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Refining trigonometric inequalities by using Padé approximant

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Abstract

A two-point Padé approximant method is presented for refining some remarkable trigonometric inequalities including the Jordan inequality, Kober inequality, Becker–Stark inequality, and Wu–Srivastava inequality. Simple proofs are provided. It shows to achieve better approximation results than those of prevailing methods.

Keywords: Trigonometric inequalities; Padé approximant; Two-sided bounds; Rational refinement

1 Introduction

Trigonometric inequalities have caused interest of a lot of researchers, they analyzed the Wilker inequality [6–11, 14, 16–19], Jordan inequality [3, 5, 15, 20, 21], Shafer–Fink inequality [12], Becker–Stark inequalities [13], and so on.

Recently, Bercu provided a Padé-approximant-based method and obtained the following inequalities [2].

$$b_1(x)=\frac{-7x^2+60}{3x^2+60}<\frac{\sin(x)}{x}<\frac{11x^4-360x^2+2520}{60x^2+2520}=b_2(x),\quad \forall x\in(0,\pi/2);\tag{1}$$

$$\begin{aligned}b_3(x)&=\frac{17x^4-480x^2+1080}{2x^4+60x^2+1080}<\cos(x)<\frac{3x^4-56x^2+120}{4x^2+120}\\&=b_4(x),\quad \forall x\in(0,\pi/2);\end{aligned}\tag{2}$$

$$b_5(x)<\frac{\tan(x)}{x}<b_6(x),\quad \forall x\in(0,1.5701);\tag{3}$$

$$\left(\frac{x}{\sin(x)}\right)^2+\frac{x}{\tan(x)}>b_7(x),\quad \forall x\in(0,1.5701),\tag{4}$$

where $b_5(x)=\frac{-28x^4-600x^2+7200}{9x^6+12x^4-3000x^2+7200}$, $b_6(x)=\frac{22x^8-60x^6-4680x^4-237{,}600x^2+2{,}721{,}600}{1020x^6+14{,}040x^4-1{,}144{,}800x^2+2{,}721{,}600}$ and $b_7(x)=\frac{11{,}220x^{10}-205{,}560x^8-14{,}256{,}000x^6+512{,}179{,}200x^4-3{,}157{,}056{,}000x^2+13{,}716{,}864{,}000}{242x^{12}-8580x^{10}+25{,}560x^8-1{,}080{,}000x^6+103{,}680{,}000x^4-1{,}578{,}528{,}000x^2+6{,}858{,}432{,}000}$.

In this paper, we present a two-point Padé-approximant-based method [1] for refining the rational bounds of several trigonometric inequalities, and also provide a method for proving the refined bounds. By applying the new method to $\frac{\sin(x)}{x}$ and $\cos(x)$, we refine the bounds of Eq. (1) ∼ (2), for $\forall x\in[0,\pi/2]$, see also Theorems 3.1 and 3.2. Applied to $\frac{\tan(x)}{x}$ and $(\frac{x}{\sin(x)})^2+\frac{x}{\tan(x)}$, it not only provides refined two-sided bounds with better approximation effect for Eq. (3) $\sim$ (4), but also extends the interval $(0, 1.5701)$ to the interval $[0, \pi/2]$, see also the theorems and remarks in Sect. 3.

2 Find bounds by using two-point Padé approximant

Given a bounded smooth function $f(x)$, $x \in [x_0, x_1]$, let $R(x) = \frac{\sum_{i=0}^{n} c_i x^i}{1+\sum_{i=1}^{m} d_i x^i}$ be a rational polynomial interpolating derivatives of $f(x)$ at two points x_0 and x_1 such that

$$E^{(i)}(x_0) = 0, \qquad E^{(j)}(x_1) = 0, \quad i = 0, 1, \dots, k, j = 0, 1, \dots, l, \tag{5}$$

where $E(x) = (1 + \sum_{i=1}^{m} d_i x^i) \cdot f(x) - (\sum_{i=1}^{n} c_i x^i)$. There are $m + n + 2$ unknowns in Eq. (5). By selecting suitable values of k and l, we have that Eq. (5) consists of $m + n + 2$ linear equations in the unknown variables c_i and d_j, and the interpolation polynomial $R(x)$ can be determined by solving Eq. (5).

