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# Reverses of Ando's and Hölder–McCarty's inequalities

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**Abstract**
In this paper, we give some reverse-types of Ando's and Hölder–McCarthy's inequalities for positive linear maps, and positive invertible operators. For this purpose, we use a recently improved Young inequality and its reverse.

**MSC:** Primary 47A63; secondary 15A64

**Keywords:** Positive operator; Young inequality; Hölder–McCarthy's inequality; Ando's inequality

## 1 Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ be the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with the operator norm $\|\cdot\|$ and the identity $I$; also $\mathcal{M}_n(\mathcal{C})$ denotes the space of all $n \times n$ complex matrices. For an operator $A \in \mathcal{B}(\mathcal{H})$, we write $A \geq 0$ if $A$ is positive, and $A > 0$ if $A$ is positive invertible. For $A, B \in \mathcal{B}(\mathcal{H})$, we say $A \geq B$ if $A - B \geq 0$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$-isomorphism between the $C^*$-algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of a selfadjoint operator $A$ and the $C^*$-algebra generated by $A$ and $I$. A linear map $\Phi$ on $\mathcal{B}(\mathcal{H})$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$. A continuous function $f : J \to \mathcal{R}$ is operator concave if

$$f\bigl(\alpha A + (1-\alpha)B\bigr) \geq \alpha f(A) + (1-\alpha)f(B)$$

for all selfadjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ with spectra in $J$ and all $\alpha \in [0,1]$.

The well-known Young inequality says that, for positive real numbers $a$, $b$ and $0 \leq t \leq 1$, we have $a^t b^{1-t} \leq ta + (1-t)b$. Refinements and reverses of this inequality are proven in [2, 9, 14–16] and the references therein. Also Kittaneh et al. in [10] obtained the following improvement of the Young inequality for any positive definite matrices $A, B \in \mathcal{M}_n(\mathcal{C})$:

$$A^{1-t}B^t + r(A + B - 2A\sharp B) \leq (1-t)A + tB \leq A^{1-t}B^t + R(A + B - 2A\sharp B), \tag{1}$$

where $t \in [0,1]$, $r = \min\{t, 1-t\}$ and $R = \max\{t, 1-t\}$.

Zhao et al. [19] obtained a refinement of Young's inequality and its reverse as follows:

(i) for $0 < t \leq \frac{1}{2}$,

$$r_0\bigl(\sqrt[4]{ab} - \sqrt{a}\bigr)^2 + r(\sqrt{a} - \sqrt{b})^2 + a^{1-t}b^t$$

$$\begin{aligned} &\leq (1-t)a + tb \\ &\leq R(\sqrt{a}-\sqrt{b})^2 - r_0\left(\sqrt[4]{ab}-\sqrt{b}\right)^2 + a^{1-t}b^t; \end{aligned} \tag{2}$$

(ii) for $\frac{1}{2} < t < 1$,

$$\begin{aligned} &r_0\left(\sqrt[4]{ab}-\sqrt{b}\right)^2 + R(\sqrt{a}-\sqrt{b})^2 + a^{1-t}b^t \\ &\quad\leq (1-t)a + tb \\ &\quad\leq r(\sqrt{a}-\sqrt{b})^2 - r_0\left(\sqrt[4]{ab}-\sqrt{a}\right)^2 + a^{1-t}b^t, \end{aligned}$$

where $r = \min\{t, 1-t\}$, $R = \max\{t, 1-t\}$ and $r_0 = \min\{2r, 1-2r\}$.

Sababheh et al. [15, 16] established some refinements and reverses of Young's inequality as follows:

(i) for $0 \leq t \leq \frac{1}{2}$,

$$S_N(t;a,b) \leq ta + (1-t)b - a^t b^{1-t} \leq (1-t)(\sqrt{a}-\sqrt{b})^2 - S_N(2t;\sqrt{ab},a); \tag{3}$$

(ii) for $\frac{1}{2} \leq t \leq 1$,

$$S_N(t;a,b) \leq ta + (1-t)b - a^t b^{1-t} \leq t(\sqrt{a}-\sqrt{b})^2 - S_N(2-2t;\sqrt{ab},b),$$

where

$$S_N(t;a,b) = \sum_{j=1}^{N} s_j(t)\left(\sqrt[2^j]{b^{2^{j-1}-k_j}a^{k_j}} - \sqrt[2^j]{a^{k_j+1}b^{2^{j-1}-k_j-1}}\right)^2,$$

$s_j(t) = ((-1)^{r_j}2^{j-1}t + (-1)^{r_j+1}[\frac{r_j+1}{2}])$, $r_j = [2^j t]$ and $k_j = [2^{j-1}t]$. Here $[x]$ is the greatest integer less than or equal to $x$.

Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive. The operator $t$-weighted arithmetic, geometric, and harmonic means of operators $A$, $B$ are defined by $A\nabla_t B = (1-t)A + tB$, $A\sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}B \times A^{-\frac{1}{2}})^t A^{\frac{1}{2}}$ and $A!_t B = ((1-t)A^{-1} + tB^{-1})^{-1}$, respectively. In particular, for $t = \frac{1}{2}$ we get the usual operator arithmetic mean $\nabla$, the geometric mean $\sharp$ and the harmonic mean $!$.

## 2 Results and discussion

For positive real numbers $a_i$ and $b_i$ ($i = 1, 2, \ldots, n$) the Hölder inequality states that

$$\sum_{i=1}^{n} a_i^{1/p} b_i^{1/q} \leq \left(\sum_{i=1}^{n} a_i\right)^{1/p} \left(\sum_{i=1}^{n} b_i\right)^{1/q} \tag{4}$$

for $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $p = q = 2$ in (4), then we get the Cauchy–Schwarz inequality. The Hölder inequality for positive operators $A_i$ and $B_i$ ($i = 1, 2, \ldots, n$) is

$$\sum_{i=1}^{n} A_i \sharp_t B_i \leq \left(\sum_{i=1}^{n} A_i\right) \sharp_t \left(\sum_{i=1}^{n} B_i\right),$$

where $0 \le t \le 1$. In the case $t = \frac{1}{2}$, we get the operator Cauchy–Schwarz inequality. For further information as regards the Hölder and Cauchy–Schwarz inequalities we refer the reader to [3–5, 11, 12, 20] and the references therein. Ando [1] proved that if $\Phi$ is a positive linear map, then for positive operators $A, B \in \mathcal{B}(\mathcal{H})$ and $t \in [0, 1]$, we have

$$\Phi(A \sharp_t B) \le \Phi(A) \sharp_t \Phi(B).$$

Recently, some authors presented several reverse-types of Ando's inequality (see [13, 17]).

The Hölder–McCarthy's inequality says that for any positive operator $A$ and any unit vector $x \in \mathcal{H}$, we have

$$\langle A^t x, x \rangle \le \langle Ax, x \rangle^t, \quad 0 \le t \le 1. \tag{5}$$

Furuta [8] showed that this inequality is equivalent to Young's inequality.

