v - u\rangle + g(u, v) + J^\circ(u; v - u) \ge -\epsilon\alpha(v - u), \quad \forall v \in K.$$

According to the monotonicity of F, we obtain

$$\langle v^* + Tu - f, v - u\rangle + g(u, v) + J^\circ(u; v - u) \ge -\epsilon\alpha(v - u), \quad \forall v \in K, \forall v^* \in F(v),$$

which means that $u \in \Phi_\alpha(\epsilon)$. Therefore $\Omega_\alpha(\epsilon) \subset \Phi_\alpha(\epsilon)$.

Now we show that $\Phi_\alpha(\epsilon) \subset \Omega_\alpha(\epsilon)$. Indeed, take arbitrary $u \in \Phi_\alpha(\epsilon)$. Then

$$\langle v^* + Tu - f, v - u\rangle + g(u, v) + J^\circ(u; v - u) \ge -\epsilon\alpha(v - u), \quad \forall v \in K, \forall v^* \in F(v).$$

Since the set K is convex, for any $v \in K$ and $\lambda \in [0,1]$, taking $v_\lambda := \lambda v + (1-\lambda)u \in K$ in this inequality, we have

$$\langle v_\lambda^* + Tu - f, v_\lambda - u \rangle + g(u, v_\lambda) + J^\circ(u; v_\lambda - u) \geq -\epsilon\alpha(v_\lambda - u), \quad \forall v_\lambda^* \in F(v_\lambda).$$

Then by (iii), (iv), and the properties of J° we obtain

$$\langle v_\lambda^* + Tu - f, v - u \rangle + g(u, v) + J^\circ(u; v - u) \geq -\epsilon\alpha(v - u), \quad \forall v_\lambda^* \in F(v_\lambda). \tag{3.1}$$

Let $u^* \in F(u)$ be fixed, and let $v_\lambda^* \in F(v_\lambda)$ be such that $v_\lambda^* \rightharpoonup u^*$ in X^* (the existence of such a sequence is ensured by the fact that F is l.h.c.). Taking the limit as $\lambda \to 0$ in (3.1), we obtain

$$\begin{aligned} &\langle u^* + Tu - f, v - u \rangle + g(u, v) + J^\circ(u; v - u) \\ &\quad = \lim_{\lambda \to 0} \bigl[\langle v_\lambda^* + Tu - f, v - u \rangle + g(u, v) + J^\circ(u; v - u)\bigr] \\ &\quad \geq -\epsilon\alpha(v - u), \end{aligned}$$

which implies that $u \in \Omega_\alpha(\epsilon)$. The proof is complete. □

The following result is a consequence of Lemmas 3.1 and 3.2.

Theorem 3.3 *Assume that*:

(i) *K is a nonempty closed convex subset of a real Banach space X*;
(ii) *$F : K \to P(X^*)$ is l.h.c. and monotone*;
(iii) *$T : K \to X_w^*$ is continuous*;
(iv) *$g : K \times K \to R$ is u.s.c. with respect to the first variable and convex with respect to the second variable*;
(v) *$\alpha : X \to R_+$ is continuous and convex with $\alpha(tv) = t\alpha(v)$ for all $t \geq 0$ and $v \in X$.*

Then $\Omega_\alpha(\epsilon) = \Phi_\alpha(\epsilon)$ is closed in X for all $\epsilon > 0$. Moreover, $\Gamma = \Omega_0(\epsilon) = \Phi_0(\epsilon)$, that is, GVHVI is equivalent to the following problem:

Find $u \in K$ such that

$$\langle v^* + Tu - f, v - u \rangle + g(u, v) + J^\circ(u; v - u) \geq 0, \quad \forall v \in K, v^* \in F(v).$$

Theorem 3.4 *GVHVI is strongly α-well-posed if and only if Γ is nonempty and*

$$\lim_{\epsilon \to 0} \operatorname{diam}\bigl(\Omega_\alpha(\epsilon)\bigr) = 0.$$

Proof The proof is similar to that of Theorem 4.3 in [26] by the assumptions of g. □

Theorem 3.5 *Assume that all the assumptions of Theorem 3.3 are satisfied. Then GVHVI is strongly α-well-posed if and only if*

$$\Omega_\alpha(\epsilon) \neq \emptyset \quad \forall \epsilon \geq 0 \quad \textit{and} \quad \lim_{\epsilon \to 0} \operatorname{diam}\bigl(\Omega_\alpha(\epsilon)\bigr) = 0. \tag{3.2}$$

