Adell et al. *Journal of Inequalities and Applications* (2018) 2018:153
https://doi.org/10.1186/s13660-018-1747-2

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Generalized Jacobi–Weierstrass operators and Jacobi expansions

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Abstract

We present a realization for some K-functionals associated with Jacobi expansions in terms of generalized Jacobi–Weierstrass operators. Fractional powers of the operators as well as results concerning simultaneous approximation and Nikolskii–Stechkin type inequalities are also considered.

MSC: 41A35; 47D06

Keywords: Jacobi–Weierstrass operators; Abel–Cartwright means; Realization of K-functionals; Semi-groups of operators; Fractional powers; Nikolskii–Stechkin type inequality

1 Introduction

In this note, we work with two fixed real parameters α and β satisfying $\alpha \geq \beta \geq -1/2$. We use the following notations:

$$\varrho^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad x \in (-1,1), \tag{1}$$

and, for $1 \leq p < \infty$,

$$L^p_{(\alpha,\beta)} = \left\{ f : [-1,1] \to \mathbb{R} : \|f\|_p = \left(\int_{-1}^{1} |f(x)|^p \varrho^{\alpha,\beta}(x)\, dx \right)^{1/p} < \infty \right\}.$$

Moreover, for each $n \in \mathbb{N}_0$, $\mathbb{P}_n$ is the family of all algebraic polynomials of degree not greater than n,

$$w_n^{\alpha,\beta} = \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)}{\Gamma(n+\beta+1)\Gamma(n+1)(\Gamma(\alpha+1))^2} \tag{2}$$

(Γ stands for the gamma function) and

$$\lambda_n = n(n+\alpha+\beta+1). \tag{3}$$

Since α and β are fixed, we set X for one of the spaces $C[-1,1]$ or $L^p_{(\alpha,\beta)}$.

For $n \in \mathbb{N}$, the Jacobi polynomial $R_n^{(\alpha,\beta)}$ is the unique polynomial of degree n which satisfies

$$R_n^{(\alpha,\beta)}(1) = 1 \quad \text{and} \quad \int_{-1}^{1} Q_{n-1}(x) R_n^{(\alpha,\beta)}(x) \varrho^{\alpha,\beta}(x)\,dx = 0$$

for all $Q_{n-1} \in \mathbb{P}_{n-1}$. We also take $R_0^{(\alpha,\beta)}(x) = 1$.

For $f \in X$, the Fourier–Jacobi coefficients are defined by

$$\langle f, R_n^{(\alpha,\beta)} \rangle = \int_{-1}^{1} f(x) R_n^{(\alpha,\beta)}(x) \varrho^{\alpha,\beta}(x)\,dx, \quad n \in \mathbb{N}_0,$$

and the associated expansion is

$$f(x) \sim \sum_{n=0}^{\infty} \langle f, R_n^{(\alpha,\beta)} \rangle w_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(x). \tag{4}$$

It is known that each $f \in L^1_{(\alpha,\beta)}$ is completely determined a.e. by its Fourier–Jacobi coefficients.

Definition 1.1 For fixed $\gamma > 0$ and $t > 0$, the generalized Jacobi–Weierstrass kernel is defined by

$$W_{t,\gamma}(x) = \sum_{n=0}^{\infty} e^{-t\lambda_n^{\gamma}} w_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(x), \quad x \in [-1,1]. \tag{5}$$

For $f \in X$, the generalized Jacobi–Weierstrass (or Abel–Cartwright) operator is defined by

$$C_{t,\gamma}(f,x) = \int_{-1}^{1} \tau_y(f,x) W_{t,\gamma}(y) \varrho^{\alpha,\beta}(y)\,dy, \quad x \in [-1,1], \tag{6}$$

where $\tau_y(f,x)$ is the translation given in Theorem 2.1 below.

Of course the kernel $W_{t,\gamma}$ and the operator $C_{t,\gamma}$ also depend on α and β but, for simplicity, we omit these indexes. The (classical) Jacobi–Weierstrass operators correspond to $\gamma = 1$.

The generalized Jacobi–Weierstrass operators have been studied in different papers, but only for parameters satisfying $0 < \gamma \le 1$. This restriction was considered because in such a case the kernels $W_{t,\gamma}$ are positive and the family $\{C_{t,\gamma}\}$ can be considered as formed by positive operators (see [2, 3], [7], pp. 96–97) and/or as a semigroup of contractions (see [2], pp. 49–52, and [18]). For $\gamma > 1$, one cannot expect the positivity of $W_{t,\gamma}$. For instance, it is known that the analogous generalized Weierstrass kernels for trigonometric expansion are not positive when $\gamma > 1$ (see [6], p. 263).

In this paper we will prove that the operators $C_{t,\gamma}$ can be used as a realization of some K-functionals which usually appear in some approximation problems related to Jacobi expansions.