## 3 Conclusions

In this paper, we establish a reverse of Ando's inequality for positive (non-unital) linear maps and positive definite matrices by using an inequality due to Sababheh. We obtain some reverses of the matrix Hölder and Cauchy–Schwarz inequalities and a reverse of inequality (5) for $t \in (0, \frac{1}{2}]$ as follows:

$$\begin{aligned}
&\langle Tx, x \rangle^t - \langle T^t x, x \rangle \\
&\quad \le 2R\left( \langle Tx, x \rangle^{\frac{1}{2}} - \langle T^{\frac{1}{2}} x, x \rangle \right) - r_0\left( \langle T^{\frac{1}{2}} x, x \rangle + \langle Tx, x \rangle^{\frac{1}{2}} - 2\langle T^{\frac{1}{4}} x, x \rangle \langle Tx, x \rangle^{\frac{1}{4}} \right).
\end{aligned}$$

## 4 Methods

We use the properties of inner product and the inequalities obtained in [16] and [19].

## 5 Main results

To prove our first result, we need the following lemmas.

**Lemma 1** ([16]) *Let $A, B \in \mathcal{M}_n(\mathcal{C})$ be positive definite matrices and $t \in [0, 1]$. Then*

$$\sum_{j=1}^{N} s_j(t)\left(A \sharp_{\alpha_j(t)} B + A \sharp_{2^{1-j} + \alpha_j(t)} B - 2A \sharp_{2^{-j} + \alpha_j(t)} B\right) + A \sharp_t B \le A \nabla_t B, \tag{6}$$

*where $\alpha_j(t) = \frac{k_j(t)}{2^{j-1}}$.*

For $N = 2$, we have the following lemma, which is shown in [19] for positive invertible operators.

**Lemma 2** *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive invertible operators and $t \in [0, 1]$.*

(i) *If $0 < t \le \frac{1}{2}$, then*

$$r_0(A \sharp B - 2A \sharp_{\frac{1}{4}} B + A) + 2t(A \nabla B - A \sharp B) + A \sharp_t B \le A \nabla_t B. \tag{7}$$

(ii) *If* $\frac{1}{2} < t < 1$, *then*

$$r_0(A\sharp B - 2A\sharp_{\frac{3}{4}}B + B) + 2(1-t)(A\nabla B - A\sharp B) + A\sharp_t B \leq A\nabla_t B, \tag{8}$$

*where* $r = \min\{\nu, 1-\nu\}$ *and* $r_0 = \min\{2r, 1-2r\}$.

**Lemma 3** ([16]) *Let* $A, B \in \mathcal{M}_n(\mathcal{C})$ *be positive definite matrices and* $t \in [0,1]$.

(i) *If* $0 \leq t \leq \frac{1}{2}$, *then*

$$\begin{aligned} A\nabla_t B &\leq A\sharp_t B + 2(1-t)(A\nabla B - A\sharp B) \\ &\quad - \sum_{j=1}^{N} s_j(2t)(A\sharp_{1-\beta_j(t)}B + A\sharp_{1+2^{-j}-\beta_j(t)}B - 2A\sharp_{1-2^{-j-1}-\beta_j(t)}B). \end{aligned} \tag{9}$$

(ii) *If* $\frac{1}{2} \leq t \leq 1$, *then*

$$\begin{aligned} A\nabla_t B &\leq A\sharp_t B + 2t(A\nabla B - A\sharp B) \\ &\quad - \sum_{j=1}^{N} s_j(2-2t)(A\sharp_{\gamma_j(t)}B + A\sharp_{\gamma_j(t)+2^{1-j}}B - 2A\sharp_{\gamma_j(t)+2^{-j}}B), \end{aligned}$$

*where* $\beta_j(t) = 2^{-j}k_j(2t)$ *and* $\gamma_j(t) = 2^{1-j}k_j(2-2t)$.

*Remark* 4 By using functional calculus and numerical inequalities in [10, 16], we can extend inequality (1), Lemmas 1 and 3 for positive invertible operators.

For $N = 2$, we have the following lemma, which is shown in [19] for positive invertible operators.

**Lemma 5** *Let* $A, B \in \mathcal{B}(\mathcal{H})$ *be positive invertible operators and* $t \in [0,1]$.

(i) *If* $0 < t \leq \frac{1}{2}$, *then*

$$A\nabla_t B \leq A\sharp_t B + 2(1-t)(A\nabla B - A\sharp B) - r_0(A\sharp B - 2A\sharp_{\frac{3}{4}}B + B). \tag{10}$$

(ii) *If* $\frac{1}{2} < t < 1$, *then*

$$A\nabla_t B \leq A\sharp_t B + 2t(A\nabla B - A\sharp B) - r_0(A\sharp B - 2A\sharp_{\frac{1}{4}}B + A), \tag{11}$$

*where* $r = \min\{\nu, 1-\nu\}$ *and* $r_0 = \min\{2r, 1-2r\}$.

Now, we obtain a reverse of Ando's inequality for positive invertible operators as follows.

**Theorem 6** *Let* $A, B \in \mathcal{B}(\mathcal{H})$ *be positive invertible operators*, $\Phi$ *be a positive linear map and* $t \in [0,1]$.

(i) *If* $0 \leq t \leq \frac{1}{2}$, *then*

$$\Phi(A)\sharp_t\Phi(B) - \Phi(A\sharp_t B)$$

$$\begin{aligned}
&\le 2R\biggl(\Phi(A)\sharp\Phi(B) - \Phi(A\sharp B) + \frac{1}{2}\bigl(\Phi(A) + \Phi(B) - 2\Phi(A)\sharp\Phi(B)\bigr)\biggr) \\
&\quad - \sum_{j=1}^{N} s_j(2t)\bigl(\Phi(A\sharp_{1-\beta_j(t)}B) + \Phi(A\sharp_{1+2^{-j}-\beta_j(t)}B) - 2\Phi(A\sharp_{1-2^{-j-1}-\beta_j(t)}B)\bigr) \\
&\quad - \sum_{j=1}^{N} s_j(t)\bigl(\Phi(A)\sharp_{\alpha_j(t)}\Phi(B) + \Phi(A)\sharp_{2^{1-j}+\alpha_j(t)}\Phi(B) \\
&\quad - 2\Phi(A)\sharp_{2^{-j}+\alpha_j(t)}\Phi(B)\bigr).
\end{aligned} \tag{12}$$

(ii) *If* $\frac{1}{2} \le t \le 1$, *then*

$$\begin{aligned}
&\Phi(A)\sharp_t\Phi(B) - \Phi(A\sharp_t B) \\
&\quad\le 2R\biggl(\Phi(A)\sharp\Phi(B) - \Phi(A\sharp B) + \frac{1}{2}\bigl(\Phi(A) + \Phi(B) - 2\Phi(A)\sharp\Phi(B)\bigr)\biggr) \\
&\quad\quad - \sum_{j=1}^{N} s_j(2-2t)\bigl(\Phi(A\sharp_{\gamma_j(t)}B) + \Phi(A\sharp_{\gamma_j(t)+2^{1-j}}B) - 2\Phi(A\sharp_{\gamma_j(t)+2^{-j}}B)\bigr) \\
&\quad\quad - \sum_{j=1}^{N} s_j(t)\bigl(\Phi(A)\sharp_{\alpha_j(t)}\Phi(B) + \Phi(A)\sharp_{2^{1-j}+\alpha_j(t)}\Phi(B) \\
&\quad\quad - 2\Phi(A)\sharp_{2^{-j}+\alpha_j(t)}\Phi(B)\bigr),
\end{aligned} \tag{13}$$

*where* $\alpha_j(t) = \frac{k_j(t)}{2^{j-1}}$, $\beta_j(t) = 2^{-j}k_j(2t)$, $\gamma_j(t) = 2^{1-j}k_j(2-2t)$ *and* $R = \max\{t, 1-t\}$.

