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The inverses of tails of the Riemann zeta function

Donggyun Kim[1] and Kyunghwan Song[1*]

*Correspondence:
heroesof@korea.ac.kr
[1]Department of Mathematics, Korea University, Seoul, Republic of Korea

Abstract

We present some bounds of the inverses of tails of the Riemann zeta function on $0 < s < 1$ and compute the integer parts of the inverses of tails of the Riemann zeta function for $s = \frac{1}{2}, \frac{1}{3}$, and $\frac{1}{4}$.

MSC: Riemann zeta function; Tails of Riemann zeta function; Inverses of tails of the Riemann zeta function

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1 Introduction

The Riemann zeta function $\zeta(s)$ in the real variable s was introduced by Euler [2] in connection with questions about the distribution of prime numbers. Later Riemann [6] derived deeper results about a dual correspondence between the distribution of prime numbers and the complex zeros of $\zeta(s)$ in the complex variable s. In these developments, he asserted that all the non-trivial zeros of $\zeta(s)$ are on the line $\mathrm{Re}(s) = \frac{1}{2}$, and this has been one of the most important unsolved problems in mathematics, called the Riemann hypothesis. A vast amount of research on calculation of $\zeta(s)$ on the line $\mathrm{Re}(s) = \frac{1}{2}$, which is called the critical line, and on the strip $0 < \mathrm{Re}(s) < 1$, which is called the critical strip, has been conducted using various methods [1].

The *Riemann zeta function* and a *tail of the Riemann zeta function from n* for an integer $n \geq 1$ are defined, respectively, by: for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \quad \text{and} \quad \zeta_n(s) = \sum_{k=n}^{\infty} \frac{1}{k^s},$$

and for $0 < \mathrm{Re}(s) < 1$,

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^s} \quad \text{and} \quad \zeta_n(s) = \frac{1}{1-2^{1-s}} \sum_{k=n}^{\infty} \frac{(-1)^{k+1}}{k^s}.$$

To understand the values of $\zeta(s)$, it would be helpful to understand the values of tails of $\zeta(s)$, for example, the integer parts of their inverses $[\zeta_n(s)^{-1}]$, where $[x]$ denotes the greatest integer that is less than or equal to x.

Some values of $[\zeta_n(s)^{-1}]$ for small positive integers s have become known recently. Xin [7] showed that for $s = 2$ and 3,

$$\left[\zeta_n(2)^{-1}\right] = n - 1 \quad \text{and} \quad \left[\zeta_n(3)^{-1}\right] = 2n(n-1).$$

For $s = 4$, Xin and Xiaoxue [8] showed that

$$\left[\zeta_n(4)^{-1}\right] = 3n^3 - 5n^2 + 4n - 1 + \left[\frac{(2n+1)(n-1)}{4}\right]$$

for any integer $n \geq 2$, and Xu [9] showed that for $s = 5$,

$$\left[\zeta_n(5)^{-1}\right] = 4n^4 - 8n^3 + 9n^2 - 5n + \left[\frac{(n+1)(n-2)}{3}\right]$$

for any integer $n \geq 4$. Hwang and Song [3] provided an alternative proof of the case when $s = 5$ and a formula when $s = 6$ as follows. For an integer n, write n_{48} for the remainder when n is divided by 48, then

$$\begin{aligned}&\left[\zeta_n(6)^{-1}\right]\\ &\quad = \begin{cases} 5n^5 - \frac{25}{2}n^4 + \frac{75}{4}n^3 - \frac{125}{8}n^2 + \frac{185}{48}n - \frac{5n_{48}}{48} - [\frac{35-5n_{48}}{48}], & \text{if } n \text{ is even},\\ 5n^5 - \frac{25}{2}n^4 + \frac{75}{4}n^3 - \frac{125}{8}n^2 + \frac{185}{48}n - \frac{5n_{48}+18}{48} - [\frac{17-5n_{48}}{48}], & \text{if } n \text{ is odd}\end{cases}\end{aligned}$$

for any integer $n \geq 829$. For the integer s greater than 6, no such a formula is known.

There are other interesting results related to this theme such as bounds of $\zeta(3)$ in greater precision in [4] and [5].

