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Simple form of a projection set in hybrid iterative schemes for non-linear mappings, application of inequalities and computational experiments

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Abstract

Some relaxed hybrid iterative schemes for approximating a common element of the sets of zeros of infinite maximal monotone operators and the sets of fixed points of infinite weakly relatively non-expansive mappings in a real Banach space are presented. Under mild assumptions, some strong convergence theorems are proved. Compared to recent work, two new projection sets are constructed, which avoids calculating infinite projection sets for each iterative step. Some inequalities are employed sufficiently to show the convergence of the iterative sequences. A specific example is listed to test the effectiveness of the new iterative schemes, and computational experiments are conducted. From the example, we can see that although we have infinite choices to choose the iterative sequences from an interval, different choice corresponds to different rate of convergence.

Keywords: Iterative scheme; Lyapunov functional; Metric projection; Maximal monotone operator; Weakly relatively non-expansive mapping

1 Introduction

Throughout this paper, let X be a real Banach space with norm $\|\cdot\|$ and X^* be the dual space of X. Let K be a non-empty closed and convex subset of X. Let $\langle x,f\rangle$ be the value of $f\in X^*$ at $x\in X$. We write $x_n\to x$ to denote that $\{x_n\}$ converges strongly to x and $x_n\rightharpoonup x$ to denote that $\{x_n\}$ converges weakly to x.

Suppose that A is a multi-valued operator from X into X^*. A is said to be monotone [1] if for $\forall v_i\in Au_i$, $i=1,2$, one has $\langle u_1-u_2,v_1-v_2\rangle\ge 0$. The monotone operator A is called maximal monotone if $R(J+kA)=X^*$, for $k>0$, where $J:X\to 2^{X^*}$ is the normalized duality mapping defined by

$$J(x)=\bigl\{f\in X^*:\langle x,f\rangle=\|x\|^2=\|f\|^2\bigr\},\quad \forall x\in X.$$

A point $x\in D(A)$ is called a zero of A if $Ax=0$. The set of zeros of A is denoted by $A^{-1}0$.

Suppose that the Lyapunov functional $\phi:X\times X\to(0,+\infty)$ is defined as follows:

$$\phi(x,y)=\|x\|^2-2\bigl\langle x,j(y)\bigr\rangle+\|y\|^2,\quad \forall x,y\in X,j(y)\in J(y).$$

Let T be a single-valued mapping of K into itself.

(1) If $Tp = p$, then p is called a fixed point of T. And $\mathrm{Fix}(T)$ denotes the set of fixed points of T;

(2) If there exists a sequence $\{x_n\} \subset K$ which converges weakly to $p \in K$ such that $x_n - Tx_n \to 0$, as $n \to \infty$, then p is called an asymptotic fixed point of T [2]. And $\widehat{\mathrm{Fix}}(T)$ denotes the set of asymptotic fixed points of T;

(3) If there exists a sequence $\{x_n\} \subset K$ which converges strongly to $p \in K$ such that $x_n - Tx_n \to 0$, as $n \to \infty$, then p is called a strong asymptotic fixed point of T [2]. And $\widetilde{\mathrm{Fix}}(T)$ denotes the set of strong asymptotic fixed points of T;

(4) T is called strongly relatively non-expansive [2] if $\widehat{\mathrm{Fix}}(T) = \mathrm{Fix}(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p,x)$ for $x \in K$ and $p \in \mathrm{Fix}(T)$;

(5) T is called weakly relatively non-expansive [2] if $\widetilde{\mathrm{Fix}}(T) = \mathrm{Fix}(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p,x)$ for $x \in K$ and $p \in \mathrm{Fix}(T)$.

If X is a real reflexive and strictly convex Banach space and K is a non-empty closed and convex subset of X, then for each $x \in X$ there exists a unique point $x_0 \in K$ such that $\|x - x_0\| = \inf\{\|x - y\| : y \in K\}$. In this case, we can define the metric projection mapping $P_K : X \to K$ by $P_K x = x_0$ for $\forall x \in X$ [3].

If X is a real reflexive, strictly convex, and smooth Banach space and K is a non-empty closed and convex subset of X, then for $\forall x \in X$, there exists a unique point $x_0 \in K$ such that $\phi(x_0, x) = \inf\{\phi(y,x) : y \in K\}$. In this case, we can define the generalized projection mapping $\Pi_K : X \to K$ by $\Pi_K x = x_0$ for $\forall x \in X$ [3].

Note that if X is a Hilbert space H, then P_K and Π_K are coincidental.

Since maximal monotone operators and weakly (or strongly) relatively non-expansive mappings have close connection with practical problems, one has a good reason to study them. During past years, much work has been done in designing iterative schemes to approximate a common element of the set of zeros of maximal monotone operators and the set of fixed points of weakly (or strongly) relatively non-expansive mappings. Among them, a projection iterative scheme is considered as one of the effective iterative schemes which almost always generates strongly convergent iterative sequences (see [4–8] and the references therein). Next, we list some recent closely related work.

Klin-eam et al. [5] presented the following projection iterative scheme for maximal monotone operator A and two strongly relatively non-expansive mappings B and C in a real uniformly convex and uniformly smooth Banach space X.

$$\begin{cases} v_n = J^{-1}[\beta_n Jx_n + (1-\beta_n)JC(J + r_n A)^{-1}Jx_n], \\ y_n = J^{-1}[\alpha_n Jx_n + (1-\alpha_n)JBv_n], \\ H_n = \{p \in K : \phi(p, y_n) \leq \phi(p, x_n)\}, \\ V_n = \{p \in K : \langle p - x_n, Jx_1 - Jx_n\rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n \cap V_n}(x_1), \quad n \in N. \end{cases} \tag{1.1}$$

Then $\{x_n\}$ generated by (1.1) converges strongly to $\Pi_{A^{-1}0 \cap \mathrm{Fix}(B) \cap \mathrm{Fix}(C)}(x_1)$.

Compared to (1.1), the following so-called monotone projection iterative scheme for maximal monotone operator A and strongly relatively non-expansive mapping B in a real uniformly convex and uniformly smooth Banach space X is presented [4].

$$\begin{cases} x_1 \in X, \qquad r_1 > 0, \\ y_n = (J + r_n A)^{-1} J(x_n + e_n), \\ z_n = J^{-1}[\alpha_n Jx_n + (1-\alpha_n)Jy_n], \\ u_n = J^{-1}[\beta_n Jx_n + (1-\beta_n)JBz_n], \\ H_1 = \{p \in X : \phi(p, z_1) \le \alpha_1 \phi(p, x_1) + (1-\alpha_1)\phi(p, x_1 + e_1)\}, \\ V_1 = \{p \in X : \phi(p, u_1) \le \beta_1 \phi(p, x_1) + (1-\beta_1)\phi(p, z_1)\}, \\ W_1 = X, \\ H_n = \{p \in H_{n-1} \cap V_{n-1} \cap W_{n-1} : \phi(p, z_n) \le \alpha_n \phi(p, x_n) + (1-\alpha_n)\phi(p, x_n + e_n)\}, \\ V_n = \{p \in H_{n-1} \cap V_{n-1} \cap W_{n-1} : \phi(p, u_n) \le \beta_n \phi(p, x_n) + (1-\beta_n)\phi(p, z_n)\}, \\ W_n = \{p \in H_{n-1} \cap V_{n-1} \cap W_{n-1} : \langle p - x_n, Jx_1 - Jx_n \rangle \le 0\}, \\ x_{n+1} = \Pi_{H_n \cap V_n \cap W_n}(x_1), \quad n \in N. \end{cases} \tag{1.2}$$

Then $\{x_n\}$ generated by (1.2) converges strongly to $\Pi_{A^{-1}0 \cap \mathrm{Fix}(B)}(x_1)$.

In recent work, Wei et al. [8] extended the corresponding topic to the case for infinite maximal monotone operators A_i and infinite weakly relatively non-expansive mappings B_i.

$$\begin{cases} x_1 \in X, \qquad r_{1,i} \in (0, +\infty), \quad i \in N, \\ y_{n,i} = (J + r_{n,i} A_i)^{-1} J(x_n + e_n), \quad i \in N, \\ z_{n,i} = J^{-1}[\alpha_n Jx_n + (1-\alpha_n)JB_i y_{n,i}], \quad i \in N, \\ V_1 = X = W_1, \\ V_{n+1,i} = \{p \in X : \langle y_{n,i} - p, J(x_n + e_n) - Jy_{n,i} \rangle \ge 0\}, \quad i \in N, \\ V_{n+1} = (\bigcap_{i=1}^{\infty} V_{n+1,i}) \cap V_n, \\ W_{n+1,i} = \{p \in V_{n+1,i} : \phi(p, z_{n,i}) \le \alpha_n \phi(p, x_n) + (1-\alpha_n)\phi(p, y_{n,i})\}, \quad i \in N, \\ W_{n+1} = (\bigcap_{i=1}^{\infty} W_{n+1,i}) \cap W_n, \\ U_{n+1} = \{p \in W_{n+1} : \|x_1 - p\|^2 \le \|P_{W_{n+1}}(x_1) - x_1\|^2 + \lambda_{n+1}\}, \\ x_{n+1} \in U_{n+1}, \quad n \in N. \end{cases} \tag{1.3}$$

Then $\{x_n\}$ generated by (1.3) converges strongly to

$$P_{\bigcap_{n=1}^{\infty} W_n}(x_1) \in \left(\bigcap_{i=1}^{\infty} A_i^{-1}0\right) \cap \left(\bigcap_{i=1}^{\infty} \mathrm{Fix}(B_i)\right).$$

Compared to traditional (monotone) projection iterative schemes (e.g., (1.1) and (1.2)), some different ideas in (1.3) can be seen. (1) Metric projection mapping $P_{W_{n+1}}$ instead of generalized projection mapping Π is involved in (1.3). (2) The iterative item x_{n+1} can be chosen arbitrarily in the set U_{n+1}, while x_{n+1} in both (1.1) and (1.2) and some others are needed to be the unique value of generalized projection mapping Π. (3) $\{x_n\}$ in (1.3) converges strongly to the unique value of metric projection mapping P, while $\{x_n\}$ in both (1.1) and (1.2) converges strongly to the unique value of the generalized projection mapping Π.

A special case of (1.3) is presented as Corollary 2.13 in [8]. Now, we rewrite it as follows:

$$\begin{cases} x_1 \in H, \qquad e_1 \in H, \\ y_n = (I + r_n A)^{-1}(x_n + e_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n) B y_n, \\ U_1 = H = V_1, \\ U_{n+1} = \{p \in U_n : (y_n - p)(x_n + e_n - y_n) \geq 0, \\ \|p - z_n\|^2 \leq \alpha_n \|p - x_n\|^2 + (1 - \alpha_n)\|p - y_n\|^2\}, \\ V_{n+1} = \{p \in U_{n+1} : \|x_1 - p\|^2 \leq \|P_{U_{n+1}}(x_1) - x_1\|^2 + \lambda_{n+1}\}, \\ x_{n+1} \in V_{n+1}, \quad n \in N. \end{cases} \tag{1.4}$$

Based on iterative scheme (1.4), an iterative sequence is defined as follows after taking $H = (-\infty, +\infty)$, $Ax = 2x$, $Bx = x$ for $x \in (-\infty, +\infty)$, $e_n = \alpha_n = \lambda_n = \frac{1}{n}$, and $r_n = 2^{n-1}$:

$$\begin{cases} x_1 = 1, \\ y_n = \frac{x_n + e_n}{1 + 2r_n}, \quad n \in N, \\ x_{n+1} = \frac{x_1 + y_n - \sqrt{(x_1 - y_n)^2 + \lambda_{n+1}}}{2}, \quad n \in N. \end{cases} \tag{1.5}$$

A computational experiment based on (1.5) is conducted in [8], from which we can see the effectiveness of iterative scheme (1.4).

Inspired by the work of [8], three questions come to our mind. (1) In iterative scheme (1.3), in each iterative step n, countable sets $V_{n+1,i}$ and $W_{n+1,i}$ are needed to be evaluated. It is formidable. Can we avoid it? (2) x_{n+1} in either (1.3) or (1.4) can be chosen arbitrarily in a set, can a different choice of x_{n+1} in V_{n+1} lead to a different rate of convergence? (3) Which one is better, our new one or those in [8]? In this paper, we shall answer the questions, construct new simple projection sets in theoretical sense, and do computational experiments for some special cases.