*Proof* The proof of inequality (13) is similar to the proof of inequality (12). Thus, we only prove inequality (12).

Let $0 \le t \le \frac{1}{2}$. Applying inequalities (10) and (9), we have

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(t)(A\sharp_{\alpha_j(t)}B + A\sharp_{2^{1-j}+\alpha_j(t)}B - 2A\sharp_{2^{-j}+\alpha_j(t)}B) \\
&\quad\le A\nabla_t B - A\sharp_t B \\
&\quad\le 2R(A\nabla B - A\sharp B) - \sum_{j=1}^{N} s_j(2t)(A\sharp_{1-\beta_j(t)}B + A\sharp_{1+2^{-j}-\beta_j(t)}B - 2A\sharp_{1-2^{-j-1}-\beta_j(t)}B).
\end{aligned} \tag{14}$$

Now, using the positive linear map $\Phi$ on (14), we get

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(t)\bigl(\Phi(A\sharp_{\alpha_j(t)}B) + \Phi(A\sharp_{2^{1-j}+\alpha_j(t)}B) - 2\Phi(A\sharp_{2^{-j}+\alpha_j(t)}B)\bigr) + \Phi(A\sharp_t B) \\
&\quad\le \Phi(A)\nabla_t\Phi(B) \\
&\quad\le 2R\bigl(\Phi(A)\nabla\Phi(B) - \Phi(A\sharp B)\bigr) + \Phi(A\sharp_t B) \\
&\quad\quad - \sum_{j=1}^{N} s_j(2t)\bigl(\Phi(A\sharp_{1-\beta_j(t)}B) + \Phi(A\sharp_{1+2^{-j}-\beta_j(t)}B) - 2\Phi(A\sharp_{1-2^{-j-1}-\beta_j(t)}B)\bigr).
\end{aligned} \tag{15}$$

Moreover, if we replace $A$ and $B$ by $\Phi(A)$ and $\Phi(B)$ in inequality (14), respectively, then

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(t)\bigl(\Phi(A)\sharp_{\alpha_j(t)}\Phi(B) + \Phi(A)\sharp_{2^{1-j}+\alpha_j(t)}\Phi(B) - 2\Phi(A)\sharp_{2^{-j}+\alpha_j(t)}\Phi(B)\bigr) \\
&\qquad + \Phi(A)\sharp_t\Phi(B) \\
&\quad \le \Phi(A)\nabla_t\Phi(B) \\
&\quad \le 2R\bigl(\Phi(A)\nabla\Phi(B) - \Phi(A\sharp B)\bigr) + \Phi(A)\sharp_t\Phi(B) \\
&\qquad - \sum_{j=1}^{N} s_j(2t)\bigl(\Phi(A)\sharp_{1-\beta_j(t)}\Phi(B) + \Phi(A)\sharp_{1+2^{-j}-\beta_j(t)}\Phi(B) \\
&\qquad - 2\Phi(A)\sharp_{1-2^{-j-1}-\beta_j(t)}\Phi(B)\bigr).
\end{aligned} \tag{16}$$

From the first inequality of (16) and the second inequality of (15), we have

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(t)\bigl(\Phi(A)\sharp_{\alpha_j(t)}\Phi(B) + \Phi(A)\sharp_{2^{1-j}+\alpha_j(t)}\Phi(B) - 2\Phi(A)\sharp_{2^{-j}+\alpha_j(t)}\Phi(B)\bigr) \\
&\qquad + \Phi(A)\sharp_t\Phi(B) \\
&\quad \le \Phi(A)\nabla_t\Phi(B) \\
&\quad \le 2R\bigl(\Phi(A)\nabla\Phi(B) - \Phi(A\sharp B)\bigr) + \Phi(A\sharp_t B) \\
&\qquad - \sum_{j=1}^{N} s_j(2t)\bigl(\Phi(A\sharp_{1-\beta_j(t)}B) + \Phi(A\sharp_{1+2^{-j}-\beta_j(t)}B) - 2\Phi(A\sharp_{1-2^{-j-1}-\beta_j(t)}B)\bigr),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(t)\bigl(\Phi(A)\sharp_{\alpha_j(t)}\Phi(B) + \Phi(A)\sharp_{2^{1-j}+\alpha_j(t)}\Phi(B) - 2\Phi(A)\sharp_{2^{-j}+\alpha_j(t)}\Phi(B)\bigr) \\
&\qquad + \Phi(A)\sharp_t\Phi(B) \\
&\quad \le 2R\bigl(\Phi(A)\nabla\Phi(B) - \Phi(A\sharp B)\bigr) + \Phi(A\sharp_t B) \\
&\qquad - \sum_{j=1}^{N} s_j(2t)\bigl(\Phi(A\sharp_{1-\beta_j(t)}B) + \Phi(A\sharp_{1+2^{-j}-\beta_j(t)}B) - 2\Phi(A\sharp_{1-2^{-j-1}-\beta_j(t)}B)\bigr).
\end{aligned}$$

Therefore, applying inequality (1), we get

$$\begin{aligned}
&\Phi(A)\sharp_t\Phi(B) - \Phi(A\sharp_t B) \\
&\quad \le 2R\bigl(\Phi(A)\nabla\Phi(B) - \Phi(A\sharp B)\bigr) \\
&\qquad - \sum_{j=1}^{N} s_j(2t)\bigl(\Phi(A\sharp_{1-\beta_j(t)}B) + \Phi(A\sharp_{1+2^{-j}-\beta_j(t)}B) - 2\Phi(A\sharp_{1-2^{-j-1}-\beta_j(t)}B)\bigr) \\
&\qquad - \sum_{j=1}^{N} s_j(t)\bigl(\Phi(A)\sharp_{\alpha_j(t)}\Phi(B) + \Phi(A)\sharp_{2^{1-j}+\alpha_j(t)}\Phi(B) - 2\Phi(A)\sharp_{2^{-j}+\alpha_j(t)}\Phi(B)\bigr)
\end{aligned}$$

$$\leq 2R\Big(\Phi(A)\sharp\Phi(B)-\Phi(A\sharp B)+\frac{1}{2}\big(\Phi(A)+\Phi(B)-2\Phi(A)\sharp\Phi(B)\big)\Big)$$
$$-\sum_{j=1}^{N}s_j(2t)\big(\Phi(A\sharp_{1-\beta_j(t)}B)+\Phi(A\sharp_{1+2^{-j}-\beta_j(t)}B)-2\Phi(A\sharp_{1-2^{-j-1}-\beta_j(t)}B)\big)$$
$$-\sum_{j=1}^{N}s_j(t)\big(\Phi(A)\sharp_{\alpha_j(t)}\Phi(B)+\Phi(A)\sharp_{2^{1-j}+\alpha_j(t)}\Phi(B)-2\Phi(A)\sharp_{2^{-j}+\alpha_j(t)}\Phi(B)\big). \qquad \square$$

Similarly for $N = 2$ by applying Lemma 2 and Lemma 5, we can obtain a reverse of Ando's inequality for positive invertible operators.