We study the inverses of tails of the Riemann zeta function $\zeta_n(s)^{-1}$ for s on the critical strip $0 < s < 1$. The following notation is needed to explain our results.

Definition 1 For any positive integer n and real number s with $0 < s < 1$, we define

$$A_{n,s} = \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) + \left(\frac{1}{(n+2)^s} - \frac{1}{(n+3)^s}\right) + \cdots$$

and

$$B_{n,s} = \left(-\frac{1}{n^s} + \frac{1}{(n+1)^s}\right) + \left(-\frac{1}{(n+2)^s} + \frac{1}{(n+3)^s}\right) + \cdots.$$

Now the tail of the Riemann zeta function for $0 < s < 1$ can be written as follows:

$$\zeta_n(s) = \begin{cases} -\frac{1}{1-2^{1-s}}A_{n,s}, & \text{if } n \text{ is even},\\ -\frac{1}{1-2^{1-s}}B_{n,s}, & \text{if } n \text{ is odd}.\end{cases} \tag{1}$$

In this paper, we present the bounds of $A_{n,s}^{-1}$ and $B_{n,s}^{-1}$, hence the bounds of the inverses of tails of the Riemann zeta function $\zeta_n(s)^{-1}$ for $0 < s < 1$ in Sect. 2.1, and compute the values $[A_{n,s}^{-1}]$ and $[B_{n,s}^{-1}]$, hence the values of the inverses of tails of the Riemann zeta function $[\frac{1}{1-2^{1-s}}\zeta_n(s)^{-1}]$ for $s = \frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ in Sect. 2.2.

2 Main results

2.1 The bounds of the inverses of $\zeta_n(s)$ for $0 < s < 1$

In this section, we present the bounds of $A_{n,s}^{-1}$ and $B_{n,s}^{-1}$ in Definition 1, hence the bounds of the inverses of tails of the Riemann zeta function $\zeta_n(s)^{-1}$ for $0 < s < 1$.

Proposition 1 *Let s be a real number with* $0 < s < 1$. *Then, for any positive even number n,*

$$2(n-1)^s < A_{n,s}^{-1} < 2n^s,$$

and for any positive odd number n,

$$-2n^s < B_{n,s}^{-1} < -2(n-1)^s.$$

Proof Let n be a positive even number. For every positive integer k, it is easy to see that

$$\begin{aligned}
&\left(\frac{1}{(n+1+2k)^s} - \frac{1}{(n+2+2k)^s}\right) \\
&\quad < \left(\frac{1}{(n+2k)^s} - \frac{1}{(n+1+2k)^s}\right) \\
&\quad < \left(\frac{1}{(n-1+2k)^s} - \frac{1}{(n+2k)^s}\right).
\end{aligned}$$

The summations of each term over k give

$$A_{n+1,s} < A_{n,s} < A_{n-1,s}$$

and

$$\frac{1}{2}(A_{n+1,s} + A_{n,s}) < A_{n,s} < \frac{1}{2}(A_{n-1,s} + A_{n,s}).$$

Therefore, we have

$$\frac{1}{2n^s} < A_{n,s} < \frac{1}{2(n-1)^s},$$

which gives the first statement.

The second statement can be shown similarly. □

Since every proof of the case when n is an odd number is analogous to that of the case when n is an even number, we omit all the proofs of the odd number cases in this paper.

Now we find tighter bounds for $A_{n,s}^{-1}$ and $B_{n,s}^{-1}$.

Proposition 2 *Let s be a real number with* $0 < s < 1$. *Then, for any positive even number n,*

$$2\left(n - \frac{1}{2}\right)^s < A_{n,s}^{-1},$$

and for any positive odd number n,

$$B_{n,s}^{-1} < -2\left(n - \frac{1}{2}\right)^s.$$

Proof Let n be a positive even number. We will show that

$$A_{n,s} < \frac{1}{2(n-\frac{1}{2})^s}.$$

Rewriting each of the both sides as a series

$$A_{n,s} = \sum_{k=\frac{n}{2}}^{\infty}\left(\frac{1}{(2k)^s} - \frac{1}{(2k+1)^s}\right)$$

and

$$\frac{1}{2(n-\frac{1}{2})^s} = \sum_{k=\frac{n}{2}}^{\infty}\left(\frac{1}{2(2k-\frac{1}{2})^s} - \frac{1}{2(2k+\frac{3}{2})^s}\right),$$

we will show that for any positive integer k,

$$\frac{1}{(2k)^s} - \frac{1}{(2k+1)^s} < \frac{1}{2(2k-\frac{1}{2})^s} - \frac{1}{2(2k+\frac{3}{2})^s}.$$

