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Majorization involving the cyclic moving average

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Abstract
We solve an open problem on some majorization inequalities involving the cyclic moving average.

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1 Introduction

We first recall two definitions.

Definition 1.1 ([1]) For fixed $n \geq 2$, let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ be two n-tuples of real numbers.

(i) x is said to be majorized by y (in symbols, $x \prec y$) if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \quad \text{for } k = 1, 2, \dots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i,$$

where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of x and y in descending order.

(ii) Let $\Omega \subset \mathbb{R}^n$. A function $\varphi : \Omega \to \mathbb{R}$ is said to be a Schur-convex function (shortly, an S-convex function) if

$$x \prec y \Rightarrow \varphi(x) \leq \varphi(y).$$

For example:

$$a^{(2)} = \left(\frac{a_1 + a_2}{2}, \frac{a_2 + a_3}{2}, \dots, \frac{a_n + a_1}{2} \right),$$
$$a^{(3)} = \left(\frac{a_1 + a_2 + a_3}{3}, \frac{a_2 + a_3 + a_4}{3}, \dots, \frac{a_n + a_1 + a_2}{3} \right).$$

In 2006, I. Olkin, one of the authors of the book [1], wrote a letter to K. Z. Guan, referring to the following interesting question: is it true that

$$a^{(k+1)} \prec a^{(k)}, \quad 1 \leq k \leq n-1? \tag{1}$$

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However, a proof for $a^{(k+1)} \prec a^{(k)}$ remains elusive (see [1], p. 63).

In 2010, Shi [2] proved that (1) holds when $n = 4$, $k = 2$ and $n = 5$, $k = 3$. In this paper, we prove that (1) holds for any $n \geq 2$ and $1 \leq k \leq n - 1$.

For any $1 \leq k \leq n$, let

$$a_{[1]}^{(k)} \geq a_{[2]}^{(k)} \geq \cdots \geq a_{[n]}^{(k)}$$

be the ordered component of the sequence $a_1^{(k)}, a_2^{(k)}, \dots, a_n^{(k)}$. We denote

$$S_h^{(k)} = \sum_{i=1}^{h} a_{[i]}^{(k)}, \quad 1 \leq h \leq n.$$

2 Lemmas and corollaries

For proving our main results, we need the following lemmas.

Lemma 2.1 *Let $n \geq 2$ and $1 \leq k \leq n - 1$. Then*

$$\begin{cases} a_1^{(k)} \geq a_2^{(k)} \geq \cdots \geq a_{n-k+1}^{(k)}, \\ a_n^{(k)} \geq a_{n-1}^{(k)} \geq \cdots \geq a_{n-k+1}^{(k)}, \\ a_1^{(k)} \geq a_n^{(k)}. \end{cases} \tag{2}$$

Proof For any $1 \leq k \leq n - 1$ and $1 \leq i \leq n$, we have

$$k\bigl(a_i^{(k)} - a_{i+1}^{(k)}\bigr) = (a_i + a_{i+1} + \cdots + a_{i+k-1}) - (a_{i+1} + a_{i+2} + \cdots + a_{i+k}) = a_i - a_{i+k}.$$

Note that $a_1 \geq a_2 \geq \cdots \geq a_n$, so we can induce that

(1) if $i + k \leq n$, that is, $1 \leq i \leq n - k$, then $a_i \geq a_{i+k}$. It follows that $a_i^{(k)} \geq a_{i+1}^{(k)}$.

(2) if $i + k > n$, that is, $n - k < i \leq n$, then $a_{i+k} = a_{n+(i+k-n)} = a_{i+k-n}$. Since $1 \leq k \leq n - 1$, we have $n \geq i > i + k - n$. So $a_i \geq a_{i+k-n} = a_{i+k}$. It follows that $a_i^{(k)} \geq a_{i+1}^{(k)}$.