**Corollary 7** *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive invertible operators, $\Phi$ be a positive linear map and $t \in [0, 1]$.*

(i) *If $0 < t \leq \frac{1}{2}$, then*

$$\begin{aligned}
&\Phi(A)\sharp_t\Phi(B)-\Phi(A\sharp_t B)\\
&\quad\leq 2R\Big(\Phi(A)\sharp\Phi(B)-\Phi(A\sharp B)+\frac{1}{2}\big(\Phi(A)+\Phi(B)-2\Phi(A)\sharp\Phi(B)\big)\Big)\\
&\qquad - r_0\big(\Phi(A\sharp B)+\Phi(B)-2\Phi(A\sharp_{\frac{3}{4}}B)\big)\\
&\qquad - r_0\big(\Phi(A)\sharp\Phi(B)+\Phi(A)-2\big(\Phi(A)\sharp_{\frac{1}{4}}\Phi(B)\big)\big)\\
&\quad\leq 2R\Big(\Phi(A)\sharp\Phi(B)-\Phi(A\sharp B)+\frac{1}{2}\big(\Phi(A)+\Phi(B)-2\Phi(A)\sharp\Phi(B)\big)\Big);
\end{aligned}\tag{17}$$

(ii) *if $\frac{1}{2} < t < 1$, then*

$$\begin{aligned}
&\Phi(A)\sharp_t\Phi(B)-\Phi(A\sharp_t B)\\
&\quad\leq 2R\Big(\Phi(A)\sharp\Phi(B)-\Phi(A\sharp B)+\frac{1}{2}\big(\Phi(A)+\Phi(B)-2\Phi(A)\sharp\Phi(B)\big)\Big)\\
&\qquad - r_0\big(\Phi(A)\sharp\Phi(B)+\Phi(B)-2\big(\Phi(A)\sharp_{\frac{3}{4}}\Phi(B)\big)\big)\\
&\qquad - r_0\big(\Phi(A\sharp B)+\Phi(A)-2\Phi(A\sharp_{\frac{1}{4}}B)\big)\\
&\quad\leq 2R\Big(\Phi(A)\sharp\Phi(B)-\Phi(A\sharp B)+\frac{1}{2}\big(\Phi(A)+\Phi(B)-2\Phi(A)\sharp\Phi(B)\big)\Big),
\end{aligned}\tag{18}$$

*where $r = \min\{t, 1-t\}$, $R = \max\{t, 1-t\}$ and $r_0 = \min\{2r, 1-2r\}$.*

We want to establish some inequalities for positive invertible operators.

**Theorem 8** *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive invertible. If $t \in [0, 1]$ and $\Phi$, $\Psi$ are two unital positive linear maps, then for any unit vector $x \in \mathcal{H}$*

(i) *for $0 < t \leq \frac{1}{2}$,*

$$\begin{aligned}
&2r\big(\langle\Phi(A)x,x\rangle\nabla\langle\Psi(B)x,x\rangle-\langle\Phi\big(A^{\frac{1}{2}}\big)x,x\rangle\langle\Psi\big(B^{1/2}\big)x,x\rangle\big)\\
&\qquad + r_0\big(\langle\Phi\big(A^{\frac{1}{2}}\big)x,x\rangle\big)\langle\Psi\big(B^{\frac{1}{2}}\big)x,x\rangle+\langle\Phi(A)x,x\rangle-2\langle\Phi\big(A^{\frac{3}{4}}\big)x,x\rangle\langle\Psi\big(B^{\frac{1}{4}}\big)x,x\rangle\big)
\end{aligned}$$

$$\begin{aligned}
&\leq (1-t)\langle\Phi(A)x,x\rangle + t\langle\Psi(B)x,x\rangle - \langle\Psi(B^t)x,x\rangle\langle\Phi(A^{1-t})x,x\rangle \\
&\leq 2R(\langle\Phi(A)x,x\rangle\nabla\langle\Psi(B)x,x\rangle - \langle\Phi(A^{\frac{1}{2}})x,x\rangle\langle\Psi(B^{\frac{1}{2}})x,x\rangle) \\
&\quad - r_0(\langle\Phi(A^{\frac{1}{2}})x,x\rangle\langle\Psi(B^{\frac{1}{2}})x,x\rangle + \langle\Psi(B)x,x\rangle - 2\langle\Phi(A^{\frac{1}{4}})x,x\rangle\langle\Psi(B^{\frac{3}{4}})x,x\rangle); \quad (19)
\end{aligned}$$

(ii) *for* $\frac{1}{2} < t < 1$,

$$\begin{aligned}
&R(\langle\Phi(A)x,x\rangle + \langle\Psi(B)x,x\rangle - 2\langle\Phi(A^{\frac{1}{2}})x,x\rangle\langle\Psi(B^{1/2})x,x\rangle) \\
&\quad + r_0(\langle\Phi(A^{\frac{1}{2}})x,x\rangle)\langle\Psi(B^{\frac{1}{2}})x,x\rangle + \langle\Phi(A)x,x\rangle - 2\langle\Phi(A^{\frac{1}{4}})x,x\rangle\langle\Psi(B^{\frac{3}{4}})x,x\rangle) \\
&\leq (1-t)\langle\Phi(A)x,x\rangle + t\langle\Psi(B)x,x\rangle - \langle\Psi(B^t)x,x\rangle\langle\Phi(A^{1-t})x,x\rangle \\
&\leq r(\langle\Phi(A)x,x\rangle + \langle\Psi(B)x,x\rangle - 2\langle\Phi(A^{\frac{1}{2}})x,x\rangle\langle\Psi(B^{\frac{1}{2}})x,x\rangle) \\
&\quad - r_0(\langle\Phi(A^{\frac{1}{2}})x,x\rangle\langle\Psi(B^{\frac{1}{2}})x,x\rangle + \langle\Phi(A)x,x\rangle - 2\langle\Phi(A^{\frac{3}{4}})x,x\rangle\langle\Psi(B^{\frac{1}{4}})x,x\rangle),
\end{aligned}$$

*where* $r = \min\{t, 1-t\}$, $R = \max\{t, 1-t\}$, $r_0 = \min\{2r, 1-2r\}$.

*Proof* The proof of part (ii) is similar to the proof of part (i). Thus we only prove (i).