For this, we let

$$f(x) = \left(\frac{1}{2(2x-\frac{1}{2})^s} - \frac{1}{2(2x+\frac{3}{2})^s}\right) - \left(\frac{1}{(2x)^s} - \frac{1}{(2x+1)^s}\right)$$

and will show that $f(x)$ is positive for $x \geq 1$ and $0 < s < 1$. With

$$g(x) = \frac{1}{2(2x-\frac{1}{2})^s} + \frac{1}{2(2x+\frac{1}{2})^s} - \frac{1}{(2x)^s},$$

we have $f(x) = g(x) - g(x+\frac{1}{2})$. Consider the derivative of $g(x)$:

$$g'(x) = -2s\left(\frac{1}{2(2x-\frac{1}{2})^{s+1}} + \frac{1}{2(2x+\frac{1}{2})^{s+1}} - \frac{1}{(2x)^{s+1}}\right).$$

Since the function $\frac{1}{x^{s+1}}$ is convex, we obtain that

$$\frac{1}{2(2x-\frac{1}{2})^{s+1}} + \frac{1}{2(2x+\frac{1}{2})^{s+1}} - \frac{1}{(2x)^{s+1}} \geq 0,$$

and therefore $g'(x)$ is negative, that is, $g(x)$ is decreasing. We conclude that $f(x)$ is positive, which gives the statement. □

Proposition 3 *Let s be a real number with* $0 < s < 1$. *Then, for any positive even number n,*

$$A_{n,s}^{-1} < 2\left(n - \frac{1}{4}\right)^s,$$

and for any positive odd number n,

$$-2\left(n - \frac{1}{4}\right)^s < B_{n,s}^{-1}.$$

Proof Let n be a positive even number. We will show that

$$\frac{1}{2(n-\frac{1}{4})^s} < A_{n,s}.$$

Rewriting each of the both sides as a series

$$A_{n,s} = \sum_{k=\frac{n}{2}}^{\infty}\left(\frac{1}{(2k)^s} - \frac{1}{(2k+1)^s}\right)$$

and

$$\frac{1}{2(n-\frac{1}{4})^s} = \sum_{k=\frac{n}{2}}^{\infty}\left(\frac{1}{2(2k-\frac{1}{4})^s} - \frac{1}{2(2k+\frac{7}{4})^s}\right),$$

we need to show that for any positive integer k,

$$\frac{1}{2(2k-\frac{1}{4})^s} - \frac{1}{2(2k+\frac{7}{4})^s} < \frac{1}{(2k)^s} - \frac{1}{(2k+1)^s}.$$

For this, we let

$$f(x) = \left(\frac{1}{(2x)^s} - \frac{1}{(2x+1)^s}\right) - \left(\frac{1}{2(2x-\frac{1}{4})^s} - \frac{1}{2(2x+\frac{7}{4})^s}\right).$$

We check that $f(1) > 0$ and now we will show that $f(x)$ is positive for $x \geq 2$ and $0 < s < 1$. With

$$g(x) = \frac{1}{(2x)^s} - \left(\frac{1}{2(2x-\frac{1}{4})^s} + \frac{1}{2(2x+\frac{3}{4})^s}\right),$$

we have $f(x) = g(x) - g(x+\frac{1}{2})$, so we only need to show that $g(x)$ is decreasing. Consider the derivative of $g(x)$:

$$\begin{aligned} g'(x) &= s\left(-\frac{2}{(2x)^{s+1}} + \left(\frac{1}{(2x-\frac{1}{4})^{s+1}} + \frac{1}{(2x+\frac{3}{4})^{s+1}}\right)\right) \\ &= s\left(\left(\frac{1}{(2x-\frac{1}{4})^{s+1}} - \frac{1}{(2x)^{s+1}}\right) - \left(\frac{1}{(2x)^{s+1}} - \frac{1}{(2x+\frac{3}{4})^{s+1}}\right)\right). \end{aligned}$$