(3) $k(a_1^{(k)} - a_n^{(k)}) = (a_1 + a_2 + \cdots + a_k) - (a_n + a_1 + \cdots + a_{k-1}) = a_k - a_n \geq 0$. Therefore (2) holds. □

From the proof of Lemma 2.1 it is easy to deduce the following:

Corollary 2.2 *Let $n \geq 2$, $1 \leq k \leq n - 1$, and let $\sum_{i=0}^{-1} a_{n-i}^{(k)} = 0$. For any $1 \leq h \leq n - 1$, there exist $1 \leq h_1 \leq n - k$ and $-1 \leq h_2 \leq k - 2$ such that $h = h_1 + h_2 + 1$ and*

$$S_h^{(k)} = \sum_{i=1}^{h_1} a_i^{(k)} + \sum_{i=0}^{h_2} a_{n-i}^{(k)}.$$

Lemma 2.3 *Let $n \geq 4$, $2 \leq k \leq n - 2$, and $0 \leq r \leq k - 2$.*

(i) *If*

$$a_2^{(k)} \leq a_{n-r}^{(k)} \leq a_1^{(k)}, \tag{3}$$

then

$$a_2^{(k+1)} \leq a_{n-r}^{(k+1)} \leq a_1^{(k+1)}.$$

(ii) *If*

$$a_{n-k+1}^{(k)} \leq a_{n-r}^{(k)} \leq a_{n-k}^{(k)}, \tag{4}$$

then

$$a_{n-k}^{(k+1)} \leq a_{n-r-1}^{(k+1)} \leq a_{n-k-1}^{(k+1)}.$$

(iii) *For* $2 \leq m \leq n-k-1$, *if*

$$a_{m+1}^{(k)} \leq a_{n-r}^{(k)} \leq a_{m}^{(k)}, \tag{5}$$

then

$$a_{m+1}^{(k+1)} \leq a_{n-r}^{(k+1)}, \qquad a_{n-r-1}^{(k+1)} \leq a_{m-1}^{(k+1)}. \tag{6}$$

(iv) *If*

$$a_{n}^{(k)} \leq a_{n-k}^{(k)}, \tag{7}$$

then

$$a_{n-1}^{(k+1)} \leq a_{n-k-1}^{(k+1)}.$$

(v) *For* $2 \leq m \leq n-k-1$, *if*

$$a_{n}^{(k)} \leq a_{m}^{(k)}, \tag{8}$$

then

$$a_{n-1}^{(k+1)} \leq a_{m}^{(k+1)}.$$

(vi) *For* $2 \leq m \leq n-k$, *if*

$$a_{n-r-1}^{(k)} \leq a_{m}^{(k)} \leq a_{n-r}^{(k)}, \tag{9}$$

then

$$a_{n-r-2}^{(k+1)} \leq a_{m-1}^{(k+1)}, \qquad a_{m}^{(k+1)} \leq a_{n-r}^{(k+1)}.$$

Proof

(i) By Lemma 2.1 we have

$$\begin{cases} a_1^{(k+1)} \geq a_2^{(k+1)} \geq \cdots \geq a_{n-k}^{(k+1)}, \\ a_n^{(k+1)} \geq a_{n-1}^{(k+1)} \geq \cdots \geq a_{n-k}^{(k+1)}, \\ a_1^{(k+1)} \geq a_n^{(k+1)}. \end{cases}$$

It follows that

$$a_1^{(k+1)} = \max\{a_1^{(k+1)}, a_2^{(k+1)}, \dots, a_n^{(k+1)}\}.$$

Thus we have

$$a_{n-r}^{(k+1)} \le a_1^{(k+1)}.$$

By the left inequality of (3) we have

$$a_2 + \cdots + a_{k+1} \le a_{n-r} + \cdots + a_{n-r+k-1}.$$

Note that $a_{k+2} \le a_{k-r} = a_{n+k-r}$, so we have

$$a_2 + \cdots + a_{k+1} + a_{k+2} \le a_{n-r} + \cdots + a_{n-r+k-1} + a_{n+k-r}.$$