Proof Suppose that GVHVI is strongly α-well-posed. Then GVHVI has a unique solution $u \in K$, and thus $\Gamma \neq \emptyset$. Now, we prove that (3.2) holds. Clearly, $\Omega_\alpha(\epsilon) \supset \Gamma \neq \emptyset$. For the second part of (3.2), arguing by contradiction, let us assume that $\operatorname{diam}(\Omega_\alpha(\epsilon))$ does not tend to 0 as $\epsilon \to 0$. Thus for any nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$, there exists a constant $\beta > 0$ such that, for each $n \in N$, there exist $u_n^{(1)}, u_n^{(2)} \in \Omega_\alpha(\epsilon_n)$ satisfying

$$\left\| u_n^{(1)} - u_n^{(2)} \right\| > \beta > 0. \tag{3.3}$$

Since $u_n^{(1)}, u_n^{(2)} \in \Omega_\alpha(\epsilon_n)$, we know that the sequences $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ are both α-approximating sequences of GVHVI, and thus

$$\lim_{n\to} u_n^{(1)} = \lim_{n\to} u_n^{(2)} = u. \tag{3.4}$$

From (3.3) and (3.4) we have

$$0 < \beta < \left\| u_n^{(1)} - u_n^{(2)} \right\| \le \left\| u_n^{(1)} - u \right\| + \left\| u_n^{(2)} - u \right\| \to 0,$$

which is a contradiction.

Conversely, assume that condition (3.2) holds. Let $\{u_n\}$ in K be an α-approximating sequence for GVHVI. Then, there exist $\{u_n^*\}$ in X^* with $u_n^* \in F(u_n)$ and a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that, for every $n \in N$,

$$\left\langle u_n^* + Tu_n - f, v - u_n \right\rangle + g(u_n, v) + J^\circ(u_n; v - u_n) \ge -\epsilon_n \alpha(v - u_n), \quad \forall v \in K,$$

that is, $u_n \in \Omega_\alpha(\epsilon_n)$ for all $n \in N$. By condition (3.2) we deduce that the sequence $\{u_n\}$ is a Cauchy sequence, and so $\{u_n\}$ converges strongly to some point $u \in K$. Let us show that $u \in K$ is a solution for GVHVI. By the monotonicity of F we obtain that, for every $n \in N$,

$$\begin{aligned} &\left\langle v^* + Tu_n - f, v - u_n \right\rangle + g(u_n, v) + J^\circ(u_n; v - u_n) \\ &\quad \ge \left\langle u_n^* + Tu_n - f, v - u_n \right\rangle + g(u_n, v) + J^\circ(u_n; v - u_n) \\ &\quad \ge -\epsilon_n \alpha(v - u_n), \quad \forall v \in K, v^* \in F(v). \end{aligned}$$

By the assumptions we obtain that

$$\begin{aligned} &\left\langle v^* + Tu - f, v - u \right\rangle + g(u, v) + J^\circ(u; v - u) \\ &\quad \ge \limsup_{n\to\infty} \left[\left\langle v^* + Tu_n - f, v - u_n \right\rangle + g(u_n, v) + J^\circ(u_n; v - u_n) \right] \\ &\quad \ge \limsup_{n\to\infty} -\epsilon_n \alpha(v - u_n) \\ &\quad = \limsup_{n\to\infty} \alpha\left(-\epsilon_n (v - u_n)\right) \\ &\quad = 0, \end{aligned}$$

which implies that

$$\left\langle v^* + Tu - f, v - u \right\rangle + g(u, v) + J^\circ(u; v - u) \ge 0, \quad \forall v \in K, \forall v^* \in F(v).$$

It follows from Theorem 3.3 that there exists $u^* \in F(u)$ such that

$$\langle u^* + Tu - f, v - u\rangle + g(u,v) + J^\circ(u; v-u) \geq 0, \quad \forall v \in K.$$

Then $u \in K$ is a solution of GVHVI.