Since the function $\frac{1}{x^{s+1}}$ is decreasing and convex, by comparing slopes at $(2x - \frac{1}{4})$ and $(2x + \frac{3}{4})$, we obtain

$$\frac{1}{(2x-\frac{1}{4})^{s+1}} - \frac{1}{(2x)^{s+1}} < \frac{1}{4}(s+1)\frac{1}{(2x-\frac{1}{4})^{s+2}}$$

and

$$\frac{1}{(2x)^{s+1}} - \frac{1}{(2x+\frac{3}{4})^{s+1}} > \frac{1}{4}(s+1)\frac{3}{(2x+\frac{3}{4})^{s+2}}.$$

Therefore,

$$g'(x) < \frac{1}{4}s(s+1)\left(\frac{1}{(2x-\frac{1}{4})^{s+2}} - \frac{3}{(2x+\frac{3}{4})^{s+2}}\right).$$

Consider $h(x,s) := \frac{1}{3}(\frac{2x+3/4}{2x-1/4})^{s+2}$, which is the ratio of two terms on the right-hand side of the above expression. We check that $h(x,s) < 1$ for $x \geq 2$ and $0 < s < 1$. Since $h(2,1) = 6859/10{,}125$ and $\lim_{x\to\infty} h(x,s) = \frac{1}{3}$ for $0 < s < 1$, we obtain that $g'(x)$ is negative and, therefore, $g(x)$ is decreasing, which gives the statement. □

We combine the results of Proposition 2 and Proposition 3.

Theorem 1 *Let s be a real number with* $0 < s < 1$. *Then, for any positive even number n,*

$$2\left(n-\frac{1}{2}\right)^s < A_{n,s}^{-1} < 2\left(n-\frac{1}{4}\right)^s,$$

and for any positive odd number n,

$$-2\left(n-\frac{1}{4}\right)^s < B_{n,s}^{-1} < -2\left(n-\frac{1}{2}\right)^s.$$

We express these bounds in terms of $\zeta_n(s)$ using expression (1).

Corollary 1 *Let s be a real number with* $0 < s < 1$. *Then, for any positive even number n,*

$$2\left(1-2^{1-s}\right)\left(n-\frac{1}{4}\right)^s < \zeta_n(s)^{-1} < 2\left(1-2^{1-s}\right)\left(n-\frac{1}{2}\right)^s,$$

and for any positive odd number n,

$$-2\left(1-2^{1-s}\right)\left(n-\frac{1}{2}\right)^s < \zeta_n(s)^{-1} < -2\left(1-2^{1-s}\right)\left(n-\frac{1}{4}\right)^s.$$

Furthermore, we have tighter bounds of $A_{n,s}^{-1}$ and $B_{n,s}^{-1}$ for a sufficiently large number n.

Theorem 2 *For any positive number* ϵ *and any real number s with* $0 < s < 1$,

$$2\left(n-\frac{1}{2}\right)^s < A_{n,s}^{-1} < 2\left(n-\frac{1}{2}+\epsilon\right)^s$$

for a sufficiently large even number n and

$$-2\left(n-\frac{1}{2}+\epsilon\right)^s < B_{n,s}^{-1} < -2\left(n-\frac{1}{2}\right)^s$$

for a sufficiently large odd number n.

Proof From Theorem 1, it suffices to show that for a sufficiently large even number n,

$$\frac{1}{2(n-\frac{1}{2}+\epsilon)^s} < A_{n,s}.$$

Rewriting each of the both sides as a series

$$A_{n,s} = \sum_{k=\frac{n}{2}}^{\infty}\left(\frac{1}{(2k)^s}-\frac{1}{(2k+1)^s}\right)$$

and

$$\frac{1}{2(n-\frac{1}{2}+\epsilon)^s} = \sum_{k=\frac{n}{2}}^{\infty}\left(\frac{1}{2(2k-\frac{1}{2}+\epsilon)^s}-\frac{1}{2(2k+\frac{3}{2}+\epsilon)^s}\right),$$

we need to show that for a sufficiently large even number n and every integer $k \geq \frac{n}{2}$,

$$\frac{1}{2(2k-\frac{1}{2}+\epsilon)^s}-\frac{1}{2(2k+\frac{3}{2}+\epsilon)^s} < \frac{1}{(2k)^s}-\frac{1}{(2k+1)^s}.$$