Applying inequality (2) for any positive real numbers $k$, $s$, we have

$$\begin{aligned}
&r(k+s-2\sqrt{ks}) + r_0(k^{\frac{1}{2}}s^{\frac{1}{2}} + k - 2k^{\frac{3}{4}}s^{\frac{1}{4}}) \\
&\quad \leq (1-t)k + ts - k^{1-t}s^t \\
&\quad \leq R(k+s-2\sqrt{ks}) - r_0(k^{\frac{1}{2}}s^{\frac{1}{2}} + s - 2k^{\frac{1}{4}}s^{\frac{3}{4}}). \quad (20)
\end{aligned}$$

Fix $s$ in (20). Then applying functional calculus to the operator $A$, we have

$$\begin{aligned}
&r(A + sI - 2\sqrt{s}A^{\frac{1}{2}}) + r_0(A^{\frac{1}{2}}s^{\frac{1}{2}} + A - 2A^{\frac{3}{4}}s^{\frac{1}{4}}) \\
&\quad \leq (1-t)A + tsI - s^tA^{1-t} \\
&\quad \leq R(A + sI - 2\sqrt{s}A^{\frac{1}{2}}) - r_0(A^{\frac{1}{2}}s^{\frac{1}{2}} + sI - 2A^{\frac{1}{4}}s^{\frac{3}{4}}). \quad (21)
\end{aligned}$$

If we apply the positive linear map $\Phi$ and inner product for $x \in \mathcal{H}$ with $\|x\| = 1$ in inequality (21), we have

$$\begin{aligned}
&r(\langle\Phi(A)x,x\rangle + s - 2\sqrt{s}\langle\Phi(A^{\frac{1}{2}})x,x\rangle) \\
&\qquad + r_0(\langle\Phi(A^{\frac{1}{2}})x,x\rangle)s^{\frac{1}{2}} + \langle\Phi(A)x,x\rangle - 2\langle\Phi(A^{\frac{3}{4}})x,x\rangle s^{\frac{1}{4}}) \\
&\quad \leq (1-t)\langle\Phi(A)x,x\rangle + ts - s^t\langle\Phi(A^{1-t})x,x\rangle \\
&\quad \leq R(\langle\Phi(A)x,x\rangle + s - 2\sqrt{s}\langle\Phi(A^{\frac{1}{2}})x,x\rangle) - r_0(\langle\Phi(A^{\frac{1}{2}})x,x\rangle s^{\frac{1}{2}} + s - 2\langle\Phi(A^{\frac{1}{4}})x,x\rangle s^{\frac{3}{4}}).
\end{aligned}$$

Now, using the functional calculus to the operator B, we have

$$\begin{aligned}
&r(\langle\Phi(A)x,x\rangle + B - 2\langle\Phi(A^{\frac{1}{2}})x,x\rangle B^{1/2}) \\
&\qquad + r_0(\langle\Phi(A^{\frac{1}{2}})x,x\rangle)B^{\frac{1}{2}} + \langle\Phi(A)x,x\rangle - 2\langle\Phi(A^{\frac{3}{4}})x,x\rangle B^{\frac{1}{4}})
\end{aligned}$$

$$\le (1-t)\langle \Phi(A)x,x\rangle + tB - B^t\langle \Phi(A^{1-t})x,x\rangle$$
$$\le R\big(\langle \Phi(A)x,x\rangle + B - 2\langle \Phi(A^{\frac{1}{2}})x,x\rangle B^{\frac{1}{2}}\big) - r_0\big(\langle \Phi(A^{\frac{1}{2}})x,x\rangle B^{\frac{1}{2}} + B - 2\langle \Phi(A^{\frac{1}{4}})x,x\rangle B^{\frac{3}{4}}\big).$$

Taking the positive linear map $\Psi$ and the inner product for $y \in \mathcal{H}$ with $\|y\| = 1$, we get

$$\begin{aligned} r\big(&\langle \Phi(A)x,x\rangle + \langle \Psi(B)y,y\rangle - 2\langle \Phi(A^{\frac{1}{2}})x,x\rangle\langle \Psi(B^{1/2})x,x\rangle\big) \\ &+ r_0\big(\langle \Phi(A^{\frac{1}{2}})x,x\rangle\big)\langle \Psi(B^{\frac{1}{2}})x,x\rangle + \langle \Phi(A)x,x\rangle - 2\langle \Phi(A^{\frac{3}{4}})x,x\rangle\langle \Psi(B^{\frac{1}{4}})x,x\rangle\big) \\ \le{}& (1-t)\langle \Phi(A)x,x\rangle + t\langle \Psi(B)x,x\rangle - \langle \Psi(B^t)x,x\rangle\langle \Phi(A^{1-t})x,x\rangle \\ \le{}& R\big(\langle \Phi(A)x,x\rangle + \langle \Psi(B)x,x\rangle - 2\langle \Phi(A^{\frac{1}{2}})x,x\rangle\langle (B^{\frac{1}{2}})x,x\rangle\big) \\ &- r_0\big(\langle \Phi(A^{\frac{1}{2}})x,x\rangle\langle \Psi(B^{\frac{1}{2}})x,x\rangle + \langle \Psi(B)x,x\rangle - 2\langle \Phi(A^{\frac{1}{4}})x,x\rangle\langle \Psi(B^{\frac{3}{4}})x,x\rangle\big). \end{aligned}$$

Now, if we put $x = y$, then we get the desired result. □

**Theorem 9** *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive invertible. If $t \in [0,1]$ and $\Phi$, $\Psi$ are two unital positive linear maps, then for any unit vector $x \in \mathcal{H}$*

(i) *for* $0 < t \le \frac{1}{2}$,

$$\begin{aligned} 2r\big(&\langle \Phi(A)x,x\rangle \nabla \langle \Psi(B)x,x\rangle - \langle \Phi(A^{\frac{1}{2}})x,x\rangle\langle \Psi(B)x,x\rangle^{1/2}\big) \\ &+ r_0\big(\langle \Phi(A^{1/2})x,x\rangle\langle \Psi(B)x,x\rangle^{1/2} + \langle \Phi(A)x,x\rangle - 2\langle \Phi(A^{3/4})x,x\rangle\langle \Psi(B)x,x\rangle^{1/4}\big) \\ \le{}& (1-t)\langle \Phi(A)x,x\rangle + t\langle \Psi(B)x,x\rangle - \langle \Psi(B)x,x\rangle^t\langle \Phi(A^{1-t})x,x\rangle \\ \le{}& 2R\big(\langle \Phi(A)x,x\rangle \nabla \langle \Psi(B)x,x\rangle - \langle \Phi(A^{1/2})x,x\rangle\langle \Psi(B)x,x\rangle^{1/2}\big) \\ &- r_0\big(\langle \Phi(A^{1/2})x,x\rangle\langle \Psi(B)x,x\rangle^{1/2} + \langle \Psi(B)x,x\rangle - 2\langle \Phi(A^{1/4})x,x\rangle\langle \Psi(B)x,x\rangle^{3/4}\big); \end{aligned}$$