We give two examples. Without loss of generality, let $\Gamma = [0, \pi/2]$.

Example 1 Let $f_1(x) = \sin(x)$. By setting $n_1 = 13$, $m_1 = 0$, $n_2 = 11$, and $m_2 = 0$ and introducing the following constraints

$$f_1^{(i)}(0) = R_j^{(i)}(0), \qquad f_1(\pi/2) = R_j(\pi/2), \quad j = 1, 2, i = 0, 1, \dots, 14 - 2j, \tag{6}$$

we obtain that

$$R_1(x) = \beta_1(x) + \alpha_1 \cdot x^{13}, \qquad R_2(x) = \beta_2(x) - \alpha_2 \cdot x^{11}, \tag{7}$$

where $\alpha_1 = \frac{\pi^{11} - 440\pi^9 + 126{,}720\pi^7 - 21{,}288{,}960\pi^5 + 1{,}703{,}116{,}800\pi^3 - 40{,}874{,}803{,}200\pi + 81{,}749{,}606{,}400}{9{,}979{,}200\pi^{13}}$, $\beta_1(x) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362{,}880} - \frac{x^{11}}{39{,}916{,}800}$, $\alpha_2 = \frac{\pi^9 - 288\pi^7 + 48{,}384\pi^5 - 3{,}870{,}720\pi^3 + 92{,}897{,}280\pi - 185{,}794{,}560}{90{,}720\pi^{11}}$, $\beta_2(x) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362{,}880}$. It can be verified that $R_j(x) \geq 0, \forall x \in \Gamma, j = 1, 2$. From Eq. (6), $\forall x \in \Gamma$, there exists $\xi_j(x) \in \Gamma$ such that [4]

$$f_1(x) - R_j(x) = \frac{f_1^{(16-2j)}(\xi_j(x))}{(16 - 2j)!} \cdot (x - \pi/2) \cdot x^{15-2j}, \quad x \in \Gamma, j = 1, 2. \tag{8}$$

Note that $f_1^{(14)}(x) = -\sin(x) \leq 0$ and $f_1^{(12)}(x) = \sin(x) \geq 0$, $\forall x \in \Gamma$. Combining with Eq. (8), one obtains that

$$0 \leq R_1(x) \leq \sin(x) \leq R_2(x), \quad x \in \Gamma. \tag{9}$$

Example 2 Let $f_2(x) = \cos(x)$. By setting $n_3 = 12$, $m_3 = 0$, $n_4 = 10$, and $m_4 = 0$ and introducing the following constraints

$$f_2^{(i)}(0) = R_j^{(i)}(0), \qquad f_2(\pi/2) = R_j(\pi/2), \quad j = 3, 4, i = 0, 1, \dots, 17 - 2j, \tag{10}$$

we obtain that

$$R_3(x) = \beta_3(x) + \alpha_3 \cdot x^{12}, \qquad R_4(x) = \beta_4(x) - \alpha_4 \cdot x^{10}, \tag{11}$$

where $\alpha_3 = \frac{\pi^{10}-360\pi^8+80{,}640\pi^6-9{,}676{,}800\pi^4+464{,}486{,}400\pi^2-3{,}715{,}891{,}200}{907{,}200\pi^{12}}$, $\beta_3(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40{,}320} - \frac{x^{10}}{3{,}628{,}800}$, $\alpha_4 = \frac{10{,}321{,}920-1{,}290{,}240\pi^2+26{,}880\pi^4-224\pi^6+\pi^8}{10{,}080\pi^{10}}$, $\beta_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40{,}320}$. It can be verified that $R_j(x) \geq 0, \forall x \in \Gamma, j = 3,4$. From Eq. (10), $\forall x \in \Gamma$, there exists $\xi_j(x) \in \Gamma, j = 3,4$, such that [4]

$$f_2(x) - R_j(x) = \frac{f_2^{(19-2j)}(\xi_j(x))}{(19-2j)!} \cdot (x - \pi/2) \cdot x^{18-2j}, \quad x \in \Gamma, j = 3,4. \tag{12}$$

Note that $f_2^{(13)}(x) = -\sin(x) \leq 0$ and $f_2^{(11)}(x) = \sin(x) \geq 0$, $\forall x \in \Gamma$. Combining with Eq. (12), one obtains that

$$0 \leq R_3(x) \leq \cos(x) \leq R_4(x), \quad x \in \Gamma. \tag{13}$$

3 Main results

The main results are as follows.