For fixed real $\gamma > 0$, let $\Phi^{\gamma}(X)$ denote the family of all $f \in X$ for which there exists $\Psi^{\gamma}(f) \in X$ satisfying

$$\Psi^{\gamma}(f)(x) \sim \sum_{n=0}^{\infty} \lambda_n^{\gamma} \langle f, R_n^{(\alpha,\beta)} \rangle w_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(x).$$

The associated K-functional is defined by

$$K_{\gamma}(f,t) = K_{\gamma}(f,t)_{\alpha,\beta} = \inf_{g \in \Psi^{\gamma}(X)} \{ \|f-g\|_X + t \|\Psi^{\gamma}(g)\|_X \} \tag{7}$$

for $f \in X$ and $t > 0$. For different realizations of these K-functionals, see [8], Theorem 7.1, and [10], Lemma 2.3. We will not use the characterization of these K-functionals in terms of moduli of smoothness. We will show that, for any $\gamma > 0$,

$$\sup_{0<s\le t} \|(I - C_{s,\gamma})(f)\|_X \approx K_{\gamma}(f,t).$$

The notation $A(f,t) \approx B(f,t)$ means that there exists a positive constant C such that $C^{-1}A(f,t) \le B(f,t) \le CA(f,t)$ with C independent of f and t.

Following [19], for $\gamma > 0$, define

$$(I - C_{t,1})^{\gamma} = \sum_{j=0}^{\infty} (-1)^j \binom{\gamma}{j} C_{jt,1}, \tag{8}$$

where

$$\binom{\gamma}{0} = 1 \quad \text{and} \quad \binom{\gamma}{j} = \prod_{k=1}^{j} \frac{\gamma - k + 1}{k} \quad \text{for } j \in \mathbb{N}.$$

For these operators, we will show the relations

$$K_{\gamma}(f, t^{\gamma}) \approx \sup_{0<s\le t} \|(I - C_{s,1})^{\gamma}(f)\|_X \approx \sup_{0<s\le t^{\gamma}} \|(I - C_{s,\gamma})(f)\|_X.$$

It is known that, if Q_n is a trigonometric polynomial of degree not greater than n and $r \in \mathbb{N}$, then

$$\|Q_n^{(r)}\|_p \le \left(\frac{n}{2\sin(nh)} \right)^r \|(1 - T_h)^r(Q_n)\|_p, \quad h \in (0, \pi/n),$$

where $\|\cdot\|_p$ denotes the L^p-norm of 2π-periodic functions and T_h is the translation operator. That is, $T_h Q(x) = Q(x+h)$. These inequalities are due to Nikolskii [11] and Stechkin [13]. For similar inequalities for algebraic polynomials, see [4] and the references given there. Here we will verify an analogous inequality by considering the operators Ψ^r and the linear combination of the Jacobi–Weierstrass operators $C_{t,1}$.

In Sect. 2 we collect some definitions and results which will be needed later. The main results are given in Sect. 3, where the result concerning simultaneous approximation is also included. Finally, in Sect. 4 we present a Nikolskii–Stechkin type inequality.

2 Auxiliary results

We need a convolution structure due to Askey and Wainger (see [1]).

Theorem 2.1 *For each $h \in [-1,1)$, there exists a function $\tau_h : X \to X$ with the following properties*:

(i) *For each $f \in X$, one has*

$$\|\tau_h f\|_X \le \|f\|_X, \qquad \lim_{h \to 1-} \|\tau_h(f) - f\|_X = 0$$

and

$$\langle \tau_h(f), R_n^{(\alpha,\beta)} \rangle = R_n^{(\alpha,\beta)}(h)\langle f, R_n^{(\alpha,\beta)} \rangle, \quad n \in \mathbb{N}_0.$$

(ii) *For $f \in X$ and $g \in L^1_{\alpha,\beta}$, the integral*

$$(f * g)(x) := \int_{-1}^{1} \tau_y(f,x) g(y) \varrho^{\alpha,\beta}(y)\, dy$$

exists a.e. in $[-1.1]$,

$$f * g = g * f, \qquad f * g \in X, \qquad \|f * g\|_p \le \|g\|_1 \|f\|_X$$

and

$$\langle f * g, R_n^{(\alpha,\beta)} \rangle = \langle f, R_n^{(\alpha,\beta)} \rangle \langle g, R_n^{(\alpha,\beta)} \rangle, \quad n \in \mathbb{N}_0. \tag{9}$$

For $j > \alpha + 1/2$ and $f \in X$, let

$$S_m^j(f) = \sum_{k=0}^{m} \frac{A_{m-k}^j}{A_m^j} \langle f, R_k^{(\alpha,\beta)} \rangle w_k^{\alpha,\beta} R_k^{(\alpha,\beta)}(x), \quad A_m^j = \binom{m+j}{m},$$

be the mth Cesàro means of order j. It is known that there exists a constant C such that

$$\|S_m^j\| \le C, \tag{10}$$

and, for each $f \in X$, one has ([2], Corollary 3.3.3, or [7], Theorem A)

$$\lim_{m \to \infty} \|f - S_m^j(f)\|_X = 0. \tag{11}$$

We need some classical results related to Banach spaces.

Definition 2.2 Let Y be a real Banach space and $B(Y)$ be the Banach algebra of all bounded linear operators $B : Y \to Y$. A uniformly bounded family of operators $\{T(t) : t \ge 0\}$ in $B(Y)$ is called an equi-bounded semigroup of class (C_0) if

$$T(s)T(t) = T(s+t) \quad \text{for } s,t \ge 0, \qquad T(0) = I, \tag{12}$$

and $\lim_{t \to 0+} \|f - T(t)f\|_Y = 0$ for each $f \in Y$.