Therefore

$$a_2^{(k+1)} \le a_{n-r}^{(k+1)}.$$

(ii) By Lemma 2.1 we have

$$a_{n-k}^{(k+1)} \le a_{n-r-1}^{(k+1)}.$$

By the right inequality of (4) we get

$$a_{n-r} + \cdots + a_{n-r+k-1} \le a_{n-k} + \cdots + a_{n-1}.$$

Note that $a_{n-r-1} \le a_{n-k-1}$, so we have

$$a_{n-r} + \cdots + a_{n-r+k-1} + a_{n-r-1} \le a_{n-k} + \cdots + a_{n-1} + a_{n-k-1}.$$

This means that

$$a_{n-r-1}^{(k+1)} \le a_{n-k-1}^{(k+1)}.$$

(iii) By (5) we get

$$a_{m+1} + \cdots + a_{m+k} \le a_{n-r} + \cdots + a_{n-r+k-1} \le a_m + \cdots + a_{m+k-1}.$$

Note that $n \ge m + k + 1 \ge k - r \ge 1$ and $n \ge n - r - 1 \ge m - 1 \ge 1$, so we have

$$a_{m+k+1} \le a_{n-r+k} = a_{k-r}, \qquad a_{n-r-1} \le a_{m-1}.$$

It follows that

$$a_{m+1} + \cdots + a_{m+k} + a_{m+k+1} \le a_{n-r} + \cdots + a_{n-r+k-1} + a_{n-r+k}$$

and

$$a_{n-r-1} + a_{n-r} + \cdots + a_{n-r+k-1} \leq a_{m-1} + a_m + \cdots + a_{m+k-1}.$$

Therefore (6) holds.

(iv) By (7) we get

$$a_n + \cdots + a_{n+k-1} \leq a_{n-k} + \cdots + a_{n-1}.$$

Since $a_{n-1} \leq a_{n-k-1}$, we have

$$a_n + \cdots + a_{n+k-1} + a_{n-1} \leq a_{n-k} + \cdots + a_{n-1} + a_{n-k-1}.$$

It follows that

$$a_{n-1}^{(k+1)} \leq a_{n-k-1}^{(k+1)}.$$

(v) By (8) we have

$$a_n + \cdots + a_{n+k-1} \leq a_m + \cdots + a_{m+k-1}.$$

Note that $n-1 \geq m+k$ and $a_{n-1} \leq a_{m+k}$, so we have

$$a_n + \cdots + a_{n+k-1} + a_{n-1} \leq a_m + \cdots + a_{m+k-1} + a_{m+k}.$$

This means that

$$a_{n-1}^{(k+1)} \leq a_m^{(k+1)}.$$

(vi) By (9) we get

$$a_{n-r-1} + \cdots + a_{n-r+k-2} \leq a_m + \cdots + a_{m+k-1} \leq a_{n-r} + \cdots + a_{n-r+k-1}.$$

Note that $0 \leq r \leq k-2$ and $2 \leq m \leq n-k$, so we have

$$n \geq n-r-2 \geq n-k \geq m \geq m-1 \geq 1.$$

It follows that

$$a_{n-r-2} \leq a_{m-1}.$$

So we get

$$a_{n-r-2} + a_{n-r-1} + \cdots + a_{n-r+k-2} \leq a_{m-1} + a_m + \cdots + a_{m+k-1}.$$

Therefore

$$a_{n-r-2}^{(k+1)} \leq a_{m-1}^{(k+1)}.$$

Note that $m \le n-k$, so we have

$$k-r \le m+k \le n, \qquad a_{m+k} \le a_{k-r} = a_{n+k-r}.$$

It follows that

$$a_m + \cdots + a_{m+k-1} + a_{m+k} \le a_{n-r} + \cdots + a_{n-r+k-1} + a_{n+k-r}.$$

Therefore

$$a_m^{(k+1)} \le a_{n-r}^{(k+1)}.$$

□

Lemma 2.4 *Let $n \ge 4$, $2 \le k \le n-1$, $1 \le m \le n-k$, and $0 \le r \le k-2$.*

(i) *If $a_{m+1}^{(k)} \le a_{n-r}^{(k)} \le a_m^{(k)}$, then*

$$S_{m+r+1}^{(k)} = \sum_{i=1}^{m} a_i^{(k)} + \sum_{i=0}^{r} a_{n-i}^{(k)}. \tag{10}$$