2 Preliminaries

In this section, we list some definitions and results we need later. The modulus of convexity of X, $\delta_X : [0, 2] \to [0, 1]$, is defined as follows [9]:

$$\delta_X(\epsilon) = \sup\left\{1 - \frac{\|x + y\|}{2} : x, y \in X, \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\right\}$$

for $\forall \epsilon \in [0, 2]$. A Banach space X is called uniformly convex [9] if $\delta_X(\epsilon) > 0$ for $\forall \epsilon \in [0, 2]$. A Banach space X is called uniformly smooth [9] if the limit $\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$ is attained uniformly for $(x, y) \in X \times X$ with $\|x\| = \|y\| = 1$.

X is said to have Property (H): if for every sequence $\{x_n\} \subset X$ converging weakly to $x \in X$ and $\|x_n\| \to \|x\|$, one has $x_n \to x$, as $n \to \infty$. The uniformly convex and uniformly smooth Banach space X has Property (H).

It is well known that if X is a real uniformly convex and uniformly smooth Banach space, then the normalized duality mapping J is single-valued, surjective and $J(kx) = kJ(x)$ for $x \in X$ and $k \in (-\infty, +\infty)$. Moreover, J^{-1} is also the normalized duality mapping from X^*

into X, and both J and J^{-1} are uniformly continuous on each bounded subset of X or X^*, respectively [9].

Lemma 2.1 ([2]) *Suppose that X is a uniformly convex and uniformly smooth Banach space and K is a non-empty closed and convex subset of X. If $B : K \to K$ is weakly relatively non-expansive, then $\mathrm{Fix}(B)$ is a closed and convex subset of X.*

Lemma 2.2 ([1]) *Let $A : X \to 2^{X^*}$ be a maximal monotone operator, then*
(1) *$A^{-1}0$ is a closed and convex subset of X;*
(2) *if $x_n \to x$ and $y_n \in Ax_n$ with $y_n \rightharpoonup y$, or $x_n \rightharpoonup x$ and $y_n \in Ax_n$ with $y_n \to y$, then $x \in D(A)$ and $y \in Ax$.*

Lemma 2.3 ([8]) *Let K be a non-empty closed and convex subset of a uniformly smooth Banach space X. Let $x \in X$ and $x_0 \in K$. Then $\phi(x_0, x) = \inf_{y\in K} \phi(y, x)$ if and only if $\langle x_0 - z, Jx - Jx_0\rangle \geq 0$ for all $z \in K$.*

Lemma 2.4 ([10]) *Let X be a real uniformly smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$ as $n \to \infty$.*

Lemma 2.5 ([11]) *Let X be a real uniformly smooth and uniformly convex Banach space and $A : X \to 2^{X^*}$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$. Then, for $\forall x \in X$, $\forall y \in A^{-1}0$, and $r > 0$, one has $\phi(y, (J + rA)^{-1}Jx) + \phi((J + rA)^{-1}Jx, x) \leq \phi(y, x)$.*

Let $\{K_n\}$ be a sequence of non-empty closed and convex subsets of X. Then the strong lower limit of $\{K_n\}$, $s\text{-}\liminf K_n$, is defined as the set of all $x \in X$ such that there exists $x_n \in K_n$ for almost all n and it tends to x as $n \to \infty$ in the norm; the weak upper limit of $\{K_n\}$, $w\text{-}\limsup K_n$, is defined as the set of all $x \in X$ such that there exists a subsequence $\{K_{n_m}\}$ of $\{K_n\}$ and $x_{n_m} \in K_{n_m}$ for every n_m and it tends to x as $n_m \to \infty$ in the weak topology; the limit of $\{K_n\}$, $\lim K_n$, is the common value when $s\text{-}\liminf K_n = w\text{-}\limsup K_n$ [12].

Lemma 2.6 ([12]) *Let $\{K_n\}$ be a decreasing sequence of closed and convex subsets of X, i.e., $K_n \subset K_m$ if $n \geq m$. Then $\{K_n\}$ converges in X and $\lim K_n = \bigcap_{n=1}^{\infty} K_n$.*

Lemma 2.7 ([13]) *Suppose that X is a real uniformly convex Banach space. If $\lim K_n$ exists and is not empty, then $\{P_{K_n}x\}$ converges weakly to $P_{\lim K_n}x$ for every $x \in X$. Moreover, if X has Property* (H)*, the convergence is in norm.*

Lemma 2.8 ([14]) *Let X be a real uniformly convex Banach space and $r \in (0, +\infty)$. Then there exists a continuous, strictly increasing, and convex function $\eta : [0, 2r] \to [0, +\infty)$ with $\eta(0) = 0$ such that $\|kx + (1 - k)y\|^2 \leq k\|x\|^2 + (1 - k)\|y\|^2 - k(1 - k)\eta(\|x - y\|)$ for $k \in [0, 1], x, y \in X$ with $\|x\| \leq r$ and $\|y\| \leq r$.*

Lemma 2.9 ([15]) *Let X be the same as that in Lemma 2.8. Then there exists a continuous, strictly increasing, and convex function $\eta : [0, 2r] \to [0, +\infty)$ with $\eta(0) = 0$ such that $\| \sum_{i=1}^{\infty} k_i x_i\|^2 \leq \sum_{i=1}^{\infty} k_i\|x_i\|^2 - k_1 k_m \eta(\|x_1 - x_m\|)$ for all $\{x_n\}_{n=1}^{\infty} \subset \{x \in X : \|x\| \leq r\}$, $\{k_n\}_{n=1}^{\infty} \subset (0, 1)$ with $\sum_{n=1}^{\infty} k_n = 1$ and $m \in N$.*

3 Main results

In this section, our discussion is based on the following conditions:

(I_1) X is a real uniformly convex and uniformly smooth Banach space and $J: X \to X^*$ is the normalized duality mapping;

(I_2) $A_i : X \to X^*$ is maximal monotone and $B_i : X \to X$ is weakly relatively non-expansive for each $i \in N$. And $(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} \mathrm{Fix}(B_i)) \neq \emptyset$;

(I_3) $\{e_n\} \subset X$ is the error sequence such that $e_n \to 0$, as $n \to \infty$;

(I_4) $\{r_{n,i}\}$ and $\{\lambda_n\}$ are two real number sequences in $(0, +\infty)$ with $\inf_n r_{n,i} > 0$ for $i \in N$ and $\lambda_n \to 0$, as $n \to \infty$;

(I_5) $\{a_{n,i}\}$ and $\{b_i\}$ are two real number sequences in $(0,1)$ and $\sum_{i=1}^{\infty} a_{n,i} = 1 = \sum_{i=1}^{\infty} b_i$ for $n \in N$;

(I_6) $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $[0,1)$.

Theorem 3.1 *Let $\{x_n\}$ be generated by the following iterative scheme*:

$$\begin{cases} x_1 \in X, \qquad e_1 \in X, \\ y_n = J^{-1}[\alpha_n Jx_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i} J(J + r_{n,i}A_i)^{-1} J(x_n + e_n)], \\ z_n = J^{-1}[\beta_n Jx_n + (1-\beta_n)\sum_{i=1}^{\infty} b_i JB_i y_n], \\ U_1 = X = V_1, \\ U_{n+1} = \{v \in U_n : \phi(v, y_n) \le \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, x_n + e_n), \\ \qquad \phi(v, z_n) \le \beta_n \phi(v, x_n) + (1-\beta_n)\phi(v, y_n)\}, \\ V_{n+1} = \{v \in U_{n+1} : \|x_1 - v\|^2 \le \|P_{U_{n+1}}(x_1) - x_1\|^2 + \lambda_{n+1}\}, \\ x_{n+1} \in V_{n+1}, \quad n \in N. \end{cases} \tag{3.1}$$

If $0 \le \sup_n \alpha_n < 1$ and $0 \le \sup_n \beta_n < 1$, then $x_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} \mathrm{Fix}(B_i))$, as $n \to \infty$.

Proof We split the proof into seven steps.

Step 1. U_n is a non-empty closed and convex subset of X for each $n \in N$.

Noticing the definition of Lyapunov functional, we have

$$\begin{aligned} &\phi(v, y_n) \le \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, x_n + e_n) \\ &\quad \Longleftrightarrow \quad 2\alpha_n \langle v, Jx_n\rangle + 2(1-\alpha_n)\bigl\langle v, J(x_n + e_n)\bigr\rangle - 2\langle v, Jy_n\rangle \\ &\qquad\qquad \le \alpha_n \|x_n\|^2 + (1-\alpha_n)\|x_n + e_n\|^2 - \|y_n\|^2 \end{aligned}$$

and

$$\begin{aligned} &\phi(v, z_n) \le \beta_n \phi(v, x_n) + (1-\beta_n)\phi(v, y_n) \\ &\quad \Longleftrightarrow \quad 2\beta_n \langle v, Jx_n\rangle + 2(1-\beta_n)\langle v, Jy_n\rangle - 2\langle v, Jz_n\rangle \\ &\qquad\qquad \le \beta_n \|x_n\|^2 + (1-\beta_n)\|y_n\|^2 - \|z_n\|^2. \end{aligned}$$

Thus U_n is closed and convex for each $n \in N$.

Next, we shall prove that $(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} \mathrm{Fix}(B_i)) \subset U_n$, which implies that $U_n \neq \emptyset$.

For this, we shall use inductive method. Now, $\forall q \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} \mathrm{Fix}(B_i))$.

If $n = 1$, then $q \in U_1 = X$ is obviously true. In view of the convexity of $\|\cdot\|^2$ and Lemma 2.5, we have

$$\begin{aligned}
\phi(q, y_1) &= \|q\|^2 - 2\Big\langle q, \alpha_1 Jx_1 + (1-\alpha_1)\sum_{i=1}^{\infty} a_{1,i} J(J + r_{1,i}A_i)^{-1}J(x_1 + e_1)\Big\rangle \\
&\quad + \Big\|\alpha_1 Jx_1 + (1-\alpha_1)\sum_{i=1}^{\infty} a_{1,i} J(J + r_{1,i}A_i)^{-1}J(x_1 + e_1)\Big\|^2 \\
&\leq \|q\|^2 - 2\alpha_1\langle q, Jx_1\rangle - 2(1-\alpha_1)\sum_{i=1}^{\infty} a_{1,i}\big\langle q, J(J + r_{1,i}A_i)^{-1}J(x_1 + e_1)\big\rangle \\
&\quad + \alpha_1\|x_1\|^2 + (1-\alpha_1)\sum_{i=1}^{\infty} a_{1,i}\big\|(J + r_{1,i}A_i)^{-1}J(x_1 + e_1)\big\|^2 \\
&= \alpha_1\phi(q, x_1) + (1-\alpha_1)\sum_{i=1}^{\infty} a_{1,i}\phi\big(q, (J + r_{1,i}A_i)^{-1}J(x_1 + e_1)\big) \\
&\leq \alpha_1\phi(q, x_1) + (1-\alpha_1)\phi(q, x_1 + e_1).
\end{aligned}$$

Moreover, from the definition of weakly relatively non-expansive mapping, we have

$$\begin{aligned}
\phi(q, z_1) &\leq \|q\|^2 - 2\beta_1\langle q, Jx_1\rangle - 2(1-\beta_1)\sum_{i=1}^{\infty} b_i\langle q, JB_i y_1\rangle \\
&\quad + \beta_1\|x_1\|^2 + (1-\beta_1)\sum_{i=1}^{\infty} b_i\|B_i y_1\|^2 \\
&= \beta_1\phi(q, x_1) + (1-\beta_1)\sum_{i=1}^{\infty} b_i\phi(q, B_i y_1) \leq \beta_1\phi(q, x_1) + (1-\beta_1)\phi(q, y_1).
\end{aligned}$$

Thus $q \in U_2$.