Let Y, $B(Y)$ and $\{T(t) : t > 0\}$ be an equi-bounded semigroup as in Definition 2.2. Let $D(Q)$ be the family of all $g \in Y$, for which there exists $Q(g) \in Y$ such that

$$Q(g) = \lim_{t\to 0+} \frac{1}{t}\bigl[T(t) - I\bigr]g \tag{13}$$

(the limit is considered in the norm of Y). The operator $Q : D(Q) \to Y$ is called the infinitesimal generator of the semigroup $\{T(t) : t \geq 0\}$. It is known that Q is a closed linear operator and $D(Q)$ is dense in Y. For properties of semigroups of operators, see [5].

For $r \in \mathbb{N}$, set

$$D\bigl(Q^{r+1}\bigr) = \bigl\{f \in Y : f \in D\bigl(Q^r\bigr) \text{ and } Q^r(f) \in D(Q)\bigr\}$$

and, for $f \in D(Q^{r+1})$,

$$Q^{r+1}(f) = Q\bigl(Q^r(f)\bigr). \tag{14}$$

A family of operators $S = \{S_t, : t > 0\}$, $S_t \in B(Y)$ for each $t > 0$ is called a (commutative) strong approximation process for Y if, for all $f \in Y$ and $s, t > 0$,

$$S_s\bigl(S_t(f)\bigr) = S_t\bigl(S_s(f)\bigr), \qquad \bigl\|S_t(f)\bigr\|_Y \leq \Lambda \|f\|_Y \quad \text{and} \quad \lim_{t\to 0+} \bigl\|f - S_t(f)\bigr\|_Y = 0,$$

where Λ is a constant. In such a case, we set

$$\theta_S(f,t) = \sup_{0<s\leq t} \bigl\|f - S_s(f)\bigr\|_Y.$$

Let $\phi : [0,1) \to \mathbb{R}^+$ be a positive increasing function, $\phi(t) \to 0$ as $t \to 0$, and Y_0 be a subspace of Y. We say that S is saturated with order ϕ and with trivial subspace Y_0 if every $f \in Y$ satisfying

$$\lim_{t\to 0+} \frac{\theta_S(f,t)}{\phi(t)} = 0$$

belongs to Y_0 and there exists $f \in Y \setminus Y_0$ satisfying $\theta_S(f,t) \leq C(f)\phi(t)$. The following assertion is known (for instance, see [2], Theorem 2.4.2).

Theorem 2.3 *Assume that Y is a Banach space, $D(B)$ is a dense subspace of Y, and $B : D(B) \to Y$ is a closed linear operator. Let $S = \{S_t : t > 0\}$ be a strong approximation process in Y satisfying $S_t(f) \in D(B)$ for any $f \in Y$ and each $t > 0$. If there exists a constant γ_0 such that, for all $g \in D(B)$,*

$$\lim_{t\to 0+} \left\| \frac{S_t(g) - g}{t^{\gamma_0}} - B(g) \right\|_Y = 0, \tag{15}$$

then the strong approximation process S is saturated with order t^{γ_0} and the trivial space is the kernel of B.

3 The operators $C_{t,\gamma}$ as a semigroup

In fact, it is known that, for $x \in (-1,1)$, $|R_n^{(\alpha,\beta)}(x)| < 1$, [14], pp. 163–164, and there exists a constant C such that, for each $n \in \mathbb{N}_0$,

$$w_n^{(\alpha,\beta)} \le Cn^{2\alpha+1}. \tag{16}$$

These relations can be used to prove that the series in (5) converges absolutely and uniformly in $[-1,1]$. Thus $W_{t,\gamma} \in L^1_{(\alpha,\beta)}$ and, for each $f \in L^1_{(\alpha,\beta)}$, the series $C_{t,\gamma}(f)$ converges absolutely and uniformly in $[-1,1]$. Moreover,

$$C_{t,\gamma}(f,x) = (W_{t,\gamma} * f)(x) = \sum_{n=0}^{\infty} e^{-t\lambda_n^{\gamma}} \langle f, R_n^{(\alpha,\beta)} \rangle w_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(x).$$

For these assertions, see [2], p. 30.

Our first result seems to be known. For convenience of the reader, we include a proof.

Theorem 3.1 *For each $\gamma > 0$, the family of operators $\{C_{t,\gamma} : t > 0\}$ is an equi-bounded semigroup of operators in X.*

Proof. It follows from Theorem 3.9 of [15] that the family of operators $\{C_{t,\gamma} : t > 0\}$ is uniformly bounded.

Condition (12) is derived from the properties of the convolution. In fact, it follows from (9) that, for each $f \in X$ and $k \in \mathbb{N}_0$,

$$\begin{aligned} \langle C_{s+t}(f), R_k^{(\alpha,\beta)} \rangle &= e^{-(s+t)\lambda_n^{\gamma}} \langle f, R_k^{(\alpha,\beta)} \rangle = e^{-s\lambda_n^{\gamma}} \langle C_{t,\gamma}(f), R_k^{(\alpha,\beta)} \rangle \\ &= \langle C_{s,\gamma}(C_{t,\gamma}(f)), R_k^{(\alpha,\beta)} \rangle \end{aligned}$$

and this implies $C_{s+t}(f) = (C_{s,\gamma} \circ C_{t,\gamma})(f)$.