Finally, we prove that the solution u is unique. If there exists another solution $u' \in K$, then $u, u_1 \in \Omega_\alpha(\epsilon)$ for all $\epsilon > 0$, and

$$0 < \|u - u'\| \leq \operatorname{diam}(\Omega_\alpha(\epsilon)) \to 0 \quad \text{as } \epsilon \to 0,$$

which is a contradiction. This completes the proof. □

Theorem 3.6 *Assume that*:

(i) *K is a nonempty closed convex subset of a real reflexive Banach space X*;
(ii) *$F : K \to P(X^*)$ is l.h.c. and monotone*;
(iii) *$T : K \to X^*$ is compact*;
(iv) *$g : K \times K \to R$ is weakly u.s.c. with respect to the first variable and convex with respect to the second variable*;
(v) *$\limsup_{n\to\infty} J^\circ(u_n; v - u_n) \leq J^\circ(u; v-u)$ for all $v \in X$ whenever $u_n \rightharpoonup u$ as $n \to \infty$*;
(vi) *$\alpha : X \to R_+$ is a continuous and convex functional with $\alpha(tv) = t\alpha(v)$ for all $t \geq 0$ and $v \in X$.*

Then GVHVI is weakly α-well-posed if and only if GVHVI has a unique solution and there exists $\epsilon_0 > 0$ such that $\Omega_\alpha(\epsilon_0)$ is nonempty and bounded.

Proof The necessity is obvious. We now prove the sufficiency. Let $\{u_n\}$ be an α-approximating sequence for GVHVI. Then, there exist $\{u_n^*\}$ in X^* with $u_n^* \in F(u_n)$ and a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that, for every $n \in N$,

$$\langle u_n^* + Tu_n - f, v - u_n\rangle + g(u_n, v) + J^\circ(u_n; v - u_n) \geq -\epsilon_n\alpha(v - u_n)$$

for all $v \in K$. We claim that the sequence $\{u_n\}$ is bounded in X. Indeed, since $\Omega_\alpha(\epsilon_0)$ is bounded and $\Omega_\alpha(\epsilon) \subset \Omega_\alpha(\epsilon_0)$ for all $\epsilon \in (0, \varepsilon_0)$, there exists $n_0 \in N$ such that $\epsilon_{n_0} \in (0, \varepsilon_0)$ and $u_n \in \Omega_\alpha(\epsilon_0)$ for all $n \geq n_0$, which shows that $\{u_n\}$ is bounded in X.

Since the Banach space X is reflexive, we can choose a subsequence of $\{u_n\}$, denoted by $\{u_n\}$ again, such that $u_n \rightharpoonup \overline{u}$ as $n \to \infty$ for some $\overline{u} \in X$. Let us show that $\overline{u} \in K$ is a solution for GVHVI. Obviously, $\overline{u} \in K$. By the monotonicity of F we obtain that

$$\begin{aligned}
&\langle v^* + Tu_n - f, v - u_n\rangle + g(u_n, v) + J^\circ(u_n; v - u_n) \\
&\quad \geq \langle u_n^* + Tu_n - f, v - u_n\rangle + g(u_n, v) + J^\circ(u_n; v - u_n) \\
&\quad \geq -\epsilon_n\alpha(v - u_n), \quad \forall v \in K, v^* \in F(v), \forall n \in N.
\end{aligned}$$

By the assumptions, we obtain that

$$\begin{aligned}
&\langle v^* + T\overline{u} - f, v - \overline{u}\rangle + g(\overline{u}, v) + J^\circ(\overline{u}; v - \overline{u}) \\
&\quad \geq \limsup_{n\to\infty}\bigl[\langle v^* + Tu_n - f, v - u_n\rangle + g(u_n, v) + J^\circ(u_n; v - u_n)\bigr]
\end{aligned}$$

$$\geq \limsup_{n\to\infty} -\epsilon_n \alpha(v - u_n)$$
$$= \limsup_{n\to\infty} \alpha\bigl(-\epsilon_n(v - u_n)\bigr)$$
$$= 0,$$

which implies that

$$\bigl\langle v^* + T\overline{u} - f, v - \overline{u}\bigr\rangle + g(\overline{u}, v) + J^\circ(\overline{u}; v - \overline{u}) \geq 0, \quad \forall v \in K, \forall v^* \in F(v).$$

It follows from Theorem 3.3 that there exists $\overline{u}^* \in F(\overline{u})$ such that

$$\bigl\langle u^* + T\,\overline{u} - f, v - \overline{u}\bigr\rangle + g(\overline{u}, v) + J^\circ(\overline{u}; v - \overline{u}) \geq 0, \quad \forall v \in K,$$

Therefore $\overline{u} \in K$ is a solution to problem GVHVI, and so we get that GVHVI is weakly α-well-posed by the uniqueness of the solution to problem GVHVI. This completes the proof. □

Remark 3.7 In the theorem, condition (v) can be found in [30], and the condition that there exists $\epsilon_0 > 0$ such that $\Omega_\alpha(\epsilon_0)$ is nonempty and bounded can be replaced by the conditions that K is bounded or that there exists $n_0 \in N$ such that, for every $u \in K \setminus B_{n_0}$, there exists $v \in K$ with $\|v\| < \|u\|$ such that

$$\sup_{u^* \in F(u)} \bigl\langle u^* + Tu - f, v - u\bigr\rangle + g(u, v) + J^\circ(u; v - u) \leq -\frac{1}{n_0}.$$

See [34, 36, 39] for more detail.