(ii) *for* $\frac{1}{2} < t < 1$,

$$\begin{aligned} R\big(&\langle \Phi(A)x,x\rangle + \langle \Psi(B)x,x\rangle - 2\langle \Phi(A^{1/2})x,x\rangle\langle \Psi(B)x,x\rangle^{1/2}\big) \\ &+ r_0\big(\langle \Phi(A^{1/2})x,x\rangle\big)\langle \Psi(B)x,x\rangle^{1/2} + \langle \Phi(A)x,x\rangle - 2\langle \Phi(A^{1/4})x,x\rangle\langle \Psi(B)x,x\rangle^{3/4}\big) \\ \le{}& (1-t)\langle \Phi(A)x,x\rangle + t\langle \Psi(B)x,x\rangle - \langle \Psi(B)x,x\rangle^t\langle \Phi(A^{1-t})x,x\rangle \\ \le{}& r\big(\langle \Phi(A)x,x\rangle + \langle \Psi(B)x,x\rangle - 2\langle \Phi(A^{1/2})x,x\rangle\langle \Psi(B)x,x\rangle^{1/2}\big) \\ &- r_0\big(\langle \Phi(A^{1/2})x,x\rangle\langle \Psi(B)x,x\rangle^{1/2} + \langle \Phi(A)x,x\rangle - 2\langle \Phi(A^{3/4})x,x\rangle\langle \Psi(B)x,x\rangle^{1/4}\big), \end{aligned}$$

*where* $r = \min\{t, 1-t\}$, $R = \max\{t, 1-t\}$, $r_0 = \min\{2r, 1-2r\}$.

*Proof* The proof of part (ii) is similar to the proof of part (i). Thus we just prove (i). For any positive real number $k$ and any unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned} r\big(&k + \langle \Psi(B)x,x\rangle - 2\sqrt{k}\langle \Psi(B)x,x\rangle\big) + r_0\big(k^{1/2}\langle \Psi(B)x,x\rangle^{1/2} + k - 2k^{3/4}\langle \Psi(B)x,x\rangle^{1/4}\big) \\ \le{}& (1-t)k + t\langle \Psi(B)x,x\rangle - k^{1-t}\langle \Psi(B)x,x\rangle^t \\ \le{}& R\big(k + \langle \Psi(B)x,x\rangle - 2\sqrt{k}\langle \Psi(B)x,x\rangle^{1/2}\big) \\ &- r_0\big(k^{1/2}\langle \Psi(B)x,x\rangle^{1/2} + \langle \Psi(B)x,x\rangle - 2k^{1/4}\langle \Psi(B)x,x\rangle^{3/4}\big). \end{aligned} \tag{22}$$

Applying inequality (22) and the functional calculus for the operator $A$, we have

$$\begin{aligned} &r\big(A + \langle\Psi(B)x,x\rangle I_{\mathcal{H}} - 2\sqrt{A}\langle\Psi(B)x,x\rangle\big) \qquad (23)\\ &\quad + r_0\big(A^{1/2}\langle\Psi(B)x,x\rangle^{1/2} + A - 2A^{3/4}\langle\Psi(B)x,x\rangle^{1/4}\big)\\ &\quad \le (1-t)A + t\langle\Psi(B)x,x\rangle I_{\mathcal{H}} - A^{1-t}\langle\Psi(B)x,x\rangle^{t}\\ &\quad \le R\big(A + \langle\Psi(B)x,x\rangle I_{\mathcal{H}} - 2\sqrt{A}\langle\Psi(B)x,x\rangle^{1/2}\big)\\ &\quad - r_0\big(A^{1/2}\langle\Psi(B)x,x\rangle^{1/2} + \langle\Psi(B)x,x\rangle I_{\mathcal{H}} - 2A^{1/4}\langle\Psi(B)x,x\rangle^{3/4}\big). \qquad (24) \end{aligned}$$

Now, using the unital positive operator $\Phi$ and the inner product for $y \in \mathcal{H}$ with $\|y\| = 1$ in inequality (23), we get

$$\begin{aligned} &r\big(\langle\Phi(A)y,y\rangle + \langle\Psi(B)x,x\rangle I_{\mathcal{H}} - 2\langle\Phi(A)y,y\rangle^{1/2}\langle\Psi(B)x,x\rangle\big)\\ &\quad + r_0\big(\langle\Phi(A^{1/2})y,y\rangle\langle\Psi(B)x,x\rangle^{1/2} + \langle\Phi(A)y,y\rangle - 2\langle\Phi(A^{3/4})y,y\rangle\langle\Psi(B)x,x\rangle^{1/4}\big)\\ &\quad \le (1-t)\langle\Phi(A)y,y\rangle + t\langle\Psi(B)x,x\rangle I_{\mathcal{H}} - \langle\Phi(A^{1-t})y,y\rangle\langle\Psi(B)x,x\rangle^{t}\\ &\quad \le R\big(\langle\Phi(A)y,y\rangle + \langle\Psi(B)x,x\rangle I_{\mathcal{H}} - 2\langle\Phi(A)y,y\rangle^{1/2}\langle\Psi(B)x,x\rangle^{1/2}\big)\\ &\quad - r_0\big(\langle\Phi(A^{1/2})y,y\rangle\langle\Psi(B)x,x\rangle^{1/2} + \langle\Psi(B)x,x\rangle I_{\mathcal{H}} - 2\langle\Phi(A^{1/4})y,y\rangle\langle\Psi(B)x,x\rangle^{3/4}\big). \end{aligned}$$

Now, putting $y = x$, we get the desired result. □

**Corollary 10** *Let $A \in \mathcal{B}(\mathcal{H})$ be positive, $\Phi$ be a unital positive linear map and $t \in [0,1]$. Then for any unit vector $x \in \mathcal{H}$*

(i) *for* $0 < t \le \frac{1}{2}$,

$$\begin{aligned} &2r\langle\Phi(A)x,x\rangle^{t-\frac{1}{2}}\big(\langle\Phi(A)x,x\rangle^{\frac{1}{2}} - \langle\Phi(A^{\frac{1}{2}})x,x\rangle\big)\\ &\quad + r_0\langle\Phi(A)x,x\rangle^{t-\frac{1}{2}}\big(\langle\Phi(A^{1/2})x,x\rangle + \langle\Phi(A)x,x\rangle^{1/2}\\ &\quad - 2\langle\Phi(A^{3/4})x,x\rangle\langle\Phi(A)x,x\rangle^{-1/4}\big)\\ &\le \langle\Phi(A)x,x\rangle^{t} - \langle\Phi(A^{t})x,x\rangle\\ &\le 2R\langle\Phi(A)x,x\rangle^{t-\frac{1}{2}}\big(\langle\Phi(A)x,x\rangle^{\frac{1}{2}} - \langle\Phi(A^{\frac{1}{2}})x,x\rangle\big)\\ &\quad - r_0\langle\Phi(A)x,x\rangle^{t-\frac{1}{2}}\big(\langle\Phi(A^{1/2})x,x\rangle + \langle\Phi(A)x,x\rangle^{1/2}\\ &\quad - 2\langle\Phi(A^{1/4})x,x\rangle\langle\Phi(A)x,x\rangle^{1/4}\big); \end{aligned}$$

