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Boundary value problems for hypergenic function vectors

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Abstract

This article mainly studies the boundary value problems for hypergenic function vectors in Clifford analysis. Firstly, some properties of hypergenic quasi-Cauchy type integrals are discussed. Then, by the Schauder fixed point theorem the existence of the solution to the nonlinear boundary value problem is proved. Finally, using the compression mapping principle the existence and uniqueness of the solution to the linear boundary value problem are proved.

Keywords: Clifford analysis; Hypergenic function vector; Nonlinear boundary value problem; Linear boundary value problem

1 Introduction

A Clifford algebra is an associative and noncommutable algebra [1]. In 1982, Brackx, Delanghe and Sommen [2] established the theoretical basis of Clifford analysis. In recent years, Clifford analysis has been widely used in physics and in mathematics [3–5]. Eriksson [6–8], Huang [9, 10], Qiao [11, 12], Xie [13–17] and Yang [18, 19] have done a lot of work in Clifford analysis. In 1996, Huang [10] studied the nonlinear boundary value problem for biregular functions in Clifford analysis. In 2000, Cai, Huang and Qiao [20] studied the nonlinear boundary value problem for biregular functions vector in Clifford analysis. In 2003, Xie, Qiao and Jiao [20] studied a nonlinear boundary value problem for a generalized biregular function vector. In 2005, Qiao [11] discussed a linear boundary value problem for hypermonogenic functions in Clifford analysis. In 2009–2010, Eriksson and Orelma [6, 7] studied hypergenic functions in the real Clifford algebra $Cl_{n+1,0}(\mathbb{R})$ and its Cauchy integral formula was given. In 2014, Xie [14, 15] studied the Cauchy integral for dual k-hypergenic functions and the boundary properties of the hypergenic quasi-Cauchy integral in real Clifford analysis were given. In 2016, Xie, Zhang and Tang [17] discussed some properties of k-hypergenic functions.

On the basis of the above, the boundary value problems for hypergenic function vectors are proved.

2 Preliminaries

See [6]; let $Cl_{n+1,0}(\mathbb{R})$ be a real Clifford algebra and have identity element $e_\varnothing = 1$ and basis elements $e_0, e_1, \ldots, e_n; e_0e_1, \ldots, e_{n-1}e_n; \ldots; e_0e_1\cdots e_n$, and satisfy

$$\begin{cases} e_ie_j = -e_je_i, \quad i \neq j, i,j = 0,1,\ldots,n; \\ e_j^2 = +1, \quad j = 0,1,\ldots,n. \end{cases}$$

Any element in $Cl_{n+1,0}(\mathbb{R})$ has the form $a = \sum_A a_Ae_A$, $e_A = e_{\alpha_1}e_{\alpha_2}\cdots e_{\alpha_h}$ or $e_\varnothing = 1$, where $A = \{\alpha_1, \alpha_2, \ldots, \alpha_h\}$, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_h \leq n$, $a_A \in \mathbb{R}$. The norm of $a \in Cl_{n+1,0}(\mathbb{R})$ is defined as $|a| = (\sum_A |a_A|^2)^{\frac{1}{2}}$. In this paper J_i ($i = 1, 2, \ldots, 32$) is a positive constant. For any $a, b \in Cl_{n+1,0}(\mathbb{R})$, we have

$$|a + b| \leq |a| + |b|, \qquad |ab| \leq J_1|a||b|. \tag{1}$$

If $a = a_0e_0 + a_1e_1 + \cdots + a_ne_n$, it may be observed that $a^2 = |a|^2$ and when $a \neq 0$ the inverse of a is $a^{-1} = \frac{a}{|a|^2}$. See [6]; any element $a \in Cl_{n+1,0}(\mathbb{R})$ can be uniquely decomposed as $a = b + e_0c$, where $b, c \in Cl_{n,0}(\mathbb{R})$. As regards decomposition we can define the mappings $P_0 : Cl_{n+1,0} \to Cl_{n,0}$ and $Q_0 : Cl_{n+1,0} \to Cl_{n,0}$ by $P_0a = b$, $Q_0a = c$, where b, c are called the P_0 part and the Q_0 part of a, respectively.

Let Ω_0 be a nonempty open connected set in R^{n+1}. The function $f : \Omega_0 \to Cl_{n+1,0}(\mathbb{R})$ is denoted by $f(x) = \sum_A f_A(x)e_A$, where $f_A \in \mathbb{R}$. The function $f : \Omega_0 \to Cl_{n+1,0}(\mathbb{R})$ is said to be continuous on Ω_0 if and only if each component $f_A(x)$ of $f(x)$ is continuous on Ω_0. Suppose $C^r(\Omega_0, Cl_{n+1,0}(\mathbb{R})) = \{f \mid f : \Omega_0 \to Cl_{n+1,0}(\mathbb{R}), f(x) = \sum_A f_A(x)e_A$, where f_A is r-times continuously differentiable on Ω_0 and $r \in \mathbb{N}^*\}$.