Suppose the result is true for $n = k+1$. Then, if $n = k+2$, we have

$$\begin{aligned}
\phi(q, y_{k+1}) &= \|q\|^2 - 2\Big\langle q, \alpha_{k+1} Jx_{k+1} + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_{k+1,i} J(J + r_{k+1,i}A_i)^{-1}J(x_{k+1} + e_{k+1})\Big\rangle \\
&\quad + \Big\|\alpha_{k+1} Jx_{k+1} + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_{k+1,i} J(J + r_{k+1,i}A_i)^{-1}J(x_{k+1} + e_{k+1})\Big\|^2 \\
&\leq \|q\|^2 - 2\alpha_{k+1}\langle q, Jx_{k+1}\rangle \\
&\quad - 2(1-\alpha_{k+1})\sum_{i=1}^{\infty} a_{k+1,i}\big\langle q, J(J + r_{k+1,i}A_i)^{-1}J(x_{k+1} + e_{k+1})\big\rangle \\
&\quad + \alpha_{k+1}\|x_{k+1}\|^2 + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_{k+1,i}\big\|(J + r_{k+1,i}A_i)^{-1}J(x_{k+1} + e_{k+1})\big\|^2 \\
&= \alpha_{k+1}\phi(q, x_{k+1}) + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_{k+1,i}\phi\big(q, (J + r_{k+1,i}A_i)^{-1}J(x_{k+1} + e_{k+1})\big) \\
&\leq \alpha_{k+1}\phi(q, x_{k+1}) + (1-\alpha_{k+1})\phi(q, x_{k+1} + e_{k+1}).
\end{aligned}$$

Moreover,

$$\begin{aligned}\phi(q,z_{k+1}) &\le \|q\|^2 - 2\beta_{k+1}\langle q, Jx_{k+1}\rangle - 2(1-\beta_{k+1})\sum_{i=1}^{\infty} b_i\langle q, JB_i y_{k+1}\rangle \\ &\quad + \beta_{k+1}\|x_{k+1}\|^2 + (1-\beta_{k+1})\sum_{i=1}^{\infty} b_i\|B_i y_{k+1}\|^2 \\ &= \beta_{k+1}\phi(q,x_{k+1}) + (1-\beta_{k+1})\sum_{i=1}^{\infty} b_i\phi(q,B_i y_{k+1}) \\ &\le \beta_{k+1}\phi(q,x_{k+1}) + (1-\beta_{k+1})\phi(q,y_{k+1}).\end{aligned}$$

Then $q \in U_{k+2}$. Therefore, by induction, $(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} \mathrm{Fix}(B_i)) \subset U_n$ for $n \in N$.

Step 2. $P_{U_{n+1}}(x_1) \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, as $n \to \infty$.

It follows from Lemma 2.6 that $\lim U_n$ exists and $\lim U_n = \bigcap_{n=1}^{\infty} U_n \neq \emptyset$. Since X has Property (H), then Lemma 2.7 implies that $P_{U_{n+1}}(x_1) \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, as $n \to \infty$.

Step 3. $V_n \neq \emptyset$, for $N \cup \{0\}$, which ensures that $\{x_n\}$ is well defined.

Since $\|P_{U_{n+1}}(x_1) - x_1\| = \inf_{y \in U_{n+1}} \|y - x_1\|$, then for λ_{n+1}, there exists $\delta_{n+1} \in U_{n+1}$ such that $\|x_1 - \delta_{n+1}\|^2 \le (\inf_{y\in U_{n+1}} \|x_1 - y\|)^2 + \lambda_{n+1} = \|P_{U_{n+1}}(x_1) - x_1\|^2 + \lambda_{n+1}$. This ensures that $V_{n+1} \neq \emptyset$ for $n \in N \cup \{0\}$.

Step 4. Both $\{x_n\}$ and $\{P_{U_{n+1}}(x_1)\}$ are bounded.

Since $\lambda_n \to 0$, then there exists $M_1 > 0$ such that $\lambda_n < M_1$ for $n \in N$. Step 2 implies that $\{P_{U_{n+1}}(x_1)\}$ is bounded, and then there exists $M_2 > 0$ such that $\|P_{U_{n+1}}(x_1)\| \le M_2$ for $n \in N$. Set $M = (M_2 + \|x_1\|)^2 + M_1$. Since $x_{n+1} \in V_{n+1}$, then $\|x_1 - x_{n+1}\|^2 \le \|P_{U_{n+1}}(x_1) - x_1\|^2 + \lambda_{n+1} \le M$, $\forall n \in N$. Thus $\{x_n\}$ is bounded.

Step 5. $x_{n+1} - P_{U_{n+1}}(x_1) \to 0$, as $n \to \infty$.

Since $x_{n+1} \in V_{n+1} \subset U_{n+1}$ and U_n is a convex subset of X, then for $\forall k \in (0,1)$, $kP_{U_{n+1}}(x_1) + (1-k)x_{n+1} \in U_{n+1}$. Thus

$$\left\|P_{U_{n+1}}(x_1) - x_1\right\| \le \left\|kP_{U_{n+1}}(x_1) + (1-k)x_{n+1} - x_1\right\|. \tag{3.2}$$

Since $\{x_n\}$ is bounded, it follows from (3.2) and Lemma 2.8 that

$$\begin{aligned}\left\|P_{U_{n+1}}(x_1) - x_1\right\|^2 &\le \left\|kP_{U_{n+1}}(x_1) + (1-k)x_{n+1} - x_1\right\|^2 \\ &\le k\left\|P_{U_{n+1}}(x_1) - x_1\right\|^2 + (1-k)\|x_{n+1} - x_1\|^2 \\ &\quad - k(1-k)\eta\left(\left\|P_{U_{n+1}}(x_1) - x_{n+1}\right\|\right).\end{aligned}$$

Therefore, $k\eta(\|P_{U_{n+1}}(x_1) - x_{n+1}\|) \le \|x_{n+1} - x_1\|^2 - \|P_{U_{n+1}}(x_1) - x_1\|^2 \le \lambda_{n+1}$. Letting $k \to 1$ first and then $n \to \infty$, we know that $P_{U_{n+1}}(x_1) - x_{n+1} \to 0$, as $n \to \infty$.

Step 6. $x_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, $y_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$ and $z_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, as $n \to \infty$.

From Step 2 and Step 5, we know that $x_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, as $n \to \infty$. And then $x_{n+1} - x_n \to 0$, as $n \to \infty$. Since $x_{n+1} \in V_{n+1} \subset U_{n+1}$ and $e_n \to 0$, then

$$\begin{aligned}0 &\le \phi(x_{n+1}, y_n) \le \alpha_n\phi(x_{n+1}, x_n) + (1-\alpha_n)\phi(x_{n+1}, x_n + e_n) \\ &= \alpha_n\|x_{n+1}\|^2 + \alpha_n\|x_n\|^2 - 2\alpha_n\langle x_{n+1}, Jx_n\rangle\end{aligned}$$

$$
\begin{aligned}
&+ (1-\alpha_n)\|x_{n+1}\|^2 + (1-\alpha_n)\|x_n+e_n\|^2 - 2(1-\alpha_n)\langle x_{n+1}, J(x_n+e_n)\rangle \\
&= \|x_{n+1}\|^2 - \alpha_n\|x_n\|^2 - (1-\alpha_n)\|x_n+e_n\|^2 \\
&\quad + 2\alpha_n\langle x_n - x_{n+1}, Jx_n\rangle + 2(1-\alpha_n)\langle x_n+e_n-x_{n+1}, J(x_n+e_n)\rangle \\
&\le \left(\|x_{n+1}\|^2 - \|x_n+e_n\|^2\right) + \alpha_n\left(\|x_n+e_n\|^2 - \|x_n\|^2\right) + 2\alpha_n\|x_n\|\|x_{n+1}-x_n\| \\
&\quad + 2(1-\alpha_n)\|x_n+e_n\|\|x_n+e_n-x_{n+1}\| \to 0.
\end{aligned}
$$

Then Lemma 2.4 implies that $x_{n+1} - y_n \to 0$ and then $y_n \to P_{\bigcap_{m=1}^\infty U_m}(x_1)$, as $n \to \infty$.

Since $x_{n+1} \in V_{n+1} \subset U_{n+1}$ and J is uniformly continuous on each bounded subset of X, then

$$
\begin{aligned}
0 &\le \phi(x_{n+1}, z_n) \le \beta_n\phi(x_{n+1}, x_n) + (1-\beta_n)\phi(x_{n+1}, y_n) \\
&= \beta_n\left(\langle x_{n+1}, Jx_{n+1} - Jx_n\rangle + \langle x_n - x_{n+1}, Jx_n\rangle\right) + (1-\beta_n)\phi(x_{n+1}, y_n) \\
&\le \beta_n\|x_{n+1}\|\|Jx_{n+1} - Jx_n\| + \beta_n\|x_n\|\|x_{n+1}-x_n\| + (1-\beta_n)\phi(x_{n+1}, y_n) \to 0.
\end{aligned}
$$

Using Lemma 2.4 again, we have $x_{n+1} - z_n \to 0$ and then $z_n \to P_{\bigcap_{m=1}^\infty U_m}(x_1)$, as $n \to \infty$.

Step 7. $P_{\bigcap_{m=1}^\infty U_m}(x_1) \in (\bigcap_{i=1}^\infty A_i^{-1}0) \cap (\bigcap_{i=1}^\infty \mathrm{Fix}(B_i))$.

First, we shall show that $P_{\bigcap_{m=1}^\infty U_m}(x_1) \in \bigcap_{i=1}^\infty A_i^{-1}0$.

From (3.1) and Lemma 2.5, for $\forall q \in (\bigcap_{i=1}^\infty A_i^{-1}0) \cap (\bigcap_{i=1}^\infty \mathrm{Fix}(B_i))$, we have

$$
\begin{aligned}
\phi(q, y_n) &\le \alpha_n\phi(q, x_n) + (1-\alpha_n)\sum_{i=1}^\infty a_{n,i}\phi\left(q, (J + r_{n,i}A_i)^{-1}J(x_n+e_n)\right) \\
&\le \alpha_n\phi(q, x_n) \\
&\quad + (1-\alpha_n)\sum_{i=1}^\infty a_{n,i}\left[\phi(q, x_n+e_n) - \phi\left((J + r_{n,i}A_i)^{-1}J(x_n+e_n), x_n+e_n\right)\right].
\end{aligned}
$$

Then

$$
\begin{aligned}
&(1-\alpha_n)\sum_{i=1}^\infty a_{n,i}\phi\left((J + r_{n,i}A_i)^{-1}J(x_n+e_n), x_n+e_n\right) \\
&\le \alpha_n\phi(q, x_n) - \phi(q, y_n) + (1-\alpha_n)\phi(q, x_n+e_n) \\
&= \alpha_n\left[\phi(q, x_n) - \phi(q, x_n+e_n)\right] + \left[\phi(q, x_n+e_n) - \phi(q, y_n)\right] \\
&\le \|x_n\|^2 - \|x_n+e_n\|^2 + 2\|q\|\left\|J(x_n+e_n) - Jx_n\right\| \\
&\quad + \|x_n+e_n\|^2 - \|y_n\|^2 + 2\|q\|\left\|Jy_n - J(x_n+e_n)\right\|.
\end{aligned}
$$

Since $0 \le \sup_n \alpha_n < 1$, then $\sum_{i=1}^\infty a_{n,i}\phi((J + r_{n,i}A_i)^{-1}J(x_n+e_n), x_n+e_n) \to 0$, which implies from Lemma 2.4 that $(J + r_{n,i}A_i)^{-1}J(x_n+e_n) - (x_n+e_n) \to 0$, as $n \to \infty$. Thus from Step 6, $(J + r_{n,i}A_i)^{-1}J(x_n+e_n) \to P_{\bigcap_{m=1}^\infty U_m}(x_1)$, as $n \to \infty$.

Denote $u_{n,i} = (J + r_{n,i}A_i)^{-1}J(x_n+e_n)$, then $Ju_{n,i} + r_{n,i}A_iu_{n,i} = J(x_n+e_n)$. Since $u_{n,i} \to P_{\bigcap_{m=1}^\infty U_m}(x_1)$, $x_n \to P_{\bigcap_{m=1}^\infty U_m}(x_1)$, $e_n \to 0$, $\inf_n r_{n,i} > 0$ and J is uniformly continuous on each bounded subset of X, then $A_iu_{n,i} \to 0$ for $i \in N$, as $n \to \infty$. Using Lemma 2.2, $P_{\bigcap_{m=1}^\infty U_m}(x_1) \in \bigcap_{i=1}^\infty A_i^{-1}0$.

Next, we shall show that $P_{\bigcap_{m=1}^{\infty} U_m}(x_1) \in \bigcap_{i=1}^{\infty} \operatorname{Fix}(B_i)$.

Since $z_n = J^{-1}[\beta_n Jx_n + (1-\beta_n)\sum_{i=1}^{\infty} b_i JB_i y_n]$, then $Jz_n - Jx_n = (1-\beta_n)(\sum_{i=1}^{\infty} b_i JB_i y_n - Jx_n)$. Since both J and J^{-1} are uniformly continuous on each bounded subset of X, $z_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, $x_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$ and $0 \le \sup_n \beta_n < 1$, then $\sum_{i=1}^{\infty} b_i JB_i y_n - Jx_n \to 0$, which implies that $J^{-1}(\sum_{i=1}^{\infty} b_i JB_i y_n) \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, as $n \to \infty$.