Finally, for each $k \in \mathbb{N}_0$,

$$C_{t,\gamma}(R_k^{(\alpha,\beta)})(x) = e^{-t\lambda_n^{\gamma}} R_k^{(\alpha,\beta)}(x). \tag{17}$$

Hence

$$\lim_{t\to 0+} \| R_k^{(\alpha,\beta)} - C_{t,\gamma}(R_k^{(\alpha,\beta)}) \|_X = 0.$$

Since the operators $C_{t,\gamma}$ are linear and uniformly bounded and the polynomials are dense in X, the last equation holds for every $f \in X$.

Taking into account Theorem 3.1, we denote by A_γ the infinitesimal generator of $C_{t,\gamma}$ and by $D(A_\gamma) = D(A_\gamma(\alpha,\beta))$ the domain of A_γ. In the next result we give a description of the infinitesimal generator.

Theorem 3.2 *If $\gamma, t > 0$ and $A_\gamma : D(A_\gamma) \to X$ is the infinitesimal generator of $C_{t,\gamma}$, then*

$$D(A_\gamma) = \Psi^{\gamma}(X) \quad \textit{and} \quad -A_\gamma(f) = \Psi^{\gamma}(f)$$

for each $f \in \Psi^{\gamma}(X)$.

Moreover, for each $r \in \mathbb{N}$ *and* $f \in D(A_\gamma^r)$,

$$D(A_\gamma^r) = \Psi^{r\gamma}(X) \quad \textit{and} \quad (-1)^r A_\gamma^r(f) = \Psi^{r\gamma}(f), \tag{18}$$

where A_γ^r *is defined as in* (14).

Proof Since A_γ is the infinitesimal generator of the semi-group (see (13)), $A_\gamma : D(A_\gamma) \to X$ is a closed operator.

If $f \in D(A_\gamma)$, then

$$\langle A_\gamma(f), R_n^{(\alpha,\beta)} \rangle = \lim_{t\to 0+} \frac{1}{t}\left(e^{-t\lambda_n^\gamma} - 1\right)\langle f, R_n^{(\alpha,\beta)} \rangle = -\lambda_n^\gamma \langle f, R_n^{(\alpha,\beta)} \rangle. \tag{19}$$

Thus $f \in \Psi^\gamma(X)$ and

$$\Psi^\gamma(f) = -A_\gamma(f).$$

In particular, for each polynomial P, one has $P \in D(A_\gamma)$ and $\Psi^\gamma(P) = -A_\gamma(P)$.

On the other hand, fix an integer $j > \alpha + 1/2$. For $f \in \Psi^\gamma(X)$, let $S_m^j(f)$ and $S_m^j(\Psi^\gamma(f))$ be the mth Cesàro means of order j of f and $\Psi^\gamma(f)$, respectively. We know that (see (11))

$$S_m^j(f) \to f, \quad m \to \infty$$

and

$$-A_\gamma\left(S_m^j(f)\right) = \Psi^\gamma\left(S_m^j(f)\right) = S_m^j\left(\Psi^\gamma(f)\right) \to \Psi^\gamma(f).$$

Since $-A_\gamma$ is a closed operator, $f \in D(A_\gamma)$ and $-A_\gamma(f) = \Psi^\gamma(f)$.

Equations (18) can be proved by recurrence. For instance, (19) can be written as

$$\langle A_\gamma^2(f), R_n^{(\alpha,\beta)} \rangle = \langle A_\gamma\left(A_\gamma(f)\right), R_n^{(\alpha,\beta)} \rangle = -\lambda_n^\gamma \langle A_\gamma(f), R_n^{(\alpha,\beta)} \rangle = \lambda_n^{2\gamma} \langle f, R_n^{(\alpha,\beta)} \rangle. \qquad \square$$

Theorem 3.3 (i) *If for* $\gamma, t > 0$, *and* $f \in X$

$$\theta_\gamma(f,t) = \theta_\gamma(f,t)_{\alpha,\beta} = \sup_{0<s\le t} \left\|(I - C_{s,\gamma})(f)\right\|,$$

and $K_\gamma(f,t)$ *is defined by* (7), *then*

$$\theta_\gamma(f,t) \approx K_\gamma(f,t).$$

(ii) *The strong approximation process* $\{C_{t,\gamma}; t > 0\}$ *is saturated with order t and the trivial class consists of the constant functions.*

Proof (i) From Theorem 3.2 we know that $-\Psi^\gamma$ is the infinitesimal generator of $\{C_{t,\gamma}\}$ and $D(A_\gamma) = \Psi^\gamma(X)$. Thus, the result is a simple consequence of [17], Theorem 1.1, or [5], p. 192.

(ii) We will derive the result from Theorem 2.3, with $B = \Psi^\gamma$ and $D(B) = D(A_\gamma)$. We should verify that $C_{t,\gamma}(f) \in D(A_\gamma)$ for any $f \in X$ and each $t > 0$.