(ii) *If $a_n^{(k)} \le a_m^{(k)}$, then*

$$S_m^{(k)} = \sum_{i=1}^{m} a_i^{(k)}.$$

(iii) *For $2 \le m \le n-k$, if $a_{n-r-1}^{(k)} \le a_m^{(k)} \le a_{n-r}^{(k)}$, then*

$$S_{m+r+1}^{(k)} = \sum_{i=1}^{m} a_i^{(k)} + \sum_{i=0}^{r} a_{n-i}^{(k)}.$$

Proof We only prove (i). Using a similar method, we can obtain (ii) and (iii).

By Lemma 2.1 we have

$$\begin{cases} a_1^{(k)} \ge a_2^{(k)} \ge \cdots \ge a_m^{(k)} \ge a_{n-r}^{(k)} \ge a_{m+1}^{(k)}, \\ a_n^{(k)} \ge a_{n-1}^{(k)} \ge \cdots \ge a_{n-r+1}^{(k)} \ge a_{n-r}^{(k)} \ge a_{m+1}^{(k)}. \end{cases} \tag{11}$$

By Corollary 2.2 we let

$$S_{m+r+1}^{(k)} = \sum_{i=1}^{h_1} a_i^{(k)} + \sum_{i=0}^{h_2} a_{n-i}^{(k)}, \tag{12}$$

where $1 \le h_1 \le n-k+1$, $-1 \le h_2 \le k-2$, and $\sum_{i=0}^{-1} a_{n-i}^{(k)} = 0$. It is clear that

$$m+r+1 = h_1 + h_2 + 1. \tag{13}$$

Next, we prove that $h_1 = m$ and $h_2 = r$.

(1) If $h_1 \geq m+1$, which means that the right-hand side of (12) includes $a_{m+1}^{(k)}$, then by (11) the right-hand side of (12) should include $a_1^{(k)}, a_2^{(k)}, \ldots, a_{m+1}^{(k)}$ and $a_{n-r}^{(k)}, a_{n-r+1}^{(k)}, \ldots, a_n^{(k)}$, so we have

$$h_1 + h_2 + 1 \geq m + r + 2 > m + r + 1.$$

This is a contradiction with (13).

(2) If $h_1 \leq m-1$, then by (13) we have $k-2 \geq h_2 \geq r+1$. Together with (11), we get

$$a_{n-h_2}^{(k)} \leq a_{n-r-1}^{(k)} \leq a_{n-r}^{(k)} \leq a_m^{(k)}.$$

So the right-hand side of (12) must include $a_m^{(k)}$, which means that $h_1 \geq m$. This is a contradiction with $h_1 \leq m-1$.

Therefore $h_1 = m$ and $h_2 = r$. So (10) holds. □

Corollary 2.5 *Let $n \geq 4$, $2 \leq k \leq n-2$, $1 \leq m \leq n-k$, and $0 \leq r \leq k-2$, and let*

$$a_{m+1}^{(k)} \leq a_{n-r}^{(k)} \leq a_m^{(k)}. \tag{14}$$

(i) *For $m = 1$, we have*

$$S_{r+2}^{(k+1)} = a_1^{(k+1)} + \sum_{i=0}^{r} a_{n-i}^{(k+1)}. \tag{15}$$

(ii) *For $m = n-k$, we have*

$$S_{n-k+r+1}^{(k+1)} = \sum_{i=1}^{n-k-1} a_i^{(k+1)} + \sum_{i=0}^{r+1} a_{n-i}^{(k+1)}. \tag{16}$$

(iii) *For $2 \leq k \leq n-3$ and $2 \leq m \leq n-k-1$, $S_{m+r+1}^{(k+1)}$ must be one of the following two cases*:

$$S_{m+r+1}^{(k+1)} = \sum_{i=1}^{m} a_i^{(k+1)} + \sum_{i=0}^{r} a_{n-i}^{(k+1)},$$

or

$$S_{m+r+1}^{(k+1)} = \sum_{i=1}^{m-1} a_i^{(k+1)} + \sum_{i=0}^{r+1} a_{n-i}^{(k+1)}.$$

Proof

(i) If $m = 1$, by (14) we have

$$a_2^{(k)} \leq a_{n-r}^{(k)} \leq a_1^{(k)}.$$

By Lemma 2.3(i) we have

$$a_2^{(k+1)} \leq a_{n-r}^{(k+1)} \leq a_1^{(k+1)},$$

and then by Lemma 2.4(i) we can induce that (15) holds.