For $f \in C^1(\Omega_0, Cl_{n+1,0}(\mathbb{R}))$, we introduce Dirac operators as follows [6]:

$$Df = \sum_{l=0}^{n} e_l \frac{\partial f}{\partial x_l}.$$

Definition 2.1 ([15]) A Lyapunov surface S is a surface satisfying the following three conditions:

(1) Through each point in S, there is a tangent plane.

(2) There is a real constant number d such that, for any $N_0 \in S$, E is a ball with radius d, centered at N_0, and E is divided into two parts by S, the part of S lying in the interior of E is denoted by S', the other is in the exterior of S: and each straight line parallel to the normal direction of S at N_0 intersects it at one point.

(3) If the angle $\theta(N_1, N_2)$ between outward normal vectors through N_1, N_2 is an acute angle and r is the distance between N_1 and N_2, then there are two numbers β, α ($0 \leq \alpha \leq 1$, $\beta > 0$) independent of N_1, N_2 such that $\theta(N_1, N_2) \leq \beta r^\alpha$.

Definition 2.2 ([15]) The function f: $\partial\Omega_0 \longrightarrow Cl_{n+1,0}(\mathbb{R})$ is said to be Hölder continuous on Ω_0 if there exists a positive constant M_0 such that $|f(x_1) - f(x_2)| \leq M_0|x_1 - x_2|^\beta$ ($0 < \beta < 1$) holds for any $x_1, x_2 \in \partial\Omega_0$.

The set of all Hölder continuous functions which are defined on Ω_0 and valued in $Cl_{n+1,0}(\mathbb{R})$ is denoted by $H(\beta, \partial\Omega_0, Cl_{n+1,0}(\mathbb{R}))$.

For any $f \in H(\beta, \partial\Omega_0, Cl_{n+1,0}(\mathbb{R}))$, we define the norm of f as $\|f\|_\beta = C(f, \partial\Omega_0) + H(f, \partial\Omega_0, \beta)$, where

$$C(f, \partial\Omega_0) = \max_{t \in \partial\Omega_0} |f(t)|, \qquad H(f, \partial\Omega_0, \beta) = \sup_{\substack{t_1 \neq t_2 \\ t_1, t_2 \in \partial\Omega_0}} \frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|^\beta}.$$

It is easy to prove that $H(\beta, \partial\Omega_0, Cl_{n+1,0}(\mathbb{R}))$ forms a Banach space.

For any $f, g \in H(\beta, \partial\Omega_0, Cl_{n+1,0}(\mathbb{R}))$, we have

$$\|f + g\|_\beta \leq \|f\|_\beta + \|g\|_\beta, \qquad \|fg\|_\beta \leq J_2 \|f\|_\beta \|g\|_\beta. \tag{2}$$

In this paper, let Ω be a domain in $\mathbb{R}^{n+1}_+ = \{x \mid (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}, x_0 > 0\}$, and its boundary $\partial\Omega$ be a smooth compact oriented Lyapunov surface. For any $f \in C^1(\Omega, Cl_{n+1,0}(\mathbb{R}))$, we introduce a modified Dirac operator as follows [6]:

$$Hf = Df - \frac{n-1}{x_0} Q_0 f.$$

Definition 2.3 ([6]) $f(x)$ is said to be a hypergenic function on Ω if $f \in C^1(\Omega, Cl_{n+1,0}(\mathbb{R}))$ satisfies $Hf = 0$ on Ω.

In this paper, let $E_1(x,y) = \frac{x-y}{|x-y|^{n+1}|x-\widehat{y}|^{n-1}}$, $E_2(x,y) = \frac{\widehat{x}-y}{|x-y|^{n-1}|x-\widehat{y}|^{n+1}}$, and w_{n+1} is the surface area of the unit hypersphere in $\mathbb{R}^{n+1}$.

Definition 2.4 ([15])

$$\Psi_f(y) = \frac{2^{n-1} y_0^{n-1}}{w_{n+1}} \left[\int_{\partial\Omega} E_1(x,y)\, d\sigma(x) f(x) - \int_{\partial\Omega} E_2(x,y) \widehat{d\sigma(x)} \widehat{f(x)} \right] \tag{3}$$

is called a hypergenic quasi-Cauchy type integral if $f \in H(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$.

Lemma 2.1 ([14]) *If $y \notin \partial\Omega$, $f \in H(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$, the hypergenic quasi-Cauchy type integral*

$$\Psi_f(y) = \frac{2^{n-1} y_0^{n-1}}{w_{n+1}} \left[\int_{\partial\Omega} E_1(x,y)\, d\sigma(x) f(x) - \int_{\partial\Omega} E_2(x,y) \widehat{d\sigma(x)} \widehat{f(x)} \right]$$

is a hypergenic function on $\mathbb{R}^{n+1}_+ \backslash \partial\Omega$.