For this, let

$$f(x) = \left(\frac{1}{(2x)^s}-\frac{1}{(2x+1)^s}\right)-\left(\frac{1}{2(2x-\frac{1}{2}+\epsilon)^s}-\frac{1}{2(2x+\frac{3}{2}+\epsilon)^s}\right),$$

and we will show that $f(x)$ is positive for $x \geq x_0$, where x_0 is a sufficiently large number. With

$$g(x) = \frac{1}{(2x)^s}-\left(\frac{1}{2(2x-\frac{1}{2}+\epsilon)^s}+\frac{1}{2(2x+\frac{1}{2}+\epsilon)^s}\right),$$

we have that $f(x) = g(x)-g(x+\frac{1}{2})$, so we only need to show that $g(x)$ is decreasing. Consider the derivative of $g(x)$:

$$\begin{aligned} g'(x) &= s\left(-\frac{2}{(2x)^{s+1}}+\frac{1}{(2x-\frac{1}{2}+\epsilon)^{s+1}}+\frac{1}{(2x+\frac{1}{2}+\epsilon)^{s+1}}\right) \\ &= s\left(\left(\frac{1}{(2x-\frac{1}{2}+\epsilon)^{s+1}}-\frac{1}{(2x)^{s+1}}\right)-\left(\frac{1}{(2x)^{s+1}}-\frac{1}{(2x+\frac{1}{2}+\epsilon)^{s+1}}\right)\right). \end{aligned}$$

Since $\frac{1}{x^{s+1}}$ is decreasing and convex, by comparing slopes at $(2x-\frac{1}{2}+\epsilon)$ and $(2x+\frac{1}{2}+\epsilon)$, we obtain

$$\frac{1}{(2x-\frac{1}{2}+\epsilon)^{s+1}}-\frac{1}{(2x)^{s+1}} < (s+1)\frac{\frac{1}{2}-\epsilon}{(2x-\frac{1}{2}+\epsilon)^{s+2}}$$

and

$$\frac{1}{(2x)^{s+1}} - \frac{1}{(2x+\frac{1}{2}+\epsilon)^{s+1}} > (s+1)\frac{\frac{1}{2}+\epsilon}{(2x+\frac{1}{2}+\epsilon)^{s+2}}.$$

Therefore

$$g'(x) < s(s+1)\left(\frac{\frac{1}{2}-\epsilon}{(2x-\frac{1}{2}+\epsilon)^{s+2}} - \frac{\frac{1}{2}+\epsilon}{(2x+\frac{1}{2}+\epsilon)^{s+2}}\right).$$

Consider $h(x) := \frac{\frac{1}{2}-\epsilon}{\frac{1}{2}+\epsilon}(\frac{2x+\frac{1}{2}+\epsilon}{2x-\frac{1}{2}+\epsilon})^{s+2}$, which is the ratio of two terms on the right-hand side of the above expression. We need to show that $h(x) < 1$ for every $x > x_0$, where x_0 is a sufficiently large number. We check that

$$h(x) < 1 \quad \Longleftrightarrow \quad \frac{2x+\frac{1}{2}+\epsilon}{2x-\frac{1}{2}+\epsilon} < \left(\frac{\frac{1}{2}+\epsilon}{\frac{1}{2}-\epsilon}\right)^{\frac{1}{s+2}}.$$

For any $\epsilon > 0$ and $0 < s < 1$, we have that $1 < (\frac{\frac{1}{2}+\epsilon}{\frac{1}{2}-\epsilon})^{1/(s+2)}$ and $\frac{2x+\frac{1}{2}+\epsilon}{2x-\frac{1}{2}+\epsilon}$ is larger than 1, decreasing and converges to 1 as x goes to infinity, so there is x_0 such that, for every $x > x_0$, $h(x) < 1$. Therefore the proof is complete. □

We express these bounds in terms of $\zeta_n(s)$ using expression (1).