For any $f \in X$, the Fourier–Jacobi coefficients of f are bounded by $\|f\|_{L^1_{(\alpha,\beta)}}$. Taking into account (16), for every $x \in [-1,1]$,

$$\begin{aligned}
&\left| \sum_{n=1}^{\infty} \lambda_n^\gamma \exp\{-t\lambda_n\} \langle f, R_n^{(\alpha,\beta)} \rangle w_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(x) \right| \\
&\quad \le \|f\|_{L^1_{(\alpha,\beta)}} \sum_{n=1}^{\infty} \lambda_n^\gamma \exp\{-t\lambda_n^\gamma\} w_n^{(\alpha,\beta)} \\
&\quad \le C\|f\|_{L^1_{(\alpha,\beta)}} \sum_{n=1}^{\infty} \lambda_n^\gamma \exp\{-t\lambda_n^\gamma\} n^{2\alpha+1} < \infty.
\end{aligned}$$

Since the series converges absolutely and uniformly, it defines a function $g_t \in X$ satisfying

$$\langle g_t, R_n^{(\alpha,\beta)} \rangle = \lambda_n^\gamma \exp\{-t\lambda_n^\gamma\} \langle f, R_n^{(\alpha,\beta)} \rangle = \lambda_n^\gamma \langle C_{t,\gamma}(f), R_n^{(\alpha,\beta)} \rangle, \quad n \in \mathbb{N}.$$

By definition of the operator Ψ^γ, $C_{t,\gamma}(f) \in \Psi^\gamma(X)$ (Theorem 3.2) and

$$\Psi^\gamma\bigl(C_{t,\gamma}(f)\bigr) = g_t.$$

We have proved that $C_{t,\gamma}(X) \in D(A_\gamma)$.

If $g \in \Psi^\gamma(X) = D(A_\gamma)$, by definition of the infinitesimal generator,

$$\lim_{t\to 0+} \left\| \frac{C_{t,\gamma}(g) - g}{t} - A_\gamma(g) \right\|_Y = 0.$$

If $f \in \Psi^\gamma(X)$ and $A_\gamma(f) = -\Psi^\gamma(f) = 0$, then $\langle f, R_n^{(\alpha,\beta)} \rangle = 0$ for all $n \in \mathbb{N}$. Therefore f is a constant.

From part (i), if $g \in \Psi^\gamma(X)$, then

$$\theta_\gamma(g,t) \le CK_\gamma(g,t) \le Ct\bigl\|\Psi^\gamma(g)\bigr\|_X.$$

Hence, the family

$$\bigl\{f \in X : \exists C(f) \text{ such that } \theta_\gamma(f,t) \le C(f)t\bigr\}$$

contains nonconstant functions.

Now, from Theorem 2.3, we know that the strong approximation process $\{C_{t,\gamma} : t > 0\}$ is saturated with order t. □

Remark 3.4 Some characterizations of the saturation class of the strong approximation process $\{C_{t,\gamma} : t > 0\}$ can be given as in [2], Theorems 5.1.1 and 7.4.1, where the case $\gamma = 1$ was considered. When $\gamma > 0$ is not an integer, fractional derivatives should be considered. This task would lead us far from our main topic.

Remark 3.5 A relation similar to (i) in Theorem 3.3 is asserted in [16], p. 2885, for the discrete case and Gauss–Weierstrass type means

$$\widetilde{W}_{\Omega(n),\gamma}(f,x)=\sum_{n=0}^{\infty}e^{-(\Omega(k)/\Omega(n))^{\gamma}}\langle f,R_n^{(\alpha,\beta)}\rangle w_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(x),$$

with Ω varying in a specified class of functions. The proof suggested there is different from the one given here (it does not use the semi-group structure). The main argument in [16] is that some abstract Riesz means are equivalent (as approximation processes) to some Gauss–Weierstrass type means. This kind of equivalence can also be derived by using Corollary 5.4 of [9]. Anyway, the arguments of [16] and the proof given here are related because both use [15], Theorem 3.9, to obtain a uniformly bounded family of multipliers. Apart from this, other topics considered here are not connected with [16].

The arguments used in the proof of Theorem 3.2 can be used to derive similar relations concerning the fractional powers of the Jacobi–Weierstrass operators $\{C_{t,1}\}$.

Recall that $A_1 : D(A_1) \to X$ is the infinitesimal generator of $\{C_{t,1}, t > 0\}$. For $\gamma > 0$, let $D((-A_1)^{\gamma}, X)$ be the family of all $f \in X$, for which there exists an element $(-A_1)^{\gamma}(f) \in X$ satisfying

$$\lim_{t\to 0+}\left\|(-A_1)^{\gamma}(f)-\frac{1}{t^{\gamma}}(I-C_{t,1})^{\gamma}(f)\right\|_X=0, \tag{20}$$

where $(I - C_{t,1})^{\gamma}(f)$ is defined by (8). This induces a map

$$\left(-A^1\right)^{\gamma}:D\left(\left(-A^1\right)^{\gamma},X\right)\to X$$

which is called the fractional power of order γ of $-A_1$.

Proposition 3.6 *If $\gamma > 0$ and $(-A_1)^{\gamma}$ is the fractional power of order γ of $-A_1$, then*

$$D\left((-A_1)^{\gamma},X\right)=\Psi^{\gamma}(X)$$

and, for each $f \in \Psi^{\gamma}(X)$,

$$\Psi^{\gamma}(f)=\lim_{t\to 0+}\frac{1}{t^{\gamma}}(I-C_{t,1})^{\gamma}(f)=\lim_{t\to 0+}\frac{1}{t}\left(f-C_{t,\gamma}(f)\right). \tag{21}$$