Theorem 3.1 *For all* $\forall x \in \Gamma = [0, \pi/2]$, *we have that*

$$\begin{aligned}[b] c_1(x) = \frac{60{,}480 - 9240x^2 + 364x^4 - 5x^6}{840(72 + x^2)} &\leq \frac{\sin(x)}{x} \\ &\leq \frac{(166{,}320 - 22{,}260x^2 + 551x^4)}{15(11{,}088 + 364x^2 + 5x^4)} = c_2(x). \end{aligned} \tag{14}$$

Proof Eq. (14) is equivalent to

$$\begin{cases} (60{,}480 - 9240x^2 + 364x^4 - 5x^6)x - 840(72 + x^2)\sin(x) \leq 0, \\ (166{,}320 - 22{,}260x^2 + 551x^4)x - 15(11{,}088 + 364x^2 + 5x^4)\sin(x) \geq 0, \end{cases} \quad \forall x \in \Gamma. \tag{15}$$

It is well known that $\forall x \in \Gamma$,

$$\begin{aligned} \beta_1(x) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362{,}880} - \frac{x^{11}}{39{,}916{,}800} \\ \leq \sin(x) \leq \beta_1(x) + \frac{x^{13}}{6{,}227{,}020{,}800}. \end{aligned} \tag{16}$$

Combining with Eq. (16), we have that

$$\begin{aligned} &\left(60{,}480 - 9240x^2 + 364x^4 - 5x^6\right)x - 840\left(72 + x^2\right)\sin(x) \\ &\quad \leq \left(60{,}480 - 9240x^2 + 364x^4 - 5x^6\right)x - 840\left(72 + x^2\right)\beta_1(x) \\ &\quad = \frac{x^{11} \cdot (-38 + x^2)}{39{,}916{,}800} \leq 0, \quad \forall x \in \Gamma, \\ &\left(166{,}320 - 22{,}260x^2 + 551x^4\right)x - 15\left(11{,}088 + 364x^2 + 5x^4\right)\sin(x) \\ &\quad \geq \left(166{,}320 - 22{,}260x^2 + 551x^4\right)x \\ &\qquad - 15\left(11{,}088 + 364x^2 + 5x^4\right)\left(\beta_1(x) + \frac{x^{13}}{6{,}227{,}020{,}800}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{x^{11}}{6{,}227{,}020{,}800}\left(1{,}661{,}088 - 40{,}104x^2 + 416x^4 - 5x^6\right) \\
&\geq \frac{x^{11}}{6{,}227{,}020{,}800}\left(1{,}661{,}088 - 40{,}104 \cdot 2^2 - 5 \cdot 2^6\right) \geq 0, \quad \forall x \in \Gamma,
\end{aligned}
$$

which is just Eq. (15). So we have completed the proof of Eq. (14). □

Theorem 3.2 *For all* $\forall x \in [0, \pi/2]$, *we have that*

$$
\begin{aligned}
c_3(x) &= \frac{20{,}160 - 9720x^2 + 660x^4 - 13x^6}{360(x^2 + 56)} \leq \cos(x) \\
&\leq \frac{15{,}120 - 6900x^2 + 313x^4}{15{,}120 + 660x^2 + 13x^4} = c_4(x).
\end{aligned} \tag{17}
$$