Employing Lemma 2.9, for $\forall q \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} \operatorname{Fix}(B_i))$, we have

$$\begin{aligned}
&\phi\left(q, J^{-1}\left(\sum_{i=1}^{\infty} b_i JB_i y_n\right)\right) \\
&\quad= \|q\|^2 - 2\left\langle q, \sum_{i=1}^{\infty} b_i JB_i y_n\right\rangle + \left\|\sum_{i=1}^{\infty} b_i JB_i y_n\right\|^2 \\
&\quad\le \|q\|^2 - 2\sum_{i=1}^{\infty} b_i \langle q, JB_i y_n\rangle + \sum_{i=1}^{\infty} b_i \|B_i y_n\|^2 - b_1 b_k \eta\bigl(\|JB_1 y_n - JB_k y_n\|\bigr) \\
&\quad= \sum_{i=1}^{\infty} b_i \phi(q, B_i y_n) - b_1 b_k \eta\bigl(\|JB_1 y_n - JB_k y_n\|\bigr).
\end{aligned} \tag{3.3}$$

Since $Jy_n \to JP_{\bigcap_{m=1}^{\infty} U_m}(x_1)$ and $\sum_{i=1}^{\infty} b_i JB_i y_n \to JP_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, then from the definition of weakly relatively non-expansive mapping and (3.3), we have

$$\begin{aligned}
&b_1 b_k \eta\bigl(\|JB_1 y_n - JB_k y_n\|\bigr) \\
&\quad\le \sum_{i=1}^{\infty} b_i \phi(q, B_i y_n) - \phi\left(q, J^{-1}\left(\sum_{i=1}^{\infty} b_i JB_i y_n\right)\right) \\
&\quad\le \sum_{i=1}^{\infty} b_i \phi(q, y_n) - \phi\left(q, J^{-1}\left(\sum_{i=1}^{\infty} b_i JB_i y_n\right)\right) \\
&\quad= \|y_n\|^2 - 2\langle q, Jy_n\rangle + 2\sum_{i=1}^{\infty} b_i \langle q, JB_i y_n\rangle - \left\|\sum_{i=1}^{\infty} b_i JB_i y_n\right\|^2 \to 0,
\end{aligned}$$

as $n \to \infty$. This ensures that $JB_1 y_n - JB_k y_n \to 0$ for $k \ne 1$, as $n \to \infty$.

Since $y_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, then $\{y_n\}$ is bounded. Since $(\|q\| - \|B_i y_n\|)^2 \le \phi(q, B_i y_n) \le \phi(q, y_n) \le (\|q\| + \|y_n\|)^2$, then $\|B_i y_n\| \le \|q\|$ or $\|B_i y_n\| \le 2\|q\| + \|y_n\|$, $i \in N$. Set $K = \sup\{\|y_n\| : n \in N\} + 2\|q\|$, then $K < +\infty$.

Since $\sum_{i=1}^{\infty} b_i = 1$, then for $\forall \varepsilon > 0$, there exists $m_0 \in N$ such that $\sum_{i=m_0+1}^{\infty} b_i < \frac{\varepsilon}{4K}$.

Since $JB_1 y_n - JB_k y_n \to 0$, as $n \to \infty$, for $\forall k \in \{1, 2, \dots, m_0\}$, then we can choose $n_0 \in N$ such that $\|JB_1 y_n - JB_k y_n\| < \frac{\varepsilon}{2}$ for all $n \ge n_0$ and $k \in \{2, \dots, m_0\}$. Then, if $n \ge n_0$,

$$\begin{aligned}
\left\|JB_1 y_n - \sum_{i=1}^{\infty} b_i JB_i y_n\right\| &\le \sum_{i=2}^{m_0} b_i \|JB_1 y_n - JB_i y_n\| + \sum_{i=m_0+1}^{\infty} b_i \|JB_1 y_n - JB_i y_n\| \\
&< \left(\sum_{i=2}^{m_0} b_i\right)\frac{\varepsilon}{2} + \left(\sum_{i=m_0+1}^{\infty} b_i\right) 2K < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This implies that $JB_1 y_n - \sum_{i=1}^{\infty} b_i JB_i y_n \to 0$, and then $JB_1 y_n \to JP_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, as $n \to \infty$. Thus $B_1 y_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, as $n \to \infty$. Lemma 2.1 implies that $P_{\bigcap_{m=1}^{\infty} U_m}(x_1) \in \operatorname{Fix}(B_1)$.

Repeating the above process for showing $P_{\bigcap_{m=1}^{\infty} U_m}(x_1) \in \mathrm{Fix}(B_1)$, we can also prove that $P_{\bigcap_{m=1}^{\infty} U_m}(x_1) \in \mathrm{Fix}(B_k)$, $\forall k \in N$. Therefore, $P_{\bigcap_{m=1}^{\infty} U_m}(x_1) \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(B_i)$.

This completes the proof. □

Theorem 3.2 *Let $\{x_n\}$ be generated by the following iterative scheme*:

$$\begin{cases} x_1 \in X, \qquad e_1 \in X, \\ y_n = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i} J(J + r_{n,i}A_i)^{-1} J(x_n + e_n)], \\ z_n = J^{-1}[\beta_n Jx_1 + (1-\beta_n)\sum_{i=1}^{\infty} b_i JB_i y_n], \\ U_1 = X = V_1, \\ U_{n+1} = \{v \in U_n : \phi(v, y_n) \le \alpha_n \phi(v, x_1) + (1-\alpha_n)\phi(v, x_n + e_n), \\ \qquad \phi(v, z_n) \le \beta_n \phi(v, x_1) + (1-\beta_n)\phi(v, y_n)\}, \\ V_{n+1} = \{v \in U_{n+1} : \|x_1 - v\|^2 \le \|P_{U_{n+1}}(x_1) - x_1\|^2 + \lambda_{n+1}\}, \\ x_{n+1} \in V_{n+1}, \quad n \in N. \end{cases} \tag{3.4}$$

If $\alpha_n \to 0$, $\beta_n \to 0$, *then* $x_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} \mathrm{Fix}(B_i))$, *as* $n \to \infty$.

Proof Copy Steps 2, 3, 4, and 5 of Theorem 3.1, and do small changes in Steps 1, 6, and 7 in the following way.

Step 1. U_n is a non-empty closed and convex subset of X.

We notice that

$$\begin{aligned} &\phi(v, y_n) \le \alpha_n \phi(v, x_1) + (1-\alpha_n)\phi(v, x_n + e_n) \\ &\quad \Longleftrightarrow \quad 2\alpha_n \langle v, Jx_1\rangle + 2(1-\alpha_n)\bigl\langle v, J(x_n + e_n)\bigr\rangle - 2\langle v, Jy_n\rangle \\ &\qquad\qquad \le \alpha_n \|x_1\|^2 + (1-\alpha_n)\|x_n + e_n\|^2 - \|y_n\|^2 \end{aligned}$$

and

$$\begin{aligned} &\phi(v, z_n) \le \beta_n \phi(v, x_1) + (1-\beta_n)\phi(v, y_n) \\ &\quad \Longleftrightarrow \quad 2\beta_n \langle v, Jx_1\rangle + 2(1-\beta_n)\langle v, Jy_n\rangle - 2\langle v, Jz_n\rangle \\ &\qquad\qquad \le \beta_n \|x_1\|^2 + (1-\beta_n)\|y_n\|^2 - \|z_n\|^2. \end{aligned}$$

Thus U_n is closed and convex for $n \in N$.

Next, we shall prove that $(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} \mathrm{Fix}(B_i)) \subset U_n$, which ensures that $U_n \neq \emptyset$.

For this, we shall use inductive method. Now, $\forall q \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} \mathrm{Fix}(B_i))$.

If $n = 1$, $q \in U_1 = X$ is obviously true. In view of the convexity of $\|\cdot\|^2$ and Lemma 2.5, we have

$$\begin{aligned} \phi(q, y_1) = {} & \|q\|^2 - 2\Biggl\langle q, \alpha_1 Jx_1 + (1-\alpha_1)\sum_{i=1}^{\infty} a_{1,i} J(J + r_{1,i}A_i)^{-1} J(x_1 + e_1)\Biggr\rangle \\ & + \Biggl\|\alpha_1 Jx_1 + (1-\alpha_1)\sum_{i=1}^{\infty} a_{1,i} J(J + r_{1,i}A_i)^{-1} J(x_1 + e_1)\Biggr\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \|q\|^2 - 2\alpha_1 \langle q, Jx_1\rangle - 2(1-\alpha_1)\sum_{i=1}^{\infty} a_{1,i}\big\langle q, J(J + r_{1,i}A_i)^{-1}J(x_1 + e_1)\big\rangle \\
&\quad + \alpha_1\|x_1\|^2 + (1-\alpha_1)\sum_{i=1}^{\infty} a_{1,i}\big\|(J + r_{1,i}A_i)^{-1}J(x_1 + e_1)\big\|^2 \\
&= \alpha_1\phi(q, x_1) + (1-\alpha_1)\sum_{i=1}^{\infty} a_{1,i}\phi\big(q, (J + r_{1,i}A_i)^{-1}J(x_1 + e_1)\big) \\
&\leq \alpha_1\phi(q, x_1) + (1-\alpha_1)\phi(q, x_1 + e_1).
\end{aligned}$$

Moreover, from the definition of weakly relatively non-expansive mapping, we have

$$\begin{aligned}
\phi(q, z_1) &\leq \|q\|^2 - 2\beta_1\langle q, Jx_1\rangle - 2(1-\beta_1)\sum_{i=1}^{\infty} b_i\langle q, JB_iy_1\rangle \\
&\quad + \beta_1\|x_1\|^2 + (1-\beta_1)\sum_{i=1}^{\infty} b_i\|B_iy_1\|^2 \\
&= \beta_1\phi(q, x_1) + (1-\beta_1)\sum_{i=1}^{\infty} b_i\phi(q, B_iy_1) \leq \beta_1\phi(q, x_1) + (1-\beta_1)\phi(q, y_1).
\end{aligned}$$

Thus $q \in U_2$.

Suppose the result is true for $n = k+1$. Then, if $n = k+2$, we have

$$\begin{aligned}
\phi(q, y_{k+1}) &= \|q\|^2 - 2\Big\langle q, \alpha_{k+1}Jx_1 + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_{k+1,i}J(J + r_{k+1,i}A_i)^{-1}J(x_{k+1} + e_{k+1})\Big\rangle \\
&\quad + \Big\|\alpha_{k+1}Jx_1 + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_{k+1,i}J(J + r_{k+1,i}A_i)^{-1}J(x_{k+1} + e_{k+1})\Big\|^2 \\
&\leq \|q\|^2 - 2\alpha_{k+1}\langle q, Jx_1\rangle \\
&\quad - 2(1-\alpha_{k+1})\sum_{i=1}^{\infty} a_{k+1,i}\big\langle q, J(J + r_{k+1,i}A_i)^{-1}J(x_{k+1} + e_{k+1})\big\rangle \\
&\quad + \alpha_{k+1}\|x_1\|^2 + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_{k+1,i}\big\|(J + r_{k+1,i}A_i)^{-1}J(x_{k+1} + e_{k+1})\big\|^2 \\
&= \alpha_{k+1}\phi(q, x_1) + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_{k+1,i}\phi\big(q, (J + r_{k+1,i}A_i)^{-1}J(x_{k+1} + e_{k+1})\big) \\
&\leq \alpha_{k+1}\phi(q, x_1) + (1-\alpha_{k+1})\phi(q, x_{k+1} + e_{k+1}).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\phi(q, z_{k+1}) &\leq \|q\|^2 - 2\beta_{k+1}\langle q, Jx_1\rangle - 2(1-\beta_{k+1})\sum_{i=1}^{\infty} b_i\langle q, JB_iy_{k+1}\rangle \\
&\quad + \beta_{k+1}\|x_1\|^2 + (1-\beta_{k+1})\sum_{i=1}^{\infty} b_i\|B_iy_{k+1}\|^2
\end{aligned}$$

$$= \beta_{k+1}\phi(q,x_1) + (1-\beta_{k+1})\sum_{i=1}^{\infty} b_i\phi(q,B_iy_{k+1})$$
$$\leq \beta_{k+1}\phi(q,x_1) + (1-\beta_{k+1})\phi(q,y_{k+1}).$$

Then $q \in U_{k+2}$. Therefore, by induction, $\emptyset \neq (\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty}\operatorname{Fix}(B_i)) \subset U_n$, for $n \in N$.