(ii) *for* $\frac{1}{2} < t < 1$,

$$\begin{aligned} &2R\langle\Phi(A)x,x\rangle^{t-\frac{1}{2}}\big(\langle\Phi(A)x,x\rangle^{\frac{1}{2}} - \langle\Phi(A^{\frac{1}{2}})x,x\rangle\big)\\ &\quad + r_0\langle\Phi(A)x,x\rangle^{t-\frac{1}{2}}\big(\langle\Phi(A^{1/2})x,x\rangle + \langle\Phi(A)x,x\rangle^{1/2}\\ &\quad - 2\langle\Phi(A^{1/4})x,x\rangle\langle\Phi(A)x,x\rangle^{1/4}\big)\\ &\le \langle\Phi(A)x,x\rangle^{t} - \langle\Phi(A^{t})x,x\rangle \end{aligned}$$

$$\begin{aligned}
&\le 2r\langle\Phi(A)x,x\rangle^{t-\frac{1}{2}}\left(\langle\Phi(A)x,x\rangle^{\frac{1}{2}}-\langle\Phi(A^{\frac{1}{2}})x,x\rangle\right)\\
&\quad - r_0\langle\Phi(A)x,x\rangle^{t-\frac{1}{2}}\left(\langle\Phi(A^{1/2})x,x\rangle+\langle\Phi(A)x,x\rangle^{1/2}\right.\\
&\quad \left.-2\langle\Phi(A^{3/4})x,x\rangle\langle\Phi(A)x,x\rangle^{-1/4}\right),
\end{aligned}$$

*where* $r=\min\{t,1-t\}$, $R=\max\{t,1-t\}$, $r_0=\min\{2r,1-2r\}$.

*Proof* Letting $\Psi=\Phi$ and $B=A$ in Theorem 9, we get the desired inequalities. □

In the next result, we obtain a refinement of inequality (5) for $t\in(0,\frac{1}{2}]$.

**Corollary 11** *Let* $T\in\mathcal{B}(\mathcal{H})$ *be positive operator and* $x\in\mathcal{H}$ *be a unit vector. Then, for* $0<t\le\frac{1}{2}$, *we have*

$$\begin{aligned}
&\langle Tx,x\rangle^t-\langle T^tx,x\rangle\\
&\quad\le 2R\langle Tx,x\rangle^{t-\frac{1}{2}}\left(\langle Tx,x\rangle^{\frac{1}{2}}-\langle T^{\frac{1}{2}}x,x\rangle\right)\\
&\qquad - r_0\langle Tx,x\rangle^{t-\frac{1}{2}}\left(\langle T^{\frac{1}{2}}x,x\rangle+\langle Tx,x\rangle^{\frac{1}{2}}-2\langle T^{\frac{1}{4}}x,x\rangle\langle Tx,x\rangle^{\frac{1}{4}}\right)\\
&\quad\le 2R\left(\langle Tx,x\rangle^{\frac{1}{2}}-\langle T^{\frac{1}{2}}x,x\rangle\right)-r_0\left(\langle T^{\frac{1}{2}}x,x\rangle+\langle Tx,x\rangle^{\frac{1}{2}}-2\langle T^{\frac{1}{4}}x,x\rangle\langle Tx,x\rangle^{\frac{1}{4}}\right),
\end{aligned}$$

*where* $r=\min\{t,1-t\}$, $R=\max\{t,1-t\}$, $r_0=\min\{2r,1-2r\}$.

*Proof* If we replace $\Phi(A)=A$, $A\in\mathcal{B}(\mathcal{H})$ and $t$ with $1-t$ in Corollary 10, then we get the desired result. □

## 6 Some new results

In this section, we prove some difference reverse-types of the Hölder and Cauchy–Schwarz inequalities.

**Theorem 12** *Let* $A_i,B_i\in\mathcal{B}(\mathcal{H})$ $(1\le i\le n)$ *be positive invertible and* $t\in[0,1]$.

(i) *If* $0<t\le\frac{1}{2}$, *then*

$$\begin{aligned}
&\left(\sum_{i=1}^n A_i\right)\sharp_t\left(\sum_{i=1}^n B_i\right)-\left(\sum_{i=1}^n A_i\sharp_t B_i\right)\\
&\quad\le R\left(\sum_{i=1}^n A_i+\sum_{i=1}^n B_i-2\sum_{i=1}^n(A_i\sharp B_i)\right)\\
&\qquad - r_0\left(\sum_{i=1}^n(A_i\sharp B_i)+\sum_{i=1}^n B_i-2\sum_{i=1}^n(A_i\sharp_{\frac{3}{4}}B_i)\right)\\
&\qquad - r_0\left(\sum_{i=1}^n A_i\sharp\sum_{i=1}^n B_i+\sum_{i=1}^n A_i-2\left(\sum_{i=1}^n A_i\sharp_{\frac{1}{4}}\sum_{i=1}^n B_i\right)\right).
\end{aligned}$$

(ii) *If* $\frac{1}{2}<t<1$, *then*

$$\left(\sum_{i=1}^n A_i\right)\sharp_t\left(\sum_{i=1}^n B_i\right)-\left(\sum_{i=1}^n A_i\sharp_t B_i\right)$$

$$\leq R\left(\sum_{i=1}^{n} A_i + \sum_{i=1}^{n} B_i - 2\left(\sum_{i=1}^{n} A_i \sharp B_i\right)\right)$$
$$- r_0\left(\left(\sum_{i=1}^{n} A_i\right)\sharp\left(\sum_{i=1}^{n} B_i\right) + \sum_{i=1}^{n} B_i - 2\left(\left(\sum_{i=1}^{n} A_i\right)\sharp_{\frac{3}{4}}\left(\sum_{i=1}^{n} B_i\right)\right)\right)$$
$$- r_0\left(\sum_{i=1}^{n}(A_i \sharp B_i) + \sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n}(A_i \sharp_{\frac{1}{4}} B_i)\right),$$

*where* $r = \min\{t, 1-t\}$, $R = \max\{t, 1-t\}$ *and* $r_0 = \min\{2r, 1-2r\}$.

*Proof* Taking $A = \operatorname{diag}(A_1, \ldots, A_n)$, $B = \operatorname{diag}(B_1, \ldots, B_n)$ and $\Phi([C_{ij}]_{1\leq i,j\leq n}) = \sum_{i=1}^{n} C_{ii}$ in equalities (17) and (18), we get the desired inequality. □

Since the function $f(x) = x^t$ ($t \in [0,1]$) is an operator concave function, $\sum_{i=1}^{n} w_i T_i^t \leq (\sum_{i=1}^{n} w_i T_i)^t$ for positive operators $T_i$ and positive real numbers $w_i$ such that $\sum_{i=1}^{n} w_i = 1$. Now, Theorem 12 yields a reverse of this inequality as follows.