Corollary 2 *For any positive number ϵ and any real number s with $0 < s < 1$, we have*

$$2\bigl(1-2^{1-s}\bigr)\left(n-\frac{1}{2}+\epsilon\right)^s < \zeta_n(s)^{-1} < 2\bigl(1-2^{1-s}\bigr)\left(n-\frac{1}{2}\right)^s,$$

for a sufficiently large even number n and

$$-2\bigl(1-2^{1-s}\bigr)\left(n-\frac{1}{2}\right)^s < \zeta_n(s)^{-1} < -2\bigl(1-2^{1-s}\bigr)\left(n-\frac{1}{2}+\epsilon\right)^s$$

for a sufficiently large odd number n.

2.2 The value of the inverse of $\zeta_n(s)$ for $s = \frac{1}{2}, \frac{1}{3}$, and $\frac{1}{4}$

We study firstly the value of the inverse of $\zeta_n(\frac{1}{2})$, where $\zeta_n(\frac{1}{2})$ is the tail of the Riemann zeta function from n at $s = \frac{1}{2}$.

Theorem 3 *For any positive even number n,*

$$\bigl[A_{n,1/2}^{-1}\bigr] = \left[2\left(n-\frac{1}{2}\right)^{1/2}\right],$$

and for any positive odd number n,

$$\bigl[B_{n,1/2}^{-1}\bigr] = \left[-2\left(n-\frac{1}{2}\right)^{1/2}\right].$$

Proof Let n be a positive even number. By Theorem 1, we have that

$$2\left(n-\frac{1}{2}\right)^{1/2} < A_{n,1/2}^{-1} < 2\left(n-\frac{1}{4}\right)^{1/2}.$$

Note that $2(n-\frac{1}{4})^{1/2} - 2(n-\frac{1}{2})^{1/2} < 1$ for $n \geq 2$, and it implies that there is at most one integer in the open interval from $2(n-\frac{1}{2})^{1/2}$ to $2(n-\frac{1}{4})^{1/2}$. Suppose that there is an integer h in the open interval, i.e.,

$$2\left(n-\frac{1}{2}\right)^{1/2} < h < 2\left(n-\frac{1}{4}\right)^{1/2} \quad \text{or} \quad 4n-2 < h^2 < 4n-1.$$

There is, however, no integer in the open interval from $4n-2$ to $4n-1$, therefore such an integer h does not exist. This gives the statement. □

We express this result in terms of $\zeta_n(s)$ using expression (1).

Corollary 3 *For any positive integer n,*

$$\left[\frac{1}{1-2^{1/2}}\zeta_n\left(\frac{1}{2}\right)^{-1}\right] = \left[(-1)^{n+1}2\left(n-\frac{1}{2}\right)^{1/2}\right].$$

We study secondly the value of the inverse of $\zeta_n(\frac{1}{3})$, where $\zeta_n(\frac{1}{3})$ is the tail of the Riemann zeta function from n at $s=\frac{1}{3}$.

Theorem 4 *For any positive even number n,*

$$\left[A_{n,1/3}^{-1}\right] = \left[2\left(n-\frac{1}{2}\right)^{1/3}\right],$$

and for any positive odd number n,

$$\left[B_{n,1/3}^{-1}\right] = \left[-2\left(n-\frac{1}{2}\right)^{1/3}\right].$$

Proof Let n be a positive even number. By Theorem 1, we have that

$$2\left(n-\frac{1}{2}\right)^{1/3} < A_{n,1/3}^{-1} < 2\left(n-\frac{1}{4}\right)^{1/3}.$$

Note that $2(n-\frac{1}{4})^{1/3} - 2(n-\frac{1}{2})^{1/3} < 1$ for $n \geq 2$, and it implies that there is at most one integer in the open interval from $2(n-\frac{1}{2})^{1/3}$ to $2(n-\frac{1}{4})^{1/3}$. Suppose that there is an integer h in the open interval, i.e.,

$$2\left(n-\frac{1}{2}\right)^{1/3} < h < 2\left(n-\frac{1}{4}\right)^{1/3} \quad \text{or} \quad 8n-4 < h^3 < 8n-2.$$