Proof Eq. (17) is equivalent to

$$
\begin{cases}
(20{,}160 - 9720x^2 + 660x^4 - 13x^6) - 360(x^2 + 56)\cos(x) \leq 0, \\
(15{,}120 - 6900x^2 + 313x^4) - (15{,}120 + 660x^2 + 13x^4)\cos(x) \geq 0,
\end{cases} \quad \forall x \in \Gamma. \tag{18}
$$

It is well known that

$$
\begin{aligned}
\beta_3(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40{,}320} - \frac{x^{10}}{3{,}628{,}800} \leq \cos(x) \\
&\leq 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40{,}320} = \beta_4(x), \quad \forall x \in \Gamma.
\end{aligned} \tag{19}
$$

Combining with Eq. (19), we have that

$$
\begin{cases}
(20{,}160 - 9720x^2 + 660x^4 - 13x^6) - 360(x^2 + 56)\cos(x) \\
\quad \leq (20{,}160 - 9720x^2 + 660x^4 - 13x^6) - 360(x^2 + 56)\beta_3(x) \\
\quad = \frac{x^{10}}{3{,}628{,}800}(-34 + x^2) \leq 0, \quad \forall x \in [0, \pi/2], \\
(15{,}120 - 6900x^2 + 313x^4) - (15{,}120 + 660x^2 + 13x^4)\cos(x) \\
\quad \geq (15{,}120 - 6900x^2 + 313x^4) - (15{,}120 + 660x^2 + 13x^4)\beta_4(x) \\
\quad = \frac{x^{10}}{40{,}320}(68 - 13x^2) \geq 0, \quad \forall x \in [0, \pi/2].
\end{cases} \tag{20}
$$

Thus, we have completed the proof of both Eq. (18) and Eq. (17). □

Theorem 3.3 *For all* $\forall x \in \Gamma$, *we have that*

$$
\begin{aligned}
c_5(x) &= \frac{21(495 - 60x^2 + x^4)}{10{,}395 - 4725x^2 + 210x^4 - x^6} \leq \frac{\tan(x)}{x} \\
&\leq \frac{T_1(x)}{105(\pi^2 - 4x^2) \cdot T_2(x)} = c_6(x),
\end{aligned} \tag{21}
$$

where $T_1(x) = (\pi^6 - 840\pi^4 + 75{,}600\pi^2 - 665{,}280)x^6 + (210\pi^6 + 52{,}920\pi^4 - 7{,}620{,}480\pi^2 + 69{,}854{,}400)x^4 + (-17{,}955\pi^6 + 1{,}323{,}000\pi^4 + 52{,}390{,}800\pi^2 - 628{,}689{,}600)x^2 + (155{,}925(\pi^4 - 112\pi^2 + 1008))\pi^2$ *and* $T_2(x) = (26\pi^4 - 2664\pi^2 + 23{,}760)x^4 + (-666\pi^4 + 73{,}980\pi^2 - 665{,}280)x^2 + (1485\pi^4 - 166{,}320\pi^2 + 1{,}496{,}880)$.

Proof Eq. (21) is equivalent to

$$\begin{cases} H_5(x) = 21(495 - 60x^2 + x^4) \cdot x\cos(x) \\ \qquad - (10{,}395 - 4725x^2 + 210x^4 - x^6) \cdot \sin(x) \le 0; \\ H_6(x) = 105(\pi^2 - 4x^2) \cdot T_2(x) \cdot \sin(x) - T_1(x) \cdot x\cos(x) \le 0, \end{cases} \quad \forall x \in \Gamma. \tag{22}$$

It can be verified that

$$\begin{cases} \cos(x) \le 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40{,}320} - \frac{x^{10}}{3{,}628{,}800} + \frac{x^{12}}{479{,}001{,}600} = \beta_5(x), \\ \beta_1(x) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362{,}880} - \frac{x^{11}}{39{,}916{,}800} \le \sin(x), \\ 495 - 60x^2 + x^4 > 0, \qquad 10{,}395 - 4725x^2 + 210x^4 - x^6 > 0, \end{cases} \quad \forall x \in \Gamma. \tag{23}$$