Step 6. $x_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, $y_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, and $z_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, as $n \to \infty$.

Following from the results of Step 2 and Step 5, $x_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, as $n \to \infty$. And then $x_{n+1} - x_n \to 0$, as $n \to \infty$.

Since $x_{n+1} \in V_{n+1} \subset U_{n+1}$, $\alpha_n \to 0$, and $e_n \to 0$, then

$$\begin{aligned}
0 &\leq \phi(x_{n+1},y_n) \leq \alpha_n\phi(x_{n+1},x_1) + (1-\alpha_n)\phi(x_{n+1},x_n+e_n) \\
&= \alpha_n\|x_{n+1}\|^2 + \alpha_n\|x_1\|^2 - 2\alpha_n\langle x_{n+1},Jx_1\rangle \\
&\quad + (1-\alpha_n)\|x_{n+1}\|^2 + (1-\alpha_n)\|x_n+e_n\|^2 - 2(1-\alpha_n)\bigl\langle x_{n+1},J(x_n+e_n)\bigr\rangle \\
&= \|x_{n+1}\|^2 - \alpha_n\|x_1\|^2 - (1-\alpha_n)\|x_n+e_n\|^2 \\
&\quad + 2\alpha_n\langle x_1-x_{n+1},Jx_1\rangle + 2(1-\alpha_n)\bigl\langle x_n+e_n-x_{n+1},J(x_n+e_n)\bigr\rangle \\
&\leq \bigl(\|x_{n+1}\|^2 - \|x_n+e_n\|^2\bigr) + \alpha_n\bigl(\|x_n+e_n\|^2 - \|x_1\|^2\bigr) + 2\alpha_n\|x_1\|\|x_{n+1}-x_1\| \\
&\quad + 2(1-\alpha_n)\|x_n+e_n\|\|x_n+e_n-x_{n+1}\| \to 0,
\end{aligned}$$

as $n \to \infty$. Lemma 2.4 implies that $x_{n+1} - y_n \to 0$ and then $y_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$ as $n \to \infty$.

Since $x_{n+1} \in V_{n+1} \subset U_{n+1}$ and $\beta_n \to 0$, then

$$0 \leq \phi(x_{n+1},z_n) \leq \beta_n\phi(x_{n+1},x_1) + (1-\beta_n)\phi(x_{n+1},y_n) \to 0.$$

Lemma 2.4 implies that $x_{n+1} - z_n \to 0$ and then $z_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$ as $n \to \infty$.

Step 7. $P_{\bigcap_{m=1}^{\infty} U_m}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty}\operatorname{Fix}(B_i))$.

First, we shall show that $P_{\bigcap_{m=1}^{\infty} U_m}(x_1) \in \bigcap_{i=1}^{\infty} A_i^{-1}0$.

From (3.4) and Lemma 2.5, for $\forall q \in (\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty}\operatorname{Fix}(B_i))$, we have

$$\begin{aligned}
\phi(q,y_n) &\leq \alpha_n\phi(q,x_1) + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i}\phi\bigl(q,(J+r_{n,i}A_i)^{-1}J(x_n+e_n)\bigr) \\
&\leq \alpha_n\phi(q,x_1) \\
&\quad + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i}\bigl[\phi(q,x_n+e_n) - \phi\bigl((J+r_{n,i}A_i)^{-1}J(x_n+e_n),x_n+e_n\bigr)\bigr].
\end{aligned}$$

Thus

$$\begin{aligned}
&(1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i}\phi\bigl((J+r_{n,i}A_i)^{-1}J(x_n+e_n),x_n+e_n\bigr) \\
&\quad\leq \alpha_n\phi(q,x_1) - \phi(q,y_n) + (1-\alpha_n)\phi(q,x_n+e_n) \\
&\quad= \alpha_n\bigl[\phi(q,x_1) - \phi(q,x_n+e_n)\bigr] + \bigl[\phi(q,x_n+e_n) - \phi(q,y_n)\bigr] \\
&\quad\leq \alpha_n\bigl[\phi(q,x_1) - \phi(q,x_n+e_n)\bigr] + \bigl(\|x_n+e_n\|^2 - \|y_n\|^2\bigr) + 2\|q\|\bigl\|J(x_n+e_n) - Jy_n\bigr\|.
\end{aligned}$$

Since $\alpha_n \to 0$, then $\sum_{i=1}^{\infty} a_{n,i}\phi((J + r_{n,i}A_i)^{-1}J(x_n + e_n), x_n + e_n) \to 0$, which implies from Lemma 2.4 that $(J + r_{n,i}A_i)^{-1}J(x_n + e_n) - (x_n + e_n) \to 0$, as $n \to \infty$. Thus $(J + r_{n,i}A_i)^{-1}J(x_n + e_n) \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, as $n \to \infty$.

Let $u_{n,i} = (J + r_{n,i}A_i)^{-1}J(x_n + e_n)$, then $Ju_{n,i} + r_{n,i}A_iu_{n,i} = J(x_n + e_n)$. Since $u_{n,i} \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, $x_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, $e_n \to 0$, and $\inf_n r_{n,i} > 0$, then $A_iu_{n,i} \to 0$ for $i \in N$, as $n \to \infty$. Using Lemma 2.2, $P_{\bigcap_{m=1}^{\infty} U_m}(x_1) \in \bigcap_{i=1}^{\infty} A_i^{-1}0$.

Next, we shall show that $P_{\bigcap_{m=1}^{\infty} U_m}(x_1) \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(B_i)$.

Since $z_n = J^{-1}[\beta_n Jx_1 + (1 - \beta_n)\sum_{i=1}^{\infty} b_iJB_iy_n]$, then $Jz_n - Jx_n = \beta_n(Jx_1 - Jx_n) + (1 - \beta_n)(\sum_{i=1}^{\infty} b_iJB_iy_n - Jx_n)$. Since both J and J^{-1} are uniformly continuous on each bounded subset of X, $z_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, $x_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, and $\beta_n \to 0$, then $\sum_{i=1}^{\infty} b_iJB_iy_n - Jx_n \to 0$, which implies that $J^{-1}(\sum_{i=1}^{\infty} b_iJB_iy_n) \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1)$, as $n \to \infty$.

The following proof is the same as the corresponding part in Step 7 of Theorem 3.1.

This completes the proof. □

Theorem 3.3 *Suppose that* $\{x_n\}$ *is generated by the following iterative scheme*:

$$\begin{cases} x_1 \in X, \qquad e_1 \in X, \\ y_n = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i}J(J + r_{n,i}A_i)^{-1}J(x_n + e_n)], \\ z_n = J^{-1}[\beta_n Jx_n + (1-\beta_n)\sum_{i=1}^{\infty} b_iJB_iy_n], \\ U_1 = X = V_1, \\ U_{n+1} = \{v \in U_n : \phi(v, y_n) \le \alpha_n\phi(v, x_1) + (1-\alpha_n)\phi(v, x_n + e_n), \\ \qquad \phi(v, z_n) \le \beta_n\phi(v, x_n) + (1-\beta_n)\phi(v, y_n)\}, \\ V_{n+1} = \{v \in U_{n+1} : \|x_1 - v\|^2 \le \|P_{U_{n+1}}(x_1) - x_1\|^2 + \lambda_{n+1}\}, \\ x_{n+1} \in V_{n+1}, \quad n \in N. \end{cases} \tag{3.5}$$

If $0 \le \sup_n \beta_n < 1$ *and* $\alpha_n \to 0$, *then* $x_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} \mathrm{Fix}(B_i))$, *as* $n \to \infty$.

Theorem 3.4 *Suppose that* $\{x_n\}$ *is generated by the following iterative scheme*:

$$\begin{cases} x_1 \in X, \qquad e_1 \in X, \\ y_n = J^{-1}[\alpha_n Jx_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i}J(J + r_{n,i}A_i)^{-1}J(x_n + e_n)], \\ z_n = J^{-1}[\beta_n Jx_1 + (1-\beta_n)\sum_{i=1}^{\infty} b_iJB_iy_n], \\ U_1 = X = V_1, \\ U_{n+1} = \{v \in U_n : \phi(v, y_n) \le \alpha_n\phi(v, x_n) + (1-\alpha_n)\phi(v, x_n + e_n), \\ \qquad \phi(v, z_n) \le \beta_n\phi(v, x_1) + (1-\beta_n)\phi(v, y_n)\}, \\ V_{n+1} = \{v \in U_{n+1} : \|x_1 - v\|^2 \le \|P_{U_{n+1}}(x_1) - x_1\|^2 + \lambda_{n+1}\}, \\ x_{n+1} \in V_{n+1}, \quad n \in N. \end{cases} \tag{3.6}$$

If $0 \le \sup_n \alpha_n < 1$ *and* $\beta_n \to 0$, *then* $x_n \to P_{\bigcap_{m=1}^{\infty} U_m}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} \mathrm{Fix}(B_i))$, *as* $n \to \infty$.

Remark 3.5 The main difference between ours and [8] is that: in [8], in each step n, countable sets $V_{n+1,i}$ and $W_{n+1,i}$ are needed to be evaluated, but in our paper, in each step n, two sets U_{n+1} and V_{n+1} are enough. This difference leads to some different techniques for proving the main results.

Corollary 3.6 *If X reduces to a Hilbert space H, then (3.1) becomes as follows:*

$$\begin{cases} x_1 \in H, \qquad e_1 \in H, \\ y_n = \alpha_n x_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i}(I + r_{n,i}A_i)^{-1}(x_n + e_n), \\ z_n = \beta_n x_n + (1-\beta_n)\sum_{i=1}^{\infty} b_i B_i y_n, \\ U_1 = H = V_1, \\ U_{n+1} = \{v \in U_n : \|v - y_n\|^2 \le \alpha_n \|v - x_n\|^2 + (1-\alpha_n)\|v - x_n - e_n\|^2, \\ \qquad \|v - z_n\|^2 \le \beta_n \|v - x_n\|^2 + (1-\beta_n)\|v - y_n\|^2\}, \\ V_{n+1} = \{v \in U_{n+1} : \|x_1 - v\|^2 \le \|P_{U_{n+1}}(x_1) - x_1\|^2 + \lambda_{n+1}\}, \\ x_{n+1} \in V_{n+1}, \quad n \in N. \end{cases} \tag{3.7}$$

Similarly, we can get the special forms of (3.4), (3.5), and (3.6) in the frame of Hilbert space H.

Corollary 3.7 *If, further, $r_{n,i} \equiv r_n$, $A_i \equiv A$, and $B_i \equiv B$, then we can get a special case for (3.7):*

$$\begin{cases} x_1 \in H, \qquad e_1 \in H, \\ y_n = \alpha_n x_n + (1-\alpha_n)(I + r_n A)^{-1}(x_n + e_n), \\ z_n = \beta_n x_n + (1-\beta_n) B y_n, \\ U_1 = H = V_1, \\ U_{n+1} = \{v \in U_n : \|v - y_n\|^2 \le \alpha_n \|v - x_n\|^2 + (1-\alpha_n)\|v - x_n - e_n\|^2, \\ \qquad \|v - z_n\|^2 \le \beta_n \|v - x_n\|^2 + (1-\beta_n)\|v - y_n\|^2\}, \\ V_{n+1} = \{v \in U_{n+1} : \|x_1 - v\|^2 \le \|P_{U_{n+1}}(x_1) - x_1\|^2 + \lambda_{n+1}\}, \\ x_{n+1} \in V_{n+1}, \quad n \in N, \end{cases} \tag{3.8}$$

where A is maximal monotone, B is weakly relatively non-expansive, and $\{r_n\} \subset [0, +\infty)$ satisfies $\inf_n r_n > 0$.

Corollary 3.8 *If, in Corollary 3.7, $\alpha_n \equiv 0$, then (3.8) can be further simplified as follows:*

$$\begin{cases} x_1 \in H, \qquad e_1 \in H, \\ y_n = (I + r_n A)^{-1}(x_n + e_n), \\ z_n = \beta_n x_n + (1-\beta_n) B y_n, \\ U_1 = H = V_1, \\ U_{n+1} = \{v \in U_n : \|v - y_n\| \le \|v - x_n - e_n\|, \\ \qquad \|v - z_n\|^2 \le \beta_n \|v - x_n\|^2 + (1-\beta_n)\|v - y_n\|^2\}, \\ V_{n+1} = \{v \in U_{n+1} : \|x_1 - v\|^2 \le \|P_{U_{n+1}}(x_1) - x_1\|^2 + \lambda_{n+1}\}, \\ x_{n+1} \in V_{n+1}, \quad n \in N. \end{cases} \tag{3.9}$$

Remark 3.9 Comparing (3.9) and (1.4), we may find that they are different due to different construction of U_{n+1}. This indicates again that (3.1) is different from (1.3).