Next, we give some equivalence results for the strong α-posedness in the generalized sense.

Theorem 3.8 *Assume that all the assumptions of Theorem 3.5 are satisfied. Then GVHVI is strongly α-well-posed in the generalized sense if and only if Γ is nonempty compact and*

$$\lim_{\epsilon\to 0} e\bigl(\Omega_\alpha(\epsilon), \Gamma\bigr) = 0,$$

where $e(A, B) := \sup_{a\in A} d(a, B)$ with $d(a, B) := \inf_{b\in B} \|a - b\|$.

Proof The proof is similar to that of Theorem 5.1 in [26] by the assumptions of g. □

Theorem 3.9 *Assume that all the assumptions of Theorem 3.5 are satisfied. Then GVHVI is strongly α-well-posed in the generalized sense if and only if*

$$\Omega_\alpha(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0, \quad \textit{and} \quad \lim_{\epsilon\to 0} \mu\bigl(\Omega_\alpha(\epsilon)\bigr) = 0.$$

Proof The proof is similar to that of Theorem 3.2 in [3] by the assumptions of g. □

Theorem 3.10 *Assume that all the assumptions of Theorem 3.6 are satisfied. Then GVHVI is weakly α-well-posed in the generalized sense if and only if there exists $\epsilon_0 > 0$ such that $\Omega_\alpha(\epsilon_0)$ is nonempty and bounded.*

Proof The proof is similar to that of Theorem 3.6 by the assumptions of g. □

4 Well-posedness for GMEP

In this section, we consider the following generalized mixed equilibrium problem (GMEP):

Find $u \in K$ such that, for some $u^* \in F(u)$,

$$\langle u^*, \eta(u,v)\rangle + \langle Tu - f, v - u\rangle + g(u,v) + h(u,v) \geq 0, \quad \forall v \in K,$$

where $\eta : K \times K \to X$ is an operator. The existence of solutions to this problem when $T \equiv 0$ and $f \equiv 0$ can be found in [25].

To study GMEP, we introduce the concept of η-monotonicity (see [7, 8]).

Definition 4.1 Let $F : K \to P(X^*)$ be a set-valued operator. F is said to be η-monotone if there exists a function $\eta : K \times K \to X$ such that, for all $u, v \in K$,

$$\langle v^* - u^*, \eta(u,v)\rangle \geq 0, \quad \forall u^* \in F(u), \forall v^* \in F(v). \tag{4.1}$$

Remark 4.2 If $\eta(u,v) = v - u$ for all $u, v \in X$, then (4.1) becomes

$$\langle v^* - u^*, v - u\rangle \geq 0, \quad \forall u^* \in F(u), \forall v^* \in F(v),$$

that is, F is monotone.

For any $\epsilon > 0$, we define the following two sets:

$$\begin{aligned}\Omega_{\eta,\alpha}(\epsilon) = \{&u \in K : \exists u^* \in F(u) \text{ such that } \langle u^*, \eta(u,v)\rangle + \langle Tu - f, v - u\rangle + g(u,v) \\ &+ h(u,v)) \geq -\epsilon\alpha(v - u), \forall v \in K\}\end{aligned}$$

and

$$\begin{aligned}\Phi_{\eta,\alpha}(\epsilon) = \{&u \in K : \langle v^*, \eta(u,v)\rangle + \langle Tu - f, v - u\rangle + g(u,v) \\ &+ h(u,v) \geq -\epsilon\alpha(v - u), \forall v \in K, \forall v^* \in F(v)\}.\end{aligned}$$

Denote by Γ_η the set of solutions to GMEP. It is clear that $\Gamma = \Omega_0(\epsilon)$.

We can obtain similar results.