(ii) If $m = n-k$, then by (14) we have

$$a_{n-k+1}^{(k)} \leq a_{n-r}^{(k)} \leq a_{n-k}^{(k)}.$$

By Lemma 2.3(ii) we have

$$a_{n-k}^{(k+1)} \leq a_{n-r-1}^{(k+1)} \leq a_{n-k-1}^{(k+1)},$$

and then by Lemma 2.4(i) we can induce that (16) holds.

(iii) By Corollary 2.2 we let

$$S_{m+r+1}^{(k+1)} = \sum_{i=1}^{p} a_i^{(k+1)} + \sum_{i=0}^{q} a_{n-i}^{(k+1)}, \tag{17}$$

where $1 \leq p \leq n-k$, $-1 \leq q \leq k-2$, and $\sum_{i=0}^{-1} a_{n-i}^{(k)} = 0$. Then we have

$$p+q+1 = m+r+1. \tag{18}$$

Next, we prove that $p = m$ or $p = m-1$.

(1) If $p \geq m+1$, then by Lemma (2.3)(iii) we have $a_{m+1}^{(k+1)} \leq a_{n-r}^{(k+1)}$. So we get

$$a_p^{(k+1)} \leq a_{m+1}^{(k+1)} \leq a_{n-r}^{(k+1)}.$$

Thus the right-hand side of (17) includes $a_{n-r}^{(k+1)}$, which means that $q \geq r$. Therefore

$$p+q+1 \geq m+r+2 > m+r+1.$$

This is a contradiction with (18).

(2) If $1 \leq p \leq m-2$, then by (18) we have $n-q \leq n-r-2$. By Lemma 2.3(iii) we get $a_{n-r-1}^{(k+1)} \leq a_{m-1}^{(k+1)}$. It follows that

$$a_{n-q}^{(k+1)} \leq a_{n-r-2}^{(k+1)} \leq a_{n-r-1}^{(k+1)} \leq a_{m-1}^{(k+1)}.$$

So the right-hand side of (17) must include $a_{m-1}^{(k+1)}$. Therefore

$$p \geq m-1.$$

This is a contradiction with $1 \leq p \leq m-2$.

Thus $p = m$ or $p = m-1$. □

In a similar way as in Corollary 2.5, we can prove the following corollaries.

Corollary 2.6 *Let* $n \geq 4$, $2 \leq k \leq n-2$, $2 \leq m \leq n-k$, *and* $0 \leq r \leq k-2$.

(i) *If* $a_n^{(k)} \leq a_m^{(k)}$, *then* $S_m^{(k+1)}$ *must be one of the following two cases*:

$$S_m^{(k+1)} = \sum_{i=1}^{m} a_i^{(k+1)}$$

or

$$S_m^{(k+1)} = \sum_{i=1}^{m-1} a_i^{(k+1)} + a_n^{(k+1)}.$$

(ii) *If* $a_{n-r-1}^{(k)} \leq a_m^{(k)} \leq a_{n-r}^{(k)}$, *then* $S_{m+r+1}^{(k+1)}$ *must be one of the following two cases*:

$$S_{m+r+1}^{(k+1)} = \sum_{i=1}^{m} a_i^{(k+1)} + \sum_{i=0}^{r} a_{n-i}^{(k+1)}$$

or

$$S_{m+r+1}^{(k+1)} = \sum_{i=1}^{m-1} a_i^{(k+1)} + \sum_{i=0}^{r+1} a_{n-i}^{(k+1)}.$$