Remark 3.10 Choose $H = (-\infty, +\infty)$, $Ax = 2x$, and $Bx = x$ for $x \in (-\infty, +\infty)$. Let $e_n = \beta_n = \lambda_n = \frac{1}{n}$ and $r_n = 2^{n-1}$ for $n \in N$. Then A is maximal monotone and B is weakly relatively non-expansive. Moreover, $A^{-1}0 \cap \mathrm{Fix}(B) = \{0\}$.

Corollary 3.11 *Take the example in Remark 3.10. We can choose the following three iterative sequences $\{x_n\}$ among infinite choices by iterative scheme (3.9).*

$$\begin{cases} x_1 = 1, \qquad x_2 = 1 - \frac{\sqrt{2}}{2}, \\ y_n = \frac{x_n + e_n}{1 + 2r_n}, \quad n \in N, \\ w_n = \min_{m \le n}(1 + r_m)y_m, \quad n \in N, \\ x_{n+1} = x_1 - \sqrt{(x_1 - w_n)^2 + \lambda_{n+1}}, \quad n \in N \setminus \{1\}, \end{cases} \tag{3.10}$$

$$\begin{cases} x_1 = 1, \\ y_n = \frac{x_n + e_n}{1 + 2r_n}, \quad n \in N, \\ x_{n+1} = (1 + r_n)y_n, \quad n \in N, \end{cases} \tag{3.11}$$

and

$$\begin{cases} x_1 = 1, \qquad x_2 = \frac{7}{6} - \frac{\sqrt{2}}{4}, \\ y_n = \frac{x_n + e_n}{1 + 2r_n}, \quad n \in N, \\ w_n = \min_{m \le n}(1 + r_m)y_m, \quad n \in N, \\ x_{n+1} = \frac{x_1 + w_n - \sqrt{(x_1 - w_n)^2 + \lambda_{n+1}}}{2}, \quad n \in N \setminus \{1\}. \end{cases} \tag{3.12}$$

Then $\{x_n\}$ generated by (3.10), (3.11), and (3.12) converges strongly to $0 \in A^{-1}0 \cap \mathrm{Fix}(B)$, as $n \to \infty$.

Proof We can easily see from iterative scheme (3.9) that

$$y_n = \frac{x_n + e_n}{1 + 2r_n} \quad \text{for } n \in N, \tag{3.13}$$

and

$$z_n = \beta_n x_n + (1 - \beta_n) y_n \quad \text{for } n \in N. \tag{3.14}$$

From (3.14), we can see that $(v - z_n)^2 \le \beta_n (v - x_n)^2 + (1 - \beta_n)(v - y_n)^2$ is always true for $v \in (-\infty, +\infty)$. Then we can simplify U_{n+1} and V_{n+1} as follows:

$$U_{n+1} = U_n \cap \{v \in (-\infty, +\infty) : 2(x_n + e_n - y_n)v \le (x_n + e_n)^2 - y_n^2\} \quad \text{for } n \in N, \tag{3.15}$$

and

$$V_{n+1} = U_{n+1} \cap \left[x_1 - \sqrt{\left(x_1 - P_{U_{n+1}}(x_1)\right)^2 + \lambda_{n+1}}, x_1 + \sqrt{\left(x_1 - P_{U_{n+1}}(x_1)\right)^2 + \lambda_{n+1}}\right] \quad \text{for } n \in N. \tag{3.16}$$

Next, we split the proof into three parts.

Part 1. We shall show that both $\{x_n\}$ and $\{y_n\}$ generated by (3.10) converge strongly to $0 \in A^{-1}0 \cap \mathrm{Fix}(B)$, as $n \to \infty$.

By using inductive method, we first show that the following is true:

$$\begin{cases} x_1 = 1, \qquad x_2 = 1 - \frac{\sqrt{2}}{2}, \\ 0 < (1 + r_{n+1})y_{n+1} < 1, \quad n \in N, \\ U_1 = (-\infty, +\infty) = V_1, \\ U_2 = (-\infty, \frac{4}{3}], \qquad V_2 = [1 - \frac{\sqrt{2}}{2}, \frac{4}{3}], \\ U_{n+1} = (-\infty, w_n], \quad n \in N \setminus \{1\}, \\ V_{n+1} = [x_1 - \sqrt{(x_1 - w_n)^2 + \lambda_{n+1}}, w_n], \quad n \in N \setminus \{1\}, \\ \text{we may choose } x_{n+1} = x_1 - \sqrt{(x_1 - w_n)^2 + \lambda_{n+1}}, \quad n \in N \setminus \{1\}. \end{cases} \tag{3.17}$$

In fact, if $n = 1$, $y_1 = \frac{x_1+e_1}{1+2r_1} = \frac{2}{3}$. Since $(x_1 + e_1) - y_1 = 2r_1y_1 = 2y_1 = \frac{4}{3} > 0$, then from (3.15), $U_2 = (-\infty, +\infty) \cap (-\infty, (1 + r_1)y_1] = (-\infty, \frac{4}{3}]$. Thus $P_{U_2}(x_1) = x_1$. From (3.16), $V_2 = U_2 \cap [1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}] = [1 - \frac{\sqrt{2}}{2}, \frac{4}{3}]$. So, we may choose $x_2 = 1 - \frac{\sqrt{2}}{2}$.

If $n = 2$, $y_2 = \frac{x_2+e_2}{1+2r_2} = \frac{3}{10} - \frac{\sqrt{2}}{10}$ and $w_2 = \min\{(1 + r_1)y_1, (1 + r_2)y_2\} = \frac{9-3\sqrt{2}}{10} = (1 + r_2)y_2$. It is easy to see that $0 < (1 + r_2)y_2 < 1$, and then $x_2 + e_2 - y_2 = 2r_2y_2 > 0$. From (3.15), $U_3 = U_2 \cap (-\infty, 3y_2] = [-\infty, \frac{4}{3}] \cap (-\infty, \frac{9-3\sqrt{2}}{10}] = (-\infty, w_2]$, and then $P_{U_3}(x_1) = w_2 = \frac{9-3\sqrt{2}}{10}$. From (3.16), $V_3 = U_3 \cap [x_1 - \sqrt{(x_1 - w_2)^2 + \lambda_3}, x_1 + \sqrt{(x_1 - w_2)^2 + \lambda_3}] = [1 - \sqrt{(\frac{1+3\sqrt{2}}{10})^2 + \frac{1}{3}}, \frac{9-3\sqrt{2}}{10}] = [x_1 - \sqrt{(x_1 - w_2)^2 + \lambda_3}, w_2]$. Then we may choose $x_3 = x_1 - \sqrt{(x_1 - w_2)^2 + \lambda_3}$.

Suppose that (3.17) is true for $n = k$. We now begin the discussion for $n = k + 1$.

Since $0 < (1 + r_{k+1})y_{k+1} < 1$, then $x_{k+1} + e_{k+1} - y_{k+1} = 2r_{k+1}y_{k+1} > 0$. From (3.15) and (3.13), $U_{k+2} = U_{k+1} \cap (-\infty, (1 + r_{k+1})y_{k+1}] = (-\infty, w_{k+1}]$, and then $P_{U_{k+2}}(x_1) = w_{k+1}$.

Note that $w_{k+1} < 1 = x_1 < x_1 + \sqrt{(x_1 - w_{k+1})^2 + \lambda_{k+2}}$ and $\sqrt{(x_1 - w_{k+1})^2 + \lambda_{k+2}} > x_1 - w_{k+1} > 0$, then from (3.16) we know that

$$V_{k+2} = \left[x_1 - \sqrt{(x_1 - w_{k+1})^2 + \lambda_{k+2}}, w_{k+1}\right].$$

Then we may choose

$$x_{k+2} = x_1 - \sqrt{(x_1 - w_{k+1})^2 + \lambda_{k+2}}.$$

Since $y_{k+2} = \frac{x_{k+2}+e_{k+2}}{1+2r_{k+2}} = \frac{x_{k+2}}{1+2^{k+2}} + \frac{1}{(k+2)(1+2^{k+2})}$, then $(1 + r_{k+2})y_{k+2} = \frac{1+r_{k+2}}{1+2r_{k+2}}(x_{k+2} + e_{k+2})$. Note that

$$\begin{aligned} (1 + r_{k+2})y_{k+2} > 0 \quad &\Longleftrightarrow \quad x_{k+2} + e_{k+2} > 0 \\ &\Longleftrightarrow \quad 1 + \frac{1}{k+2} > \sqrt{(1 - w_{k+1})^2 + \lambda_{k+2}} \\ &\Longleftrightarrow \quad 1 + \frac{2}{k+2} + \frac{1}{(k+2)^2} > (1 - w_{k+1})^2 + \frac{1}{k+2} \\ &\Longleftrightarrow \quad 1 + \frac{1}{k+2} + \frac{1}{(k+2)^2} > (1 - w_{k+1})^2. \end{aligned}$$

This is obviously true. Then $(1 + r_{k+2})y_{k+2} > 0$. Since

$$x_{k+2} + \frac{1}{k+2} = 1 - \sqrt{(1 - w_{k+1})^2 + \frac{1}{k+2}} + \frac{1}{k+2} < w_{k+1} + \frac{1}{k+2}$$
$$< 1 + \frac{1}{k+2} < \frac{1 + 2^{k+2}}{1 + 2^{k+1}} = \frac{1 + 2r_{k+2}}{1 + r_{k+2}},$$

then $(1 + r_{k+2})y_{k+2} = \frac{1+r_{k+2}}{1+2r_{k+2}}(x_{k+2} + e_{k+2}) < 1$.

By now, we have proved that (3.17) is true.

In this part, it is left to prove that $x_n \to 0$, $y_n \to 0$, as $n \to \infty$.

From (3.17), $\{(1 + r_n)y_n\}$ is bounded, which implies that $\{w_n\}$ is bounded. Thus $\{x_n\}$ is bounded. Let $\{x_{n_i}\}$ be any subsequence of $\{x_n\}$ such that $\lim_{i\to\infty} x_{n_i} = a$. Then $w_{n_i} \to a$ and $y_{n_i} \to 0$ as $i \to \infty$. Since $0 < w_{n_i} \le (1 + r_{n_i})y_{n_i} < 1$, then $0 \le a \le \lim_{i\to\infty}(1 + r_{n_i})y_{n_i} \le 1$. That is, $0 \le a \le \lim_{i\to\infty} r_{n_i}y_{n_i} \le 1$. From the fact that $2r_ny_n = x_n + e_n - y_n$, we have $\lim_{i\to\infty}(1 + r_{n_i})y_{n_i} = \frac{a}{2}$. By now, we know that $0 \le a \le \frac{a}{2} \le 1$, then $a = 0$. This means that each strongly convergent subsequence of $\{x_n\}$ converges strongly to 0. Thus $x_n \to 0 \in A^{-1}0 \cap \mathrm{Fix}(B)$, as $n \to \infty$. And then $y_n \to 0$, $w_n \to 0$, as $n \to \infty$.

Part 2. We shall show that both $\{x_n\}$ and $\{y_n\}$ generated by (3.11) converge strongly to $0 \in A^{-1}0 \cap \mathrm{Fix}(B)$, as $n \to \infty$.

First, we shall use inductive method to show that the following is true:

$$\begin{cases} x_1 = 1, \\ 0 < (1 + r_{n+1})y_{n+1} < (1 + r_n)y_n, \quad n \in N, \\ \frac{1+2^{n+1}}{(n+2)2^{n+1}} < (1 + r_{n+1})y_{n+1} < 1, \quad n \in N \setminus \{1\}, \\ U_1 = (-\infty, +\infty) = V_1, \qquad V_2 = [1 - \frac{\sqrt{2}}{2}, \frac{4}{3}], \qquad V_3 = [1 - \frac{\sqrt{3}}{3}, \frac{11}{10}], \\ U_{n+1} = (-\infty, (1 + r_n)y_n], \quad n \in N \setminus \{1\}, \\ V_{n+1} = [x_1 - \sqrt{[x_1 - (1 + r_n)y_n]^2 + \lambda_{n+1}}, (1 + r_n)y_n], \quad n \in N \setminus \{1, 2\}, \\ \text{we may choose } x_{n+1} = (1 + r_n)y_n, \quad n \in N. \end{cases} \tag{3.18}$$

In fact, if $n = 1$, $y_1 = \frac{x_1+e_1}{1+2r_1} = \frac{2}{3}$. Since $(x_1 + e_1) - y_1 = 2r_1y_1 = 2y_1 = \frac{4}{3} > 0$, then from (3.15), $U_2 = (-\infty, +\infty) \cap (-\infty, (1 + r_1)y_1] = (-\infty, \frac{4}{3}]$. Thus $P_{U_2}(x_1) = x_1$. From (3.16), $V_2 = U_2 \cap [1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}] = [1 - \frac{\sqrt{2}}{2}, \frac{4}{3}]$. Then we may choose $x_2 = (1 + r_1)y_1 = \frac{4}{3}$.