Proof If γ is a positive integer or $|a| < 1$, the Taylor expansion gives

$$(1-a)^{\gamma}=\sum_{j=0}^{\infty}(-1)^j\binom{\gamma}{j}a^j.$$

Notice that

$$\left\langle (I-C_{t,1})^{\gamma}(f),R_n^{(\alpha,\beta)}\right\rangle=\sum_{k=0}^{\infty}(-1)^k\binom{\gamma}{k}\left\langle C_{kt,1}(f),R_n^{(\alpha,\beta)}\right\rangle$$

$$= \sum_{k=0}^{\infty} (-1)^k \binom{\gamma}{k} \langle W_{kt}, R_n^{(\alpha,\beta)} \rangle \langle f, R_n^{(\alpha,\beta)} \rangle$$
$$= \langle f, R_n^{(\alpha,\beta)} \rangle \sum_{k=0}^{\infty} (-1)^k \binom{\gamma}{k} \exp(-kt\lambda_n))$$
$$= \langle f, R_n^{(\alpha,\beta)} \rangle \big(1 - \exp(-t\lambda_n)\big)^{\gamma}. \tag{22}$$

Therefore, if $f \in D((-A_1)^{\gamma}, X)$, then

$$\langle (-A_1)^{\gamma}(f), R_n^{(\alpha,\beta)} \rangle = (\lambda_n)^{\gamma} \langle f, R_n^{(\alpha,\beta)} \rangle.$$

Hence $f \in \Psi^{\gamma}(X)$ and $(-A_1)^{\gamma}(f) = \Psi^{\gamma}(f)$.

It is clear that, for each polynomial P, one has $P \in D((-A_1)^{\gamma}, X)$ and

$$(-A_1)^{\gamma}(P) = \Psi^{\gamma}(P).$$

On the other hand, fix an integer $j > \alpha + 1/2$. For $f \in \Psi^{\gamma}(X)$, let $S_m^j(f)$ and $S_m^j(\Psi^{\gamma}(f))$ be the mth Cesàro means of order j of f and $\Psi^{\gamma}(f)$, respectively. From (11), as in the proof of Theorem 3.2, one has $\lim_{m\to\infty} \|S_m^j(f) - f\|_X = 0$ and

$$\lim_{m\to\infty} \big\| (-A^1)^{\gamma} \big(S_m^j(f)\big) - \Psi^{\gamma}(f) \big\|_X = \lim_{m\to\infty} \big\| \Psi^{\gamma} \big(S_m^j(f)\big) - \Psi^{\gamma}(f) \big\|_X$$
$$= \lim_{m\to\infty} \big\| S_m^j \big(\Psi^{\gamma}(f)\big) - \Psi^{\gamma}(f) \big\|_X = 0.$$

It was proved in [19], Theorem 4, that $D((-A_1)^{\gamma}, X)$ is dense in X and $(-A_1)^{\gamma}$ is a closed operator. Hence $f \in D((-A_1)^{\gamma}, X)$ and $(-A_1)^{\gamma}(f) = \Psi^{\gamma}(f)$.

The last equality in (21) was proved in Theorem 3.2, because Ψ^{γ} is the infinitesimal generator of $\{C_{t,\gamma}, t > 0\}$. □

Theorem 3.7 *For fixed $\gamma > 0$, one has*

$$K_{\gamma}\big(f, t^{\gamma}\big) \approx \sup_{0<s\le t} \big\| (I - C_{s,1})^{\gamma}(f) \big\|_X \approx \theta_{\gamma}\big(f, t^{\gamma}\big)$$

for each $f \in X$ and $t > 0$.

Proof From Theorems 3.1 and 3.2 we know that the family $\{C_{t,1}, t \ge 0\}$ is a semi-group of operators of class (C_0) with the infinitesimal generator $A_1 = -\Psi^1$. From Theorem 1.1 of [17], we know that, for all $f \in X$ and $t > 0$,

$$\inf_{g \in D((-A_1)^{\gamma}, X)} \big(\|f - g\|_X + t^{\gamma} \big\| (-A_1)^{\gamma}(g) \big\|_X \big) \approx \sup_{0<s\le t} \big\| (I - C_{s,1})^{\gamma}(f) \big\|_X,$$

where $(-A_1)^{\gamma}$ is given as in (20). But it was verified in Proposition 3.6 that $\Psi^{\gamma}(X) = D((-A_1)^{\gamma}, X)$ and $(-A_1)^{\gamma}(g) = \Psi^{\gamma}(g)$ for each $g \in \Psi^{\gamma}(X)$.

The equivalence with $\theta_{\gamma}(f, t^{\gamma})$ follows from Theorem 3.3. □

Remark 3.8 When γ is an integer, Theorem 3.7 is similar to the Main Theorem in [18], p. 390, but the authors assumed that the operators are positive (plus other conditions).

Remark 3.9 The results of Theorem 3.7 allow us to obtain equivalent relations between fractional powers $(I - C_{s,1})^\gamma$ and some Riesz means as in Theorem 5.1 of [9].

Some result concerning simultaneous approximation can be derived from the ones given above.

Theorem 3.10 *If γ, σ, and t are positive real numbers and $f \in \Psi^\sigma(X)$, then*

$$C_{t,\gamma}(f), (I - C_{t,1})^\gamma(f) \in \Psi^\sigma(X),$$
$$\left\| \Psi^\sigma(f) - \Psi^\sigma\left(C_{t,\gamma}(f)\right)\right\|_X \leq C\theta_\gamma\left(\Psi^\sigma(f), t\right)$$

and

$$\left\| \Psi^\sigma\left((I - C_{t,1})^\gamma(f)\right)\right\|_X \leq C\theta_\gamma\left(\Psi^\sigma(f), t^\gamma\right),$$

where the constant C is independent of f and t.