Corollary 2.7 *Let* $n \geq 4$, $2 \leq k \leq n-2$, $1 \leq m \leq n-k$, *and* $-1 \leq r \leq k-2$, *and let* $\sum_{i=0}^{-1} a_{n-i}^{(k)} = 0$. *If*

$$S_{m+r+1}^{(k)} = \sum_{i=1}^{m} a_i^{(k)} + \sum_{i=0}^{r} a_{n-i}^{(k)},$$

then we have:

(i) *if* $m = 1$, *then*

$$S_{r+2}^{(k+1)} = a_1^{(k+1)} + \sum_{i=0}^{r} a_{n-i}^{(k+1)};$$

(ii) *if* $2 \leq m \leq n-k$, *then* $S_{m+r+1}^{(k+1)}$ *must be one of the following two cases*:

$$S_{m+r+1}^{(k+1)} = \sum_{i=1}^{m} a_i^{(k+1)} + \sum_{i=0}^{r} a_{n-i}^{(k+1)}$$

or

$$S_{m+r+1}^{(k+1)} = \sum_{i=1}^{m-1} a_i^{(k+1)} + \sum_{i=0}^{r+1} a_{n-i}^{(k+1)}.$$

Lemma 2.8 *Let* $n \geq 4$, $2 \leq k \leq n-2$, $1 \leq m \leq n-k$, *and* $-1 \leq r \leq k-2$, *and let* $\sum_{i=0}^{-1} a_{n-i}^{(k)} = 0$. *If*

$$\begin{cases} S_{m+r+1}^{(k)} = \sum_{i=1}^{m} a_i^{(k)} + \sum_{i=0}^{r} a_{n-i}^{(k)}, \\ S_{m+r+1}^{(k+1)} = \sum_{i=1}^{m} a_i^{(k+1)} + \sum_{i=0}^{r} a_{n-i}^{(k+1)}, \end{cases} \tag{19}$$

then

$$S_{m+r+1}^{(k)} \geq S_{m+r+1}^{(k+1)}. \tag{20}$$

Proof By a simple calculation we obtain

$$S_{m+r+1}^{(k)} - S_{m+r+1}^{(k+1)} = \frac{1}{k+1}\left(S_{m+r+1}^{(k)} - \sum_{i=k-r}^{k+m} a_i\right).$$

By (19) we have

$$a_{m+1}^{(k)} \leq a_{n-r}^{(k)}.$$

It follows that

$$a_{m+1} + a_{m+2} + \cdots + a_{k+m} \leq a_{n-r} + a_{n-r+1} + \cdots + a_{n-r+k-1}.$$

So we have

$$a_{k-r} + a_{k-r+1} + \cdots + a_{k+m} \leq a_{n-r} + a_{n-r+1} + \cdots + a_{n+m}.$$

Note that

$$\begin{cases} a_{n-r} + a_{n-r+1} + \cdots + a_{n+m} \leq a_{n-r+j} + a_{n-r+1+j} + \cdots + a_{n+m+j}, & 0 \leq j \leq r, \\ a_{k-r} + a_{k-r+1} + \cdots + a_{k+m} \leq a_{n-r+j} + a_{n-r+1+j} + \cdots + a_{n+m+j}, & r+1 \leq j \leq k-1. \end{cases}$$

Thus we can induce that

$$S_{m+r+1}^{(k)} = \sum_{i=1}^{m} a_i^{(k)} + \sum_{i=0}^{r} a_{n-i}^{(k)} = \frac{1}{k}\sum_{i=n-r}^{n+m}\sum_{j=0}^{k-1} a_{j+i} = \frac{1}{k}\sum_{j=0}^{k-1}\sum_{i=n-r}^{n+m} a_{j+i} \geq \sum_{i=k-r}^{k+m} a_i.$$