Combining with Eq. (23), we have that

$$\begin{aligned} H_5(x) &\le 21\bigl(495 - 60x^2 + x^4\bigr) \cdot x\beta_5(x) - \bigl(10{,}395 - 4725x^2 + 210x^4 - x^6\bigr) \cdot \beta_1(x) \\ &= \frac{x^{13}}{159{,}667{,}200}\bigl(-915 - 64x^2 + 3x^4\bigr) \le 0, \quad \forall x \in \Gamma. \end{aligned} \tag{24}$$

Let $\beta_6(x) = T_1(x) + 105(\pi^2 - 4x^2) \cdot T_2'(x) - 840x \cdot T_2(x)$, $\beta_7(x) = 105(\pi^2 - 4x^2) \cdot T_2(x) - T_1'(x)$. On the other hand, it can be verified that, $\forall x \in \Gamma$,

$$\begin{aligned} & H_6'(x) = \beta_6(x) \cdot \sin(x) + \beta_7(x) \cdot \cos(x), \\ & \beta_6(x) \le 0, \qquad \beta_7(x) \ge 0, \qquad T_2(x) \ge 0, \qquad T_1(x) \ge 0, \\ & \cos(x) \ge 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40{,}320} - \frac{x^{10}}{3{,}628{,}800} + \frac{x^{12}}{479{,}001{,}600} \\ & \qquad - \frac{x^{14}}{87{,}178{,}291{,}200} = \beta_8(x), \\ & \beta_9(x) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362{,}880} - \frac{x^{11}}{39{,}916{,}800} + \frac{x^{13}}{6{,}227{,}020{,}800} \ge \sin(x). \end{aligned} \tag{25}$$

Combining Eq. (23) with Eq. (25), we have that

$$\begin{aligned} H_6(x) &\le 105\bigl(\pi^2 - 4x^2\bigr) \cdot T_2(x) \cdot \beta_9(x) - T_1(x) \cdot x\beta_8(x) \\ &= \frac{x^{13}}{9{,}153{,}720{,}576{,}000}\beta_{10}(x) \le 0, \quad \forall x \in \left[0, \frac{31\pi}{64}\right], \\ H_6'(x) &\ge \beta_6(x) \cdot \beta_1(x) + \beta_7(x) \cdot \beta_5(x) \\ &= \frac{x^{12}}{50{,}295{,}168{,}000}\beta_{11}(x) \ge 0, \quad \forall x \in \left[\frac{31\pi}{64}, \frac{\pi}{2}\right], \end{aligned} \tag{26}$$

where $\beta_{10}(x) = (18{,}063{,}360\pi^6 - 8{,}128{,}512{,}000\pi^4 + 643{,}778{,}150{,}400\pi^2 - 5{,}579{,}410{,}636{,}800) + (-634{,}725\pi^6 + 305{,}912{,}880\pi^4 - 24{,}700{,}198{,}320\pi^2 + 214{,}592{,}716{,}800)x^2 + (6069\pi^6 - 4{,}639{,}320\pi^4 + 411{,}823{,}440\pi^2 - 3{,}618{,}457{,}920)x^4 + (28\pi^6 + 52{,}920\pi^4 - 5{,}715{,}360\pi^2 + 51{,}226{,}560)x^6 + (\pi^6 - 840\pi^4 + 75{,}600\pi^2 - 665{,}280)x^8 \le 0, \forall x \in [0, \frac{31\pi}{64}]$, $\beta_{11}(x) = (-1{,}290{,}240\pi^6 + 580{,}608{,}000\pi^4 - 45{,}984{,}153{,}600\pi^2 + 398{,}529{,}331{,}200) + (54{,}405\pi^6 - 25{,}552{,}800\pi^4 + 2{,}048{,}684{,}400\pi^2 -$

$17{,}782{,}934{,}400)x^2 + (-1404\pi^6 + 556{,}920\pi^4 - 42{,}366{,}240\pi^2 + 365{,}238{,}720)x^4 + (19\pi^6 - 5040\pi^4 + 317{,}520\pi^2 - 2{,}661{,}120)x^6 \geq 0, \forall x \in [\frac{31\pi}{64}, \frac{\pi}{2}]$. Combining Eq. (26) with $H_6(\pi/2) = 0$, we obtain that

$$H_6(x) \leq 0, \quad \forall x \in \left[0, \frac{\pi}{2}\right]. \tag{27}$$

Combining Eq. (24) with Eq. (27), we have completed the proof of both Eq. (22) and Eq. (21). □

From Theorems 3.1, 3.2, and 3.3, we directly obtain the following theorem.