If $n = 2$, $y_2 = \frac{x_2+e_2}{1+2r_2} = \frac{11}{30}$. It is easy to see that $0 < (1 + r_2)y_2 = \frac{11}{10} < (1 + r_1)y_1 = \frac{4}{3}$. From (3.15), $U_3 = U_2 \cap (-\infty, 3y_2] = (-\infty, \frac{11}{10}] = (-\infty, (1 + r_2)y_2]$, and then $P_{U_3}(x_1) = x_1$. From (3.16), $V_3 = U_3 \cap [1 - \frac{\sqrt{3}}{3}, 1 + \frac{\sqrt{3}}{3}] = [1 - \frac{\sqrt{3}}{3}, \frac{11}{10}]$. Then we may choose $x_3 = (1 + r_2)y_2 = \frac{11}{10}$. Thus $y_3 = \frac{x_3+e_3}{1+2r_3} = \frac{43}{270}$. It is easy to check that $0 < (1 + r_3)y_3 = \frac{43}{54} < \frac{11}{10} = (1 + r_2)y_2$ and $\frac{1+2^3}{(2+2)2^3} < (1 + r_3)y_3 < 1$.

Suppose that (3.18) is true for $n = k$. Next, we show the result is true for $n = k + 1$.

Since $0 < (1 + r_{k+1})y_{k+1} < (1 + r_k)y_k < 1$, then (3.15) implies that $U_{k+2} = U_{k+1} \cap (-\infty, (1 + r_{k+1})y_{k+1}] = (-\infty, (1 + r_{k+1})y_{k+1}]$ and $P_{U_{k+2}}(x_1) = (1 + r_{k+1})y_{k+1}$.

Note that $(1 + r_{k+1})y_{k+1} < 1 = x_1 < x_1 + \sqrt{[x_1 - (1 + r_{k+1})y_{k+1}]^2 + \lambda_{k+2}}$ and $x_1 - \sqrt{[x_1 - (1 + r_{k+1})y_{k+1}]^2 + \lambda_{k+2}} < (1 + r_{k+1})y_{k+1}$. Then, from (3.16), we know that

$$V_{k+2} = \left[x_1 - \sqrt{\left[x_1 - (1 + r_{k+1})y_{k+1}\right]^2 + \lambda_{k+2}}, (1 + r_{k+1})y_{k+1}\right].$$

Thus we may choose

$$x_{k+2} = (1 + r_{k+1})y_{k+1}.$$

And then, $y_{k+2} = \frac{x_{k+2}+e_{k+2}}{1+2r_{k+2}} = \frac{x_{k+2}}{1+2^{k+2}} + \frac{1}{(k+2)(1+2^{k+2})}$. So $(1 + r_{k+2})y_{k+2} = \frac{1+r_{k+2}}{1+2r_{k+2}}(x_{k+2} + e_{k+2})$. Note that

$$(1 + r_{k+2})y_{k+2} > 0 \quad \Longleftrightarrow \quad x_{k+2} + e_{k+2} > 0 \quad \Longleftrightarrow \quad (1 + r_{k+1})y_{k+1} + \frac{1}{k+2} > 0,$$

which is obviously true from the assumption. Thus $(1 + r_{k+2})y_{k+2} > 0$.

Since $(1 + r_{k+1})y_{k+1} < 1$, then $\frac{1+2^{k+1}}{1+2^{k+2}}[(1 + r_{k+1})y_{k+1} + \frac{1}{k+2}] < \frac{1+2^{k+1}}{1+2^{k+2}} \frac{k+3}{k+2} < 1$. Thus

$$(1 + r_{k+2})y_{k+2} = (1 + r_{k+2})\frac{x_{k+2} + e_{k+2}}{1 + 2r_{k+2}} = \frac{1 + 2^{k+1}}{1 + 2^{k+2}}\left[(1 + r_{k+1})y_{k+1} + \frac{1}{k+2}\right] < 1.$$

Note that

$$\begin{aligned}
&(1 + r_{k+2})y_{k+2} < (1 + r_{k+1})y_{k+1} \\
&\quad\Longleftrightarrow\quad \frac{1 + r_{k+2}}{1 + 2r_{k+2}}(x_{k+2} + e_{k+2}) < (1 + r_{k+1})y_{k+1} \\
&\quad\Longleftrightarrow\quad \frac{1 + 2^{k+1}}{1 + 2^{k+2}}\left[(1 + r_{k+1})y_{k+1} + \frac{1}{k+2}\right] < (1 + r_{k+1})y_{k+1} \\
&\quad\Longleftrightarrow\quad \frac{2^{k+2} - 2^{k+1}}{1 + 2^{k+2}}(1 + r_{k+1})y_{k+1} > \frac{1 + 2^{k+1}}{(k+2)(1 + 2^{k+2})} \\
&\quad\Longleftrightarrow\quad (1 + r_{k+1})y_{k+1} > \frac{1 + 2^{k+1}}{(k+2)2^{k+1}},
\end{aligned}$$

which is true from the assumption.

Compute the following:

$$\begin{aligned}
(1 + r_{k+2})y_{k+2} &= \frac{1 + r_{k+2}}{1 + 2r_{k+2}}(x_{k+2} + e_{k+2}) \\
&= \frac{1 + 2^{k+1}}{1 + 2^{k+2}}\left[(1 + r_{k+1})y_{k+1} + \frac{1}{k+2}\right] \\
&> \frac{1 + 2^{k+1}}{1 + 2^{k+2}}\left[\frac{1 + 2^{k+1}}{(k+2)2^{k+1}} + \frac{1}{k+2}\right] \\
&= \frac{1 + 2^{k+1}}{(k+2)2^{k+1}} > \frac{1 + 2^{k+2}}{(k+3)2^{k+2}}.
\end{aligned}$$

By now, we have proved that (3.18) is true.

In this part, it is left to prove that $x_n \to 0$, $y_n \to 0$, as $n \to \infty$.

Since $\{(1 + r_n)y_n\}$ is decreasing and bounded in $(0, 1)$, then $\lim_{n\to\infty}(1 + r_n)y_n = \lim_{n\to\infty} x_n = a$. Coming back to (3.13), we know that $r_n y_n \to 0$, as $n \to \infty$. Then $y_n \to 0$, and then $x_n \to 0$, as $n \to \infty$.

Part 3. We shall show that both $\{x_n\}$ and $\{y_n\}$ generated by (3.12) converge strongly to $0 \in A^{-1}0 \cap \mathrm{Fix}(B)$, as $n \to \infty$.

First, we shall use inductive method to show that the following is true:

$$\begin{cases} x_1 = 1, \qquad x_2 = \frac{7}{6} - \frac{\sqrt{2}}{4}, \\ 0 < (1 + r_{n+1})y_{n+1} < 1, \quad n \in N, \\ U_1 = (-\infty, +\infty) = V_1, \\ U_2 = (-\infty, \frac{4}{3}], \qquad V_2 = [1 - \frac{\sqrt{2}}{2}, \frac{4}{3}], \\ U_{n+1} = (-\infty, w_n], \quad n \in N \setminus \{1\}, \\ V_{n+1} = [x_1 - \sqrt{(x_1 - w_n)^2 + \lambda_{n+1}}, w_n], \quad n \in N \setminus \{1\}, \\ \text{we may choose } x_{n+1} = \frac{x_1 - \sqrt{(x_1 - w_n)^2 + \lambda_{n+1}} + w_n}{2}, \quad n \in N \setminus \{1\}. \end{cases} \tag{3.19}$$

In fact, if $n = 1$, $y_1 = \frac{x_1 + e_1}{1 + 2r_1} = \frac{2}{3}$. Since $(x_1 + e_1) - y_1 = 2r_1 y_1 = 2y_1 = \frac{4}{3} > 0$, then from (3.15), $U_2 = (-\infty, +\infty) \cap (-\infty, (1 + r_1)y_1] = (-\infty, \frac{4}{3}]$. Then $P_{U_2}(x_1) = x_1$. From (3.16), $V_2 = U_2 \cap [1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}] = [1 - \frac{\sqrt{2}}{2}, \frac{4}{3}]$. Thus we may choose $x_2 = \frac{1 - \frac{\sqrt{2}}{2} + \frac{4}{3}}{2} = \frac{7}{6} - \frac{\sqrt{2}}{4}$.

If $n = 2$, $y_2 = \frac{x_2 + e_2}{1 + 2r_2} = \frac{1}{3} - \frac{\sqrt{2}}{20}$ and $w_2 = \min\{(1 + r_1)y_1, (1 + r_2)y_2\} = 1 - \frac{3\sqrt{2}}{20}$. It is easy to see that $0 < (1 + r_2)y_2 = 1 - \frac{3\sqrt{2}}{20} < 1$. Thus from (3.15), $U_3 = U_2 \cap (-\infty, 3y_2] = (-\infty, \frac{4}{3}] \cap (-\infty, 1 - \frac{3\sqrt{2}}{20}] = (-\infty, w_2]$, and then $P_{U_3}(x_1) = 1 - \frac{3\sqrt{2}}{20} = w_2$.

From (3.16), $V_3 = U_3 \cap [x_1 - \sqrt{(x_1 - w_2)^2 + \lambda_3}, x_1 + \sqrt{(x_1 - w_2)^2 + \lambda_3}] = [1 - \sqrt{\frac{18}{400} + \frac{1}{3}}, 1 - \frac{3\sqrt{2}}{20}] = [x_1 - \sqrt{(x_1 - w_2)^2 + \lambda_3}, w_2]$. Then we may choose $x_3 = \frac{x_1 - \sqrt{(x_1 - w_2)^2 + \lambda_3} + w_2}{2} = 1 - \frac{3\sqrt{2}}{40} - \frac{\sqrt{1362}}{120}$. We can easily check that $0 < (1 + r_3)y_3 = 5y_3 = \frac{20}{27} - \frac{9\sqrt{2} + \sqrt{1362}}{216} < 1$.

Suppose that (3.19) is true for $n = k$. Next, we shall show that (3.19) is true for $n = k + 1$.

Since $0 < (1 + r_{k+1})y_{k+1} < 1$, then $x_{k+1} + e_{k+1} - y_{k+1} = 2r_{k+1}y_{k+1} > 0$. From (3.15), $U_{k+2} = U_{k+1} \cap (-\infty, (1 + r_{k+1})y_{k+1}] = (-\infty, w_{k+1}]$, and $P_{U_{k+2}}(x_1) = w_{k+1}$. From (3.16), $V_{k+2} = U_{k+2} \cap [x_1 - \sqrt{(x_1 - w_{k+1})^2 + \lambda_{k+2}}, x_1 + \sqrt{(x_1 - w_{k+1})^2 + \lambda_{k+2}}]$.

Note that $w_{k+1} < 1 = x_1 < x_1 + \sqrt{(x_1 - w_{k+1})^2 + \lambda_{k+2}}$ and $\sqrt{(x_1 - w_{k+1})^2 + \lambda_{k+2}} > x_1 - w_{k+1} > 0$. Then $V_{k+2} = [x_1 - \sqrt{(x_1 - w_{k+1})^2 + \lambda_{k+2}}, w_{k+1}]$. Thus we may choose

$$x_{k+2} = \frac{x_1 - \sqrt{(x_1 - w_{k+1})^2 + \lambda_{k+2}} + w_{k+1}}{2}.$$

Note that

$$\begin{aligned} (1 + r_{k+2})y_{k+2} > 0 \quad &\Longleftrightarrow \quad x_{k+2} + e_{k+2} > 0 \\ &\Longleftrightarrow \quad \frac{1 - \sqrt{(1 - w_{k+1})^2 + \frac{1}{k+2}} + w_{k+1}}{2} + \frac{1}{k+2} > 0 \\ &\Longleftrightarrow \quad \frac{1 + w_{k+1}}{2} + \frac{1}{k+2} > \frac{\sqrt{(1 - w_{k+1})^2 + \frac{1}{k+2}}}{2} \\ &\Longleftrightarrow \quad \left(\frac{k+4}{k+2}\right)^2 + \frac{2(k+4)}{k+2} w_{k+1} > 1 - 2w_{k+1} + \frac{1}{k+2} \\ &\Longleftrightarrow \quad \left(\frac{k+4}{k+2}\right)^2 + \frac{12 + 4k}{k+2} w_{k+1} > \frac{k+3}{k+2}, \end{aligned}$$

which is obviously true since $(\frac{k+4}{k+2})^2 > 1 + \frac{1}{k+2}$. Then $(1 + r_{k+2})y_{k+2} > 0$.