This means that (20) holds. □

Lemma 2.9 *Let $n \geq 4$, $2 \leq k \leq n-2$, $2 \leq m \leq n-k$, and $-1 \leq r \leq k-2$, and let $\sum_{i=0}^{-1} a_{n-i}^{(k)} = 0$. If*

$$\begin{cases} S_{m+r+1}^{(k)} = \sum_{i=1}^{m} a_i^{(k)} + \sum_{i=0}^{r} a_{n-i}^{(k)}, \\ S_{m+r+1}^{(k+1)} = \sum_{i=1}^{m-1} a_i^{(k+1)} + \sum_{i=0}^{r+1} a_{n-i}^{(k+1)}, \end{cases} \tag{21}$$

then

$$S_{m+r+1}^{(k)} \geq S_{m+r+1}^{(k+1)}. \tag{22}$$

Proof Note that

$$S_{m+r+1}^{(k)} - S_{m+r+1}^{(k+1)}$$

$$= \frac{1}{k+1}\left(\sum_{i=1}^{m-1} a_i^{(k)} + \sum_{i=0}^{r} a_{n-i}^{(k)}\right) + a_m^{(k)} - a_{n-r-1}^{(k+1)} - \frac{1}{k+1}\sum_{i=k-r}^{k+m-1} a_i$$
$$= \frac{1}{k+1}\left(\sum_{i=1}^{m} a_i^{(k)} + \sum_{i=0}^{r} a_{n-i}^{(k)} - \sum_{i=n-r-1}^{n+m-1} a_i\right)$$
$$= \frac{1}{k+1}\left(S_{m+r+1}^{(k)} - \sum_{i=n-r-1}^{n+m-1} a_i\right).$$

By (21) we have

$$a_{n-r-1}^{(k)} \le a_m^{(k)}.$$

It follows that

$$a_{n-r-1} + a_{n-r} + \cdots + a_{n-r+k-2} \le a_m + a_{m+1} + \cdots + a_{m+k-1}.$$

So we can induce that

$$a_{n-r-1} + a_{n-r} + \cdots + a_{n+m-1} \le a_{k-r-1} + a_{k-r} + \cdots + a_{k+m-1}.$$

Since

$$\begin{cases} a_{n-r-1} + a_{n-r} + \cdots + a_{n+m-1} \le a_{n-r+j} + a_{n-r+1+j} + \cdots + a_{n+m+j}, & 0 \le j \le r, \\ a_{k-r-1} + a_{k-r} + \cdots + a_{k+m-1} \le a_{n-r+j} + a_{n-r+1+j} + \cdots + a_{n+m+j}, & r+1 \le j \le k-1, \end{cases}$$

we have

$$S_{m+r+1}^{(k)} = \frac{1}{k}\sum_{j=0}^{k-1}\sum_{i=n-r}^{n+m} a_{j+i} \ge \sum_{i=n-r-1}^{n+m-1} a_i.$$

This means that (22) holds. □

3 Main results

We are now in a position to prove our main results (1) in two cases: $k = 1$ and $2 \le k \le n-1$.

Theorem 3.1 *For any $n \ge 2$, we have*

$$a^{(2)} \prec a^{(1)} = a. \tag{23}$$

Proof It is clear that (23) holds if $n = 2$. Next, let $n \ge 3$. Then we have

$$\frac{a_1 + a_2}{2} = S_1^{(2)} \le S_1^{(1)} = a_1, \qquad S_n^{(2)} = S_n^{(1)}.$$

For $2 \le m \le n-1$, we prove that $S_m^{(2)} \le S_m^{(1)}$ in the following two cases:

(i) If $S_m^{(2)} = \sum_{i=1}^{m} a_i^{(2)}$, then

$$S_m^{(2)} - S_m^{(1)} = \frac{a_{m+1} - a_1}{2} \le 0.$$

(ii) If $S_m^{(2)} = a_n^{(2)} + \sum_{i=1}^{m-1} a_i^{(2)}$, then

$$S_m^{(2)} - S_m^{(1)} = \frac{a_n - a_m}{2} \leq 0.$$

So (23) holds. □

Theorem 3.2 *For any $n \geq 3$ and $2 \leq k \leq n-1$, we have*

$$a^{(k+1)} \prec a^{(k)}. \tag{24}$$

Proof It is clear that (24) holds for any $n \geq 3$, $k = n-1$ and for $n = 3$, $k = 1$.

Next, let $n \geq 4$ and $2 \leq k \leq n-2$.