Theorem 3.4 *We have that*

$$\frac{1}{c_2(x)^2} + \frac{1}{c_6(x)} \leq \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} \leq \frac{1}{c_1(x)^2} + \frac{1}{c_5(x)}, \quad \forall x \in [0, \pi/2].$$

4 Discussion and conclusions

Firstly, we compare the results of $\frac{\sin(x)}{x}$ between $b_i(x)$ in [2] and $c_i(x)$ in this paper, $i = 1, 2$. It can be verified that $c_1(x) - b_1(x) = \frac{x^6(264-5x^2)}{840(72+x^2)(x^2+20)} \geq 0$ and $c_2(x) - b_2(x) = \frac{-11x^8}{12(11{,}088+364x^2+5x^4)(x^2+42)} \leq 0, \forall x \in [0, \pi/2]$, we have that

$$b_1(x) \leq c_1(x) \leq \frac{\sin(x)}{x} \leq c_2(x) \leq b_2(x), \quad \forall x \in [0, \pi/2].$$

Secondly, we compare the approximation results of $\cos(x)$ between previous $b_i(x)$ and present $c_i(x)$, $i = 3, 4$. It can be verified that $c_3(x) - b_3(x) = \frac{x^8(270-13x^2)}{360(56+x^2)(x^4+30x^2+540)} \geq 0$ and $c_2(x) - b_2(x) = \frac{-39x^8}{4(15{,}120+660x^2+13x^4)(x^2+30)} \leq 0, \forall x \in [0, \pi/2]$, we have that

$$b_3(x) \leq c_3(x) \leq \cos(x) \leq c_4(x) \leq b_4(x), \quad \forall x \in [0, \pi/2].$$

Thirdly, we compare the approximation results of $\frac{\tan(x)}{x}$, which also shows that this paper achieves a much better result. It can be verified that $\forall x \in [0, \pi/2]$,

$$c_5(x) - b_5(x) = \frac{x^6(161x^2 - 495)(x^2 - 33)}{3(10{,}395 - 4725x^2 + 210x^4 - x^6)(x^2 + 20)(3x^4 - 56x^2 + 120)} \geq 0.$$

However, note that the denominator of $b_6(x)$ is $T_3(x) = 1020x^6 + 14{,}040x^4 - 1{,}144{,}800x^2 + 2{,}721{,}600 = 30(17x^4 - 480x^2 + 1080)(x^2 + 42)$, which has a real root ≈ 1.5701 within the interval Γ, and we have $T_3(x) > 0, \forall x \in [0, 1.5701]$. It can be verified that $c_6(x) - b_6(x) = \frac{-x^8 H_7(x)}{210 T_2(x) T_3(x)(\pi^2-4x^2)}$, where $H_7(x) = 378{,}675(\pi^4 - 112\pi^2 + 1008)\pi^2 + (-64{,}350\pi^6 + 5{,}536{,}440\pi^4 + 106{,}323{,}840\pi^2 - 1{,}526{,}817{,}600)x^2 + (1968\pi^6 + 50{,}400\pi^4 - 25{,}764{,}480\pi^2 + 247{,}484{,}160)x^4 + (-8008\pi^4 + 820{,}512\pi^2 - 7{,}318{,}080)x^6$. By using the Maple software, $H_7(x)$ has six real roots $\approx -9.16, -4.97, -2.76, 2.76, 4.97, 9.16$, and $H_7(x), T_2(x), T_3(x) > 0, \forall x \in (0, 1.5701)$, we have that

$$c_6(x) - b_6(x) \leq 0, \quad \forall x \in [0, 1.5701].$$

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Competing interests
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Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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