Theorem 4.3 *Assume that all the assumptions of Theorem 3.3 are satisfied and, in addition, $\eta : K \times K \to X$ is continuous on $K \times K$ with $\eta(u,u) = 0$ for any $u \in K$ and affine with respect to the first variable. Let $h : K \times K \to R$ be such that:*

(i) *$h(u,u) = 0$ for all $u \in X$,*
(ii) *for all $v \in K$, $h(\cdot, v)$ is u.s.c.,*
(iii) *for all $u \in K$, $h(u, \cdot)$ is convex.*

Then $\Omega_{\eta,\alpha}(\epsilon) = \Phi_{\eta,\alpha}(\epsilon)$ is closed in X for all $\epsilon > 0$. Moreover, $\Gamma_\eta = \Omega_{\eta,0}(\epsilon) = \Phi_{\eta,0}(\epsilon)$, that is, GMEP is equivalent to the following problem:

Find $u \in K$ such that

$$\langle v^* + Tu - f, \eta(u,v)\rangle + g(u,v) + h(u,v) \geq 0, \quad \forall v \in K, v^* \in F(v).$$

Theorem 4.4 *Assume that all the assumptions of Theorem 3.5 are satisfied and, in addition, $\eta : K \times K \to X$ is continuous on $K \times K$ with $\eta(u,u) = 0$ for any $u \in K$ and affine with respect to the first variable. Let $h : K \times K \to R$ be such that:*

(i) *$h(u,u) = 0$ for all $u \in X$,*
(ii) *for all $v \in K$, $h(\cdot, v)$ is u.s.c.,*
(iii) *for all $u \in K$, $h(u, \cdot)$ is convex.*

Then GMVHVI is strongly α-well-posed if and only if

$$\Omega_{\eta,\alpha}(\epsilon) \neq \emptyset, \quad \forall \epsilon \geq 0, \quad \textit{and} \quad \lim_{\epsilon \to 0} \operatorname{diam}\bigl(\Omega_{\eta,\alpha}(\epsilon)\bigr) = 0.$$

Theorem 4.5 *Assume that all the assumptions of Theorem 3.6 are satisfied and, in addition, $\eta : K \times K \to X$ is continuous on $K \times K$ with $\eta(u,u) = 0$ for any $u \in K$ and affine with respect to the first variable. Let $h : K \times K \to R$ be such that:*

(i) *$h(u,u) = 0$ for all $u \in X$,*
(ii) *for all $v \in K$, $h(\cdot, v)$ is weakly u.s.c.,*
(iii) *for all $u \in K$, $h(u, \cdot)$ is convex.*

Then GMEP is weakly α-well-posed if and only if GMEP has a unique solution and there exists $\epsilon_0 > 0$ such that $\Omega_\alpha(\epsilon_0)$ is nonempty and bounded.

Theorem 4.6 *Assume that all the assumptions of Theorem 3.5 are satisfied and, in addition, $\eta : K \times K \to X$ is continuous on $K \times K$ with $\eta(u,u) = 0$ for any $u \in K$ and affine with respect to the first variable. Let $h : K \times K \to R$ is such that:*

(i) *$h(u,u) = 0$ for all $u \in X$,*
(ii) *for all $v \in K$, $h(\cdot, v)$ is u.s.c.,*
(iii) *for all $u \in K$, $h(u, \cdot)$ is convex.*

Then GMEP is strongly α-well-posed in the generalized sense if and only if

$$\Omega_{\eta,\alpha}(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0, \quad \textit{and} \quad \lim_{\epsilon \to 0} \mu\bigl(\Omega_{\eta,\alpha}(\epsilon)\bigr) = 0.$$

Theorem 4.7 *Assume that all the assumptions of Theorem 3.6 are satisfied and, in addition, $\eta : K \times K \to X$ is continuous on $K \times K$ with $\eta(u,u) = 0$ for any $u \in K$ and affine with respect to the first variable. Let $h : K \times K \to R$ be such that:*

(i) *$h(u,u) = 0$ for all $u \in X$,*
(ii) *for all $v \in K$, $h(\cdot, v)$ is weakly u.s.c.,*
(iii) *for all $u \in K$, $h(u, \cdot)$ is convex.*

Then GMEP is weakly α-well-posed in the generalized sense if and only if there exists $\epsilon_0 > 0$ such that $\Omega_\alpha(\epsilon_0)$ is nonempty and bounded.

5 Conclusion

In this paper, inspired by the previous works, we study the well-posedness for GVHVI. Under relatively weak conditions for the data F, T, g, J (see Theorems 3.3 and 3.6), we provide some equivalence results for the strong and weak α-well-posed GVHVI in the generalized sense. Our results generalize and improve many known results and can be applied to many other problems.

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