Proof If $f \in \Psi^\sigma(X)$ and $n \in \mathbb{N}_0$, from (17) we obtain

$$\begin{aligned}\left\langle C_{t,\gamma}\left(\Psi^\sigma(f)\right), R_n^{(\alpha,\beta)}\right\rangle &= \exp\left(-t\lambda_n^\gamma\right)\left\langle \Psi^\sigma(f), R_n^{(\alpha,\beta)}\right\rangle \\ &= \lambda_n^\sigma \exp\left(-t\lambda_n^\gamma\right)\left\langle f, R_n^{(\alpha,\beta)}\right\rangle = \lambda_n^\sigma\left\langle C_{t,\gamma}(f), R_n^{(\alpha,\beta)}\right\rangle\end{aligned}$$

and from (22) one has

$$\begin{aligned}\left\langle (I - C_{t,1})^\gamma\left(\Psi^\sigma(f)\right), R_n^{(\alpha,\beta)}\right\rangle &= \left(1 - \exp(-t\lambda_n)\right)^\gamma\left\langle \Psi^\sigma(f); R_n^{(\alpha,\beta)}\right\rangle \\ &= \lambda_n^\sigma\left(1 - \exp(-t\lambda_n)\right)^\gamma\left\langle f, R_n^{(\alpha,\beta)}\right\rangle = \lambda_n^\sigma\left\langle (I - C_{t,1})^\gamma(f), R_n^{(\alpha,\beta)}\right\rangle.\end{aligned}$$

Therefore $C_{t,\gamma}(f), (I - C_{t,1})^\gamma(f) \in \Psi^\sigma(X)$,

$$\Psi^\sigma\left(C_{t,\gamma}(f)\right) = C_{t,\gamma}\left(\Psi^\sigma(f)\right) \quad \text{and} \quad \Psi^\sigma\left((I - C_{t,1})^\gamma(f)\right) = (I - C_{t,1})^\gamma\left(\Psi^\sigma(f)\right).$$

Now, from Theorem 3.3 one has

$$\left\| \Psi^\sigma(f) - \Psi^\sigma(C_{t,\gamma})\right\|_X = \left\| (I - C_{t,\gamma})\left(\Psi^\sigma(f)\right)\right\|_X \leq C\theta_\gamma\left(\Psi^\sigma(f), t\right),$$

and using Theorem 3.7 we obtain

$$\left\| \Psi^\sigma\left((I - C_{t,1})^\gamma(f)\right)\right\|_X = \left\| (I - C_{t,1})^\gamma\left(\Psi^\sigma(f)\right)\right\|_X \leq C\theta_\gamma\left(\Psi^\sigma(f), t^\gamma\right). \qquad \square$$

4 A Nikolskii–Stechkin type inequality

Theorem 4.1 *For each $r \in \mathbb{N}$, there exists a constant C, depending upon r, such that, for every $\lambda \geq 1$ and for each polynomial $P \in \mathbb{P}_{\xi(\lambda)}$,*

$$\left\| \Psi^r(P)\right\|_X \leq C\lambda^r \sup_{0<h\leq 1/\lambda} \left\| (I - C_{h,1})^r(P)\right\|_X,$$

where

$$\xi(\lambda) = \max\bigl\{k \in \mathbb{N}_0 : k(k+\alpha+\beta+1) < \lambda\bigr\}.$$

Proof In this proof the infinitesimal generator of $\{C_{t,1} : t > 0\}$ is denoted by A.

From the proof of Lemma 1 in [12] we know that, given $r \in \mathbb{N}$, there exists a constant $C_1 = C(r)$ such that, for each $f \in X$ and $t > 0$, there is $g_t \in D(A^{r+1})$ satisfying

$$\|f - g_t\|_X \leq \sup_{0<h\leq t} \bigl\|(I - C_{h,1})^r f\bigr\|_X, \tag{23}$$

$$\bigl\|A^{r+1}(g_t)\bigr\|_X \leq C_1 \frac{1}{t^{r+1}} \sup_{0<h\leq t} \bigl\|(I - C_{h,1})^r f\bigr\|_X \tag{24}$$

and

$$\bigl\|(-A)^r(g_t)\bigr\|_X \leq C_1 \frac{1}{t^r} \sup_{0<h\leq t} \bigl\|(I - C_{h,1})^r f\bigr\|_X. \tag{25}$$

As in [9], for $\lambda > 0$ and $f \in X$, consider the best approximation

$$E_\lambda(f) = \inf\bigl\{\|f - P\|_X : P \in \mathbb{P}_{\xi(\lambda)}\bigr\}.$$

It was proved there (Theorem 6.1) that there exists a constant $C_2 = C(r, \alpha, \beta)$ such that, for $\lambda > 0$ and $f \in X$,

$$E_\lambda(f) \leq C_2 K_{r+1}\bigl(f, \lambda^{-r-1}\bigr), \tag{26}$$

and (Theorem 3.2) for each $Q \in \mathbb{P}_{\xi(\lambda)}$,

$$\bigl\|\Psi^r(Q)\bigr\|_X \leq C_2 \lambda^r \|Q\|_X. \tag{27}$$

Now, fix $\lambda > 0$ and $P \in \mathbb{P}_{\xi(\lambda)}$. Let $g_t \in D(A^{r+1}) = \Psi^{r+1}(X)$ (see (18)) be given as (23)–(25) with $t = 1/\lambda$ and $f = P$.