*Example* 13 If for positive operators $T_i$ ($1 \leq i \leq n$), we take $A_i = w_i I$ and $B_i = w_i T_i$ ($1 \leq i \leq n$), in Theorem 12, where $w_i$'s are positive real numbers such that $\sum_{i=1}^{n} w_i = 1$, we obtain the following inequalities:

(i) If $0 \leq t \leq \frac{1}{2}$, then

$$\left(\sum_{i=1}^{n} w_i T_i\right)^t - \sum_{i=1}^{n} w_i T_i^t \leq R\left(I + \sum_{i=1}^{n} w_i T_i - 2\sum_{i=1}^{n} w_i T_i^{1/2}\right)$$
$$- r_0\left(\sum_{i=1}^{n} w_i T_i^{1/2} + \sum_{i=1}^{n} w_i T_i - 2\sum_{i=1}^{n} w_i T_i^{3/4}\right)$$
$$- r_0\left(\left(\sum_{i=1}^{n} w_i T_i\right)^{1/2} + I - 2\left(\sum_{i=1}^{n} w_i T_i\right)^{1/4}\right).$$

(ii) If $\frac{1}{2} < t \leq 1$, then

$$\left(\sum_{i=1}^{n} w_i T_i\right)^t - \sum_{i=1}^{n} w_i T_i^t \leq R\left(I + \sum_{i=1}^{n} w_i T_i - 2\sum_{i=1}^{n} w_i T_i^{1/2}\right)$$
$$- r_0\left(\left(\sum_{i=1}^{n} w_i T_i\right)^{1/2} + \sum_{i=1}^{n} w_i T_i - 2\left(\sum_{i=1}^{n} w_i T_i\right)^{3/4}\right)$$
$$- r_0\left(\sum_{i=1}^{n} w_i T_i^{1/2} + I - 2\sum_{i=1}^{n} w_i T_i^{1/4}\right).$$

In [18], the Tsallis relative operator entropy $T_t(A|B)$ for positive invertible operators $A$, $B$ and $0 < t \leq 1$ is defined as follows:

$$T_t(A,B) = \frac{A \sharp_t B - A}{t}.$$

For further information as regards the Tsallis relative operator entropy see [6] and the references therein. In [7, Proposition 2.3], it is shown that for any unital positive linear map $\Phi$ the following inequality holds:

$$\Phi\bigl(T_t(A|B)\bigr) \le T_t\bigl(\Phi(A)|\Phi(B)\bigr). \tag{25}$$

In (25), by similar techniques of Theorem 12, for positive operators $A_i$, $B_i$ ($i = 1, 2, \ldots, n$), we have

$$\sum_{i=1}^{n}\bigl(T_t(A_i|B_i)\bigr) \le T_t\left(\sum_{i=1}^{n} A_i \middle| \sum_{i=1}^{n} B_i\right). \tag{26}$$

In the next theorem, we show a reverse of inequality (26).

**Theorem 14** *Let $A_i, B_i \in \mathcal{B}(\mathcal{H})$ $(1 \le i \le n)$ be positive invertible and $t \in (0,1)$.*

(i) *If $0 < t \le \frac{1}{2}$, then*

$$\begin{aligned}
&T_t\left(\sum_{i=1}^{n} A_i \middle| \sum_{i=1}^{n} B_i\right) - \sum_{i=1}^{n}\bigl(T_t(A_i|B_i)\bigr) \\
&\quad \le \frac{1}{t}\left[R\left(\sum_{i=1}^{n} A_i + \sum_{i=1}^{n} B_i - 2\sum_{i=1}^{n}(A_i \sharp B_i)\right)\right. \\
&\qquad - r_0\left(\sum_{i=1}^{n}(A_i \sharp B_i) + \sum_{i=1}^{n} B_i - 2\sum_{i=1}^{n}(A_i \sharp_{\frac{3}{4}} B_i)\right) \\
&\qquad \left. - r_0\left(\sum_{i=1}^{n} A_i \sharp \sum_{i=1}^{n} B_i + \sum_{i=1}^{n} A_i - 2\left(\sum_{i=1}^{n} A_i \sharp_{\frac{1}{4}} \sum_{i=1}^{n} B_i\right)\right)\right].
\end{aligned}$$

(ii) *If $\frac{1}{2} < t < 1$, then*

$$\begin{aligned}
&T_t\left(\sum_{i=1}^{n} A_i \middle| \sum_{i=1}^{n} B_i\right) - \sum_{i=1}^{n}\bigl(T_t(A_i|B_i)\bigr) \\
&\quad \le \frac{1}{t}\left[R\left(\sum_{i=1}^{n} A_i + \sum_{i=1}^{n} B_i - 2\left(\sum_{i=1}^{n} A_i \sharp B_i\right)\right)\right. \\
&\qquad - r_0\left(\left(\sum_{i=1}^{n} A_i\right) \sharp \left(\sum_{i=1}^{n} B_i\right) + \sum_{i=1}^{n} B_i - 2\left(\sum_{i=1}^{n} A_i \sharp_{\frac{3}{4}} \sum_{i=1}^{n} B_i\right)\right) \\
&\qquad \left. - r_0\left(\sum_{i=1}^{n}(A_i \sharp B_i) + \sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n}(A_i \sharp_{\frac{1}{4}} B_i)\right)\right].
\end{aligned}$$

*Proof* Applying Theorem 12 for $0 < t \le \frac{1}{2}$, we have

$$\begin{aligned}
&T_t\left(\sum_{i=1}^{n} A_i \middle| \sum_{i=1}^{n} B_i\right) - \sum_{i=1}^{n}\bigl(T_t(A_i|B_i)\bigr) \\
&\quad = \frac{(\sum_{i=1}^{n} A_i) \sharp_t (\sum_{i=1}^{n} B_i) - \sum_{i=1}^{n} A_i}{t} - \sum_{i=1}^{n} \frac{A_i \sharp_t B_i - A_i}{t}
\end{aligned}$$

$$\leq \frac{1}{t}\left[R\left(\sum_{i=1}^{n} A_i + \sum_{i=1}^{n} B_i - 2\sum_{i=1}^{n}(A_i\sharp B_i)\right) - r_0\left(\sum_{i=1}^{n}(A_i\sharp B_i) + \sum_{i=1}^{n} B_i - 2\sum_{i=1}^{n}(A_i\sharp_{\frac{3}{4}} B_i)\right)\right.$$
$$\left. - r_0\left(\sum_{i=1}^{n} A_i\sharp\sum_{i=1}^{n} B_i + \sum_{i=1}^{n} A_i - 2\left(\sum_{i=1}^{n} A_i\sharp_{\frac{1}{4}}\sum_{i=1}^{n} B_i\right)\right)\right],$$

whence we get the first inequality. The proof of the second inequality is similar. □

*Remark* 15 We can present our results for non-invertible operators; see [6]. It is a direct consequence of the definition of the mean in the sense of Kubo–Ando [11] that $A\natural_t(B+\varepsilon)$ is a monotone increasing net. Let $B$ be a non-invertible operator and $\epsilon > 0$. It follows from the set $\{A\natural_t(B+\epsilon) : \epsilon > 0\}$ being bounded above for $0 < \varepsilon < 1$ that the limit

$$A\natural_t B = \lim_{\epsilon\downarrow 0} A\natural_t(B+\epsilon) \tag{27}$$

exists in the strong operator topology. So by (27), $A\natural_t B$ exists.

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