For any $1 \leq m \leq n-k$ and $-1 \leq r \leq k-2$, let $\sum_{i=0}^{-1} a_{n-i}^{(k)} = 0$, and let

$$S_{m+r+1}^{(k)} = \sum_{i=1}^{m} a_i^{(k)} + \sum_{i=0}^{r} a_{n-i}^{(k)}.$$

Next, we prove that

$$S_{m+r+1}^{(k)} \geq S_{m+r+1}^{(k+1)}$$

in the following two cases:

(i) If $m = 1$ and $-1 \leq r \leq k-2$, then by Corollary 2.7(i) and Lemma 2.8 we get

$$S_{r+2}^{(k)} \geq S_{r+2}^{(k+1)}.$$

(ii) If $2 \leq m \leq n-k$ and $-1 \leq r \leq k-2$, then by Corollary 2.7(ii), Lemma 2.8, and Lemma 2.9 we get

$$S_{m+r+1}^{(k)} \geq S_{m+r+1}^{(k+1)}.$$

Note that

$$S_n^{(k)} = S_n^{(k+1)},$$

so (24) holds. □

4 Discussion

In the theory of majorizations, there are two key concepts, majorizing relations and Schur-convex functions. Majorizing relations are weaker ordered relations among vectors, and Shur-convex functions are an extension of classical convex functions. Combining these two objects is an effective method of constructing inequalities.

In the theory of majorization, there are two important and fundamental objects, establishing majorizing relations among vectors and finding various Schur-convex functions. Majorizing relations deeply characterize intrinsic connections among vectors, and combining a new majorizing relation with suitable Schur-convex functions can lead to various interesting inequalities; see [3–13].

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References

1. Marshall, A.W., Olkin, I., Arnord, B.C.: Inequalities: Theory of Majorization and Its Application, 2nd edn. Springer, New York (2011)
2. Shi, H.N.: A class of majorization inequality with applications. J. Beijing Union Univ. (Nat. Sci.) **24**(1), 60–64 (2010)
3. Wu, S.H., Shi, H.N.: A relation of weak majorization and its applications to certain inequalities for means. Math. Slovaca **61**(4), 561–570 (2011)
4. Shi, H.N., Jiang, Y.M., Jiang, W.D.: Schur-convexity and Schur-geometrically concavity of Gini means. Comput. Math. Appl. **57**, 266–274 (2009)
5. Shi, H.N., Wu, S.H., Qi, F.: An alternative note on the Schur-convexity of the extended mean values. Math. Inequal. Appl. **9**, 219–224 (2006)
6. Shi, H.N.: Schur-convex functions related to Hadamard-type inequalities. J. Math. Inequal. **1**, 127–136 (2007)
7. Shi, H.N., Zhang, J.: Compositions involving Schur geometrically convex functions. J. Inequal. Appl. **2015**, 320 (2015). https://doi.org/10.1186/s13660-015-0842-x
8. Shi, H.N., Zhang, J.: Compositions involving Schur harmonically convex functions. J. Comput. Anal. Appl. **22**(5), 907–922 (2017)
9. Shi, H.N., Zhang, J.: Some new judgment theorems of Schur geometric and Schur harmonic convexities for a class of symmetric functions. J. Inequal. Appl. **2013**, 527 (2013). https://doi.org/10.1186/1029-242X-2013-527
10. Shi, H.N., Zhang, J.: Schur-convexity of dual form of some symmetric functions. J. Inequal. Appl. **2013**, 295 (2013). https://doi.org/10.1186/1029-242X-2013-295
11. Shi, H.N., Zhang, J.: Schur-convexity, Schur-geometric and harmonic convexities of dual form of a class symmetric functions. J. Math. Inequal. **8**(2), 349–358 (2014)
12. Shi, H.N., Zhang, J., Ma, Q.H.: Schur-convexity, Schur-geometric and Schur-harmonic convexity for a composite function of complete symmetric function. SpringerPlus **5**, 296 (2016). https://doi.org/10.1186/s40064-016-1940-z
13. Shi, H.N., Li, D.M., Gu, C.: Schur-convexity of a mean of convex function. Appl. Math. Lett. **22**(6), 932–937 (2009)

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