For $\varepsilon > 0$ and $k \in \mathbb{N}_0$, choose

$$q(g_t, k) \in \mathbb{P}_{\xi(2^k\lambda)} \tag{28}$$

such that

$$\bigl\|g_t - q(g_t, k)\bigr\|_X \leq (1+\varepsilon) E_{2^k\lambda}(g_t). \tag{29}$$

From (26), (18), and (24) we know that

$$\begin{aligned}\bigl\|g_t - q(g_t, k)\bigr\|_X &\leq C_2(1+\varepsilon) K_{r+1}\bigl(g_t, \bigl(2^k\lambda\bigr)^{-r-1}\bigr) \\ &\leq \frac{C_2(1+\varepsilon)}{(2^k\lambda)^{r+1}} \bigl\|\Psi^{r+1}(g_t)\bigr\|_X \\ &= \frac{C_2(1+\varepsilon)}{(2^k\lambda)^{r+1}} \bigl\|A^{r+1}(g_t)\bigr\|_X\end{aligned}$$

$$\le \frac{C_1C_2(1+\varepsilon)}{(2^k\lambda)^{r+1}}\frac{1}{t^{r+1}}\sup_{0<h\le t}\left\|(I-C_{h,1})^rP\right\|_X$$
$$=\frac{C_1C_2(1+\varepsilon)}{(2^k)^{r+1}}\sup_{0<h\le 1/\lambda}\left\|(I-C_{h,1})^rP\right\|_X.$$

On the other hand, from the identity

$$q(g_t,0)-g_t=\sum_{k=0}^{\infty}\bigl(q(g_t,k)-q(g_t,k+1)\bigr),$$

(28), (27), (29), and (26), one has

$$\begin{aligned}\left\|\Psi^r\bigl(q(g_t,0)-g_t\bigr)\right\|_X &\le \sum_{k=0}^{\infty}\left\|\Psi^r\bigl(q(g_t,k)-q(g_t,k+1)\bigr)\right\|_X\\
&\le C_2\sum_{k=0}^{\infty}\bigl(2^{k+1}\lambda\bigr)^r\left\|q(g_t,k)-q(g_t,k+1)\right\|_X\\
&\le C_2\sum_{k=0}^{\infty}\bigl(2^{k+1}\lambda\bigr)^r\bigl(\left\|q(g_t,k)-g_t\right\|_X+\left\|g_t-q(g_t,k+1)\right\|_X\bigr)\\
&\le 2C_1C_2^2(1+\varepsilon)\sup_{0<h\le 1\lambda}\left\|(I-C_{h,1})^rP\right\|_X\sum_{k=0}^{\infty}\bigl(2^{k+1}\lambda\bigr)^r\frac{1}{(2^k)^{r+1}}\\
&=2^{r+1}C_1C_2^2(1+\varepsilon)\lambda^r\sup_{0<h\le 1\lambda}\left\|(I-C_{h,1})^rP\right\|_X\sum_{k=0}^{\infty}\frac{1}{2^k}\\
&=C_3(1+\varepsilon)\lambda^r\sup_{0<h\le 1/\lambda}\left\|(I-C_{h,1})^rP\right\|_X.\end{aligned}$$

We also need the inequality (see (18) and (25))

$$\begin{aligned}\left\|\Psi^r(g_t)\right\|_X=\left\|A^r(g_t)\right\|_X &\le C_1\frac{1}{t^r}\sup_{0<h\le t}\left\|(I-C_{h,1})^rP\right\|_X\\
&=C_1\lambda^r\sup_{0<h\le 1/\lambda}\left\|(I-C_{h,1})^rP\right\|_X.\end{aligned}$$

From the inequalities given above, for $P\in\mathbb{P}_{\xi(\lambda)}$, we obtain

$$\begin{aligned}\left\|\Psi^r(P)\right\|_X &\le \left\|\Psi^r\bigl(P-q(g_t,0)\bigr)\right\|_X+\left\|\Psi^r\bigl(q(g_t,0)\bigr)\right\|_X\\
&\le C_2\lambda^r\left\|P-q(g_t,0)\right\|_X+\left\|\Psi^r(g_t)\right\|_X+\left\|\Psi^r\bigl(g_t-q(g_t,0)\bigr)\right\|_X\\
&\le C_1\lambda^r(\|P-g_t\|_X+\left\|g_t-q(g_t,0)\right\|_{X_{\alpha,\beta}}+C_4\lambda^r\sup_{0<h\le 1/\lambda}\left\|(I-C_{h,1})^rP\right\|_X\\
&\le C_5\lambda^r\sup_{0<h\le 1/\lambda}\left\|(I-C_{h,1})^rP\right\|_X.\end{aligned}$$

□

Remark 4.2 The problem of obtaining a Nikolskii–Stechkin inequality for fractional derivatives is open.

Acknowledgements
The authors would like to thank the referees for their careful reading of the manuscript and for their helpful comments.

Funding
The coauthors Adell, J.A. and Quesada, J.M. are partially supported by Research Project MTM2015-67006-P.

Competing interests
The authors declare that they have no competing interests.

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The authors read and approved the final manuscript.

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Received: 16 February 2018 Accepted: 20 June 2018 Published online: 28 June 2018

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