This shows that the integer h is of the form $h = 2(n-\frac{3}{8})^{1/3}$. If we show $A_{n,1/3}^{-1} < 2(n-\frac{3}{8})^{1/3}$ or, equivalently, $\frac{1}{2(n-\frac{3}{8})^{1/3}} < A_{n,1/3}$, then our proof will be done. Let us rewrite

$$A_{n,1/3} = \sum_{k=\frac{n}{2}}^{\infty}\left(\frac{1}{(2k)^{1/3}} - \frac{1}{(2k+1)^{1/3}}\right)$$

and

$$\frac{1}{2(n-\frac{3}{8})^{1/3}} = \sum_{k=\frac{n}{2}}^{\infty}\left(\frac{1}{2(2k-\frac{3}{8})^{1/3}} - \frac{1}{2(2k+\frac{13}{8})^{1/3}}\right).$$

Now it suffices to show that for any positive integer k,

$$\frac{1}{2(2k-\frac{3}{8})^{1/3}} - \frac{1}{2(2k+\frac{13}{8})^{1/3}} < \frac{1}{(2k)^{1/3}} - \frac{1}{2(2k+1)^{1/3}}.$$

For this, we let

$$f(x) = \left(\frac{1}{(2x)^{1/3}} - \frac{1}{(2x+1)^{1/3}}\right) - \left(\frac{1}{2(2x-\frac{3}{8})^{1/3}} - \frac{1}{2(2x+\frac{13}{8})^{1/3}}\right),$$

and we will show that $f(x)$ is positive for any positive integer x.

We check that $f(1) = 0.00053\cdots$ and $f(2) = 0.00081\cdots$, so it suffices to show $f(x) > 0$ for $x \geq 3$. With

$$g(x) = \frac{1}{(2x)^{1/3}} - \left(\frac{1}{2(2x-\frac{3}{8})^{1/3}} + \frac{1}{2(2x+\frac{5}{8})^{1/3}}\right),$$

we have that $f(x) = g(x) - g(x+\frac{1}{2})$, so we only need to show that $g(x)$ is decreasing for $x \geq 3$. Consider the derivative of $g(x)$:

$$\begin{aligned} g'(x) &= \frac{1}{3}\left(-\frac{2}{(2x)^{4/3}} + \frac{1}{(2x-\frac{3}{8})^{4/3}} + \frac{1}{(2x+\frac{5}{8})^{4/3}}\right) \\ &= \frac{1}{3}\left(\left(\frac{1}{(2x-\frac{3}{8})^{4/3}} - \frac{1}{(2x)^{4/3}}\right) - \left(\frac{1}{(2x)^{4/3}} - \frac{1}{(2x+\frac{5}{8})^{4/3}}\right)\right). \end{aligned}$$

Since $\frac{1}{x^{4/3}}$ is decreasing and convex, by comparing slopes at $(2x-\frac{3}{8})$ and $(2x+\frac{5}{8})$, we obtain

$$\frac{1}{(2x-\frac{3}{8})^{4/3}} - \frac{1}{(2x)^{4/3}} < 2\cdot\frac{3}{16}\cdot\frac{4}{3}\cdot\frac{1}{(2x-\frac{3}{8})^{7/3}}$$

and

$$\frac{1}{(2x)^{4/3}} - \frac{1}{(2x+\frac{5}{8})^{4/3}} > 2\cdot\frac{5}{16}\cdot\frac{4}{3}\cdot\frac{1}{(2x+\frac{5}{8})^{7/3}}.$$

Therefore

$$g'(x) < \frac{1}{18}\left(\frac{3}{(2x-\frac{3}{8})^{7/3}} - \frac{5}{(2x+\frac{5}{8})^{7/3}}\right).$$

Consider $h(x) := \frac{3}{5}(\frac{2x+5/8}{2x-3/8})^{7/3}$, which is the ratio of two terms of the right-hand side of the above expression. We check that $h(x) < 1$ for $x \geq 3$ because $h(3) = 0.87\cdots$ and $\lim_{x\to\infty} h(x) = \frac{3}{5}$ and $h'(x) < 0$ for $x \geq 3$. Hence we obtain that $g'(x)$ is negative and so $g(x)$ is decreasing for $x \geq 3$, which proves the statement. □

We express this result in terms of $\zeta_n(s)$ using expression (1).