Moreover,

$$
\begin{aligned}
&(1+r_{k+2})y_{k+2}<1\\
\Longleftrightarrow\quad & \frac{1+r_{k+2}}{1+2r_{k+2}}\left(x_{k+2}+\frac{1}{k+2}\right)<1\\
\Longleftrightarrow\quad & \frac{1+w_{k+1}-\sqrt{(1-w_{k+1})^2+\frac{1}{k+2}}}{2}<\frac{1+2r_{k+2}}{1+r_{k+2}}-\frac{1}{k+2}\\
\Longleftrightarrow\quad & 1+w_{k+1}<\sqrt{(1-w_{k+1})^2+\frac{1}{k+2}}+\frac{2(1+2^{k+2})}{1+2^{k+1}}-\frac{2}{k+2}\\
\Longleftrightarrow\quad & w_{k+1}<\sqrt{(1-w_{k+1})^2+\frac{1}{k+2}}+\frac{1+3\cdot 2^{k+1}}{1+2^{k+1}}-\frac{2}{k+2}
\end{aligned}
$$

which is true since $\frac{1+3\cdot 2^{k+1}}{1+2^{k+1}}-\frac{2}{k+2}>1$. Then $(1+r_{k+2})y_{k+2}=\frac{1+r_{k+2}}{1+2r_{k+2}}(x_{k+2}+e_{k+2})<1$.

By now, we have proved that (3.19) is true.

In this part, it is left to prove that $x_n\to 0$, $y_n\to 0$, as $n\to\infty$.

From (3.19), $\{(1+r_n)y_n\}$ is bounded, which implies that $\{w_n\}$ is bounded. Then we can easily check that $\{x_n\}$ is bounded. Let $\{x_{n_i}\}$ be any subsequence of $\{x_n\}$ such that $\lim_{i\to\infty}x_{n_i}=a$. Then $w_{n_i}\to a$ and $y_{n_i}\to 0$ as $i\to\infty$. Since $2r_ny_n=x_n+e_n-y_n$, then $\lim_{i\to\infty}(1+r_{n_i})y_{n_i}=\frac{a}{2}$. Since $0<w_{n_i}\le(1+r_{n_i})y_{n_i}<1$, then $0\le a\le\frac{a}{2}\le 1$. Thus $a=0$. This means that each strongly convergent subsequence of $\{x_n\}$ converges strongly to 0. Thus $x_n\to 0$, as $n\to\infty$. And then $y_n\to 0$, $w_n\to 0$, as $n\to\infty$.

This completes the proof. □

Remark 3.12 Do computational experiments on (3.10), (3.11), and (3.12) in Corollary 3.11. By using the codes of Visual Basic Six, we get Tables 1–3 and Figs. 1–3.

Table 1 Numerical results of $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ with initial $x_1=1.0$ based on (3.10)

n	x_n	y_n	w_n
1	1.000000000000000	0.666666666666667	1.33333333333333
2	0.292893218813452	0.15857864376269	0.475735931288071
3	0.220137097256371	0.061496714509967	0.307483572549836
4	0.145846031275193	0.023285060663247	0.20956554596922
5	0.091822359822189	0.008843101812794	0.150332730817491
6	0.057343575321990	0.003446311415210	0.113728276701933
7	0.036498723210567	0.001390355550912	0.090373110809311
8	0.024079369242186	0.000580075366701	0.074829722304444
9	0.016612409147652	0.000248973723701	0.063986246991232
10	0.012011262300244	0.000109279280293	0.056060270790268
11	0.009075530986527	0.000048796789603	0.050016709342611
12	0.007124586938736	0.000022079062795	0.045239999667432
13	0.005771789196202	0.000010093356050	0.041352479737665
14	0.004794674685823	0.000004652013800	0.038113949064097
15	0.004062531254230	0.000002158417954	0.035365678169425
16	0.003496425067359	0.000001007010163	0.032998716038761
17	0.003047136222354	0.000000472032117	0.030935568833118
18	0.002682885282545	0.000000222161174	0.029119331499637
19	0.002382412236492	0.000000104930661	0.027507048057261
20	0.002130999790903	0.000000049715948	0.026065524753426

Table 2 Numerical results of $\{x_n\}$ and $\{y_n\}$ with initial $x_1 = 1.0$ based on (3.11)

n	x_n	y_n
1	1.000000000000000	0.666666666666667
2	1.333333333333333	0.366666666666667
3	1.100000000000000	0.159259259259259
4	0.796296296296296	0.061546840958606
5	0.553921568627451	0.022846108140226
6	0.388383838383838	0.008539238539239
7	0.281794871794872	0.003291876082574
8	0.213971945367294	0.001318956985865
9	0.17014545117658	0.000548258406019
10	0.14090241034686	0.000235026741802
11	0.12056871854433	0.000103210253516
12	0.10579050985347	0.000046161543370
13	0.094585002365085	0.000020933489477
14	0.085764506388820	0.000009593718512
15	0.078601335767952	0.000004433092326
16	0.072636217763472	0.0000020619835782
17	0.067569139873525	0.0000009642921827
18	0.063196816788736	0.0000004530026220
19	0.059376412673457	0.0000002136378822
20	0.056004102629354	0.0000001010932937

Table 3 Numerical results of $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ with initial $x_1 = 1.0$ based on (3.12)

n	x_n	y_n	w_n
1	1.000000000000000	0.66666666666667	1.333333333333333
2	0.292893218813452	0.15857864376269	0.475735931288071
3	0.347936514272221	0.07569664973395	0.378483248669752
4	0.290404855661242	0.03178852092125	0.286096688291246
5	0.221842355801224	0.01278310169095	0.217312728746085
6	0.167276133945363	0.00513758154788	0.169540191079954
7	0.12855725127519	0.00210398755141	0.136759190841874
8	0.101961142045651	0.00088311728422	0.113922129664937
9	0.083609952177013	0.00037957322278	0.097550318255454
10	0.070649802113264	0.00016648761182	0.085408144862541
11	0.061199308074553	0.00007423543142	0.07609131720753
12	0.054067201574777	0.00003353686476	0.068717035886434
13	0.048505907532604	0.00001530928652	0.062722146871100
14	0.044042300609567	0.00000704735258	0.057738959695359
15	0.040371134215207	0.00000326643477	0.053520533658321
16	0.037290303448763	0.00000152265596	0.049895913052360
17	0.034661992662647	0.00000071323249	0.046743117653451
18	0.032389319001134	0.00000033548179	0.043972605019241
19	0.030402139089697	0.00000015837395	0.041516938205506
20	0.028648273174303	0.00000007500477	0.039324174089535

Remark 3.13 From Tables 1–3 and Figs. 1–3, we can see that for initial value $x_1 = 1$, different choices of x_{n+1} in V_{n+1} lead to different rates of convergence. It is a natural phenomenon that the larger x_{n+1} is chosen, the slower the rate of convergence is. Although x_{n+1} in (3.11) is the slowest sequence among the three, it is worth being considered because of its "nice and simple" expression compared to the other two.

Remark 3.14 Although both x_{n+1} in (3.12) and (1.5) are chosen as the mid-point of V_{n+1}, they have different rates of convergence. From Table 1 in [8], we may find that the iterative sequence in (1.5) converges more rapidly than that in (3.12). From this point view, it is not easy for us to draw the conclusion which one is better, (1.3) or (3.1).

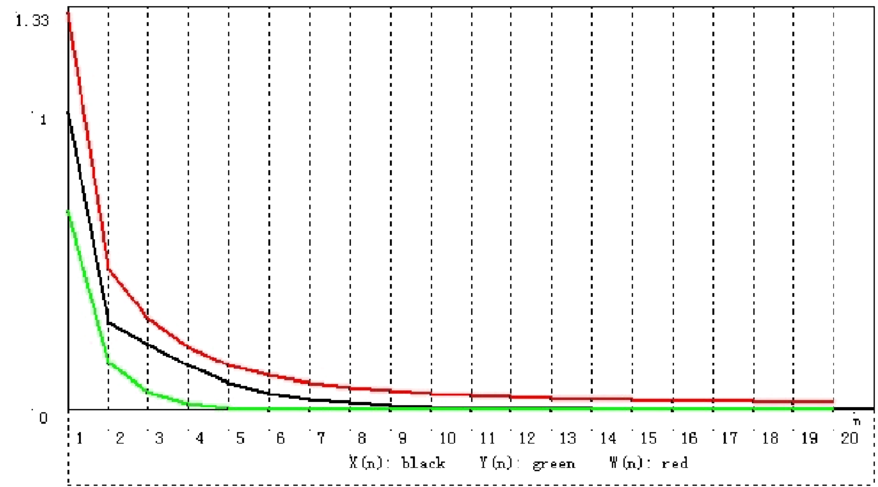

Figure 1 Convergence of $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ corresponding to Table 1

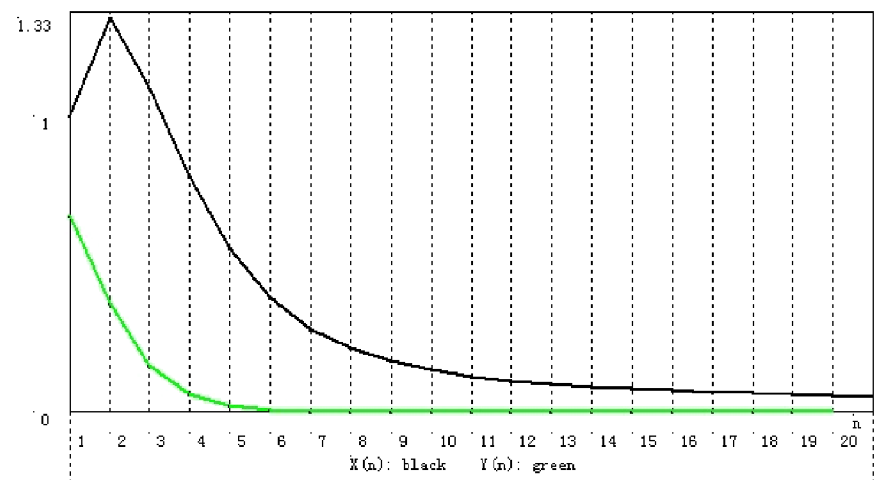

Figure 2 Convergence of $\{x_n\}$ and $\{y_n\}$ corresponding to Table 2

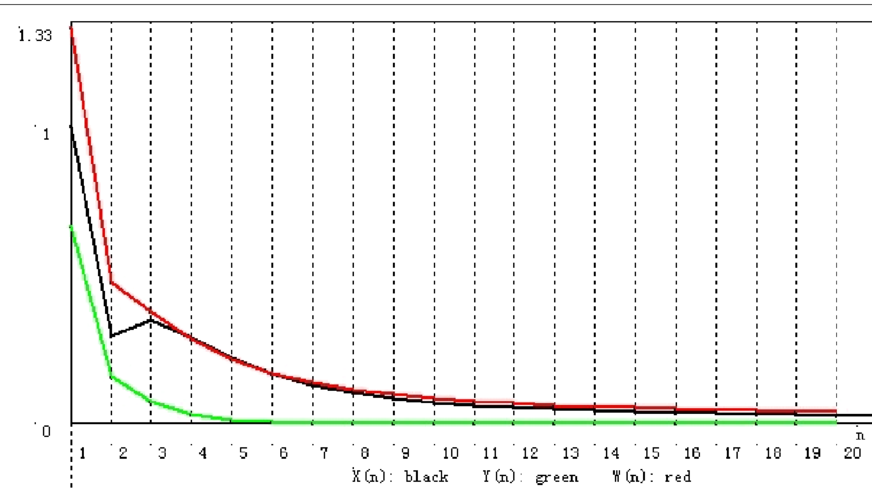

Figure 3 Convergence of $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ corresponding to Table 3

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Competing interests
The authors declare that they have no competing interests.

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