Corollary 4 *For any positive integer n,*

$$\left[\frac{1}{1-2^{2/3}}\zeta_n\left(\frac{1}{3}\right)^{-1}\right] = \left[(-1)^{n+1}2\left(n-\frac{1}{2}\right)^{1/3}\right].$$

We study lastly the value of the inverse of $\zeta_n(\frac{1}{4})$, which is the tail of the Riemann zeta function from n at $s = \frac{1}{4}$.

Theorem 5 *For any positive even number n,*

$$\left[A_{n,1/4}^{-1}\right] = \left[2\left(n-\frac{1}{2}\right)^{1/4}\right],$$

and for any positive odd number n,

$$\left[B_{n,1/4}^{-1}\right] = \left[-2\left(n-\frac{1}{2}\right)^{1/4}\right].$$

Proof Let n be a positive even number. By Theorem 1, we have that

$$2\left(n-\frac{1}{2}\right)^{1/4} < A_{n,1/4}^{-1} < 2\left(n-\frac{1}{4}\right)^{1/4}.$$

Note that $2(n-\frac{1}{4})^{1/4} - 2(n-\frac{1}{2})^{1/4} < 1$ for $n \geq 2$, and it implies that there is at most one integer in the open interval from $2(n-\frac{1}{2})^{1/4}$ to $2(n-\frac{1}{4})^{1/4}$. Suppose that there is an integer h in the open interval, i.e.,

$$2\left(n-\frac{1}{2}\right)^{1/4} < h < 2\left(n-\frac{1}{4}\right)^{1/4} \quad \text{or} \quad 16n-8 < h^4 < 16n-4.$$

This shows that the integer h^4 is one of the form $16n-7$, $16n-6$, or $16n-5$. For any integer h, however, $h^4 \equiv 0$ or $1 \pmod{16}$, hence such an integer h does not exist. Therefore this gives the statement. □

We express this result in terms of $\zeta_n(s)$ using expression (1).

Corollary 5 *For any positive integer n,*

$$\left[\frac{1}{1-2^{3/4}}\zeta_n\left(\frac{1}{4}\right)^{-1}\right] = \left[(-1)^{n+1}2\left(n-\frac{1}{2}\right)^{1/4}\right].$$

We express the results of Theorems 3, 4, and 5 in a single statement.

Theorem 6 *For $s = \frac{1}{2}, \frac{1}{3}$, or $\frac{1}{4}$, and for any positive even number n,*

$$[A_{n,s}^{-1}] = \left[2\left(n - \frac{1}{2}\right)^s\right],$$

and for any positive odd number n,

$$[B_{n,s}^{-1}] = \left[-2\left(n - \frac{1}{2}\right)^s\right].$$

We express the results of Corollaries 3, 4, and 5 in a single statement.

Corollary 6 *For any positive integer n and $s = \frac{1}{2}, \frac{1}{3}$, or $\frac{1}{4}$,*

$$\left[\frac{1}{1 - 2^{1-s}}\zeta_n(s)^{-1}\right] = \left[(-1)^{n+1}2\left(n - \frac{1}{2}\right)^s\right].$$

3 Conclusion

In this paper, we have presented the bounds of $A_{n,s}^{-1}$ and $B_{n,s}^{-1}$, hence the bounds of the inverses of tails of the Riemann zeta function $\zeta_n(s)^{-1}$ for $0 < s < 1$, and computed the values $[A_{n,s}^{-1}]$ and $[B_{n,s}^{-1}]$, hence the values of the inverses of tails of the Riemann zeta function $[\frac{1}{1-2^{1-s}}\zeta_n(s)^{-1}]$ for $s = \frac{1}{2}, \frac{1}{3}$, and $\frac{1}{4}$. For other values of s, for example $s = \frac{1}{5}$ or $\frac{2}{3}$, the values of $A_{n,s}$ and $B_{n,s}$ do not seem to have simple expressions.

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