Remark 2.1 If $y \notin \partial\Omega$, $f \in H(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$, the hypergenic quasi-Cauchy type integral

$$\Psi_f(y) = \frac{2^{n-1} y_0^{n-1}}{w_{n+1}} \left[\int_{\partial\Omega} E_1(x,y)\, d\sigma(x) f(x) - \int_{\partial\Omega} E_2(x,y) \widehat{d\sigma(x)} \widehat{f(x)} \right]$$

satisfies $\Psi_f(\infty) = 0$.

Lemma 2.2 ([15]) *If $f \in H(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$, then $\Psi_f(y)$ is Hölder continuous on $\Omega^+ \cup \partial\Omega$ and $\Omega^- \cup \partial\Omega$.*

Let $\mathbf{B}(\mathbf{y},\delta)$ be a ball with radius $\delta > 0$, centered at y when $y \in \partial\Omega$. $\partial\Omega$ is divided into two parts by $\mathbf{B}(\mathbf{y},\delta)$. The part of $\partial\Omega$ lying in the interior of $\mathbf{B}(\mathbf{y},\delta)$ is denoted by λ_δ.

Definition 2.5 ([15]) $\mathbf{I}$ is called the Cauchy principal of the singular integral value if $\lim_{\delta\to 0}\Psi_f(y) = \mathbf{I}$ exists, and we write directly $\mathbf{I} = \Phi_f(y)$.

Lemma 2.3 ([15]) *If $y \in \partial\Omega, f \in H(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$, then the Cauchy principal values of the singular integral* (3) *exist, and*

$$\begin{aligned}\Phi_f(y) = \frac{2^{n-1}y_0^{n-1}}{w_{n+1}}\biggl[\int_{\partial\Omega} E_1(x,y)\,d\sigma(x)\bigl(f(x)-f(y)\bigr) \\ - \int_{\partial\Omega} E_2(x,y)\widehat{d\sigma(x)}\bigl(\widehat{f(x)}-f(y)\bigr)\biggr] + \frac{1}{2}f(y),\end{aligned}\tag{4}$$

when $f = 1$, we have $\Phi_1(y) = \frac{1}{2}$.

Lemma 2.4 ([15]) *If $y \in \partial\Omega, f \in H(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$, then*

$$\begin{cases}\Psi_f^+(y) = \Phi_f(y) + \frac{1}{2}f(y), \\ \Psi_f^-(y) = \Phi_f(y) - \frac{1}{2}f(y).\end{cases}\tag{5}$$

Lemma 2.5 ([5]) *If Ω is a bounded domain in $\mathbb{R}_+^{n+1}$, $0 < \alpha < n+1$, for any $y \in \Omega$ we have*

$$\int_\Omega |x-y|^{-\alpha}\,dx \le M_1(\alpha,\Omega),$$

where $M_1(\alpha,\Omega)$ is a positive constant only related to α, Ω.

Definition 2.6 $F(x) = (f_1(x),\dots,f_q(x))$ is called a function vector if $f_i(x) : \Omega \to Cl_{n+1,0}(\mathbb{R})$ $(i = 1,\dots,q)$.

For $F(x) = (f_1(x),\dots,f_q(x))$, $K(x) = (k_1(x),\dots,k_q(x))$, define the addition operation and multiplication operation for function vectors as follows:

$$F \oplus K = (f_1 + k_1,\dots,f_q + k_q); F \otimes K = (f_1k_1,\dots,f_qk_q).$$

Let $L(x)$ be a function valued in Clifford algebra $Cl_{n+1,0}(\mathbb{R})$ and $F(x)$ be a function vector, then

$$LF = (Lf_1,\dots,Lf_q), \qquad FL = (f_1L,\dots,f_qL).$$

Define the model of a function vector as follows: $|F(x)| = (\sum_{i=0}^q |f_i(x)|^2)^{\frac{1}{2}}$, we have

$$|F \oplus K| \le |F| + |K|, \qquad |F \otimes K| \le J_1|F||K|..\tag{6}$$

Definition 2.7 $F(x) = (f_1(x),\dots,f_q(x))$ is called a hypergenic function vector when each component $f_i(x)$ $(i = 1,\dots,q)$ is a hypergenic function on Ω.

Definition 2.8 A hypergenic function vector F is said to be Hölder continuous on $\partial\Omega$ if there is a positive constant M_2 such that

$$\left|F(x_1)-F(x_2)\right| = \left(\sum_{i=0}^{q}\left|f_i(x_1)-f_i(x_2)\right|^2\right)^{\frac{1}{2}} \le M_2|x_1-x_2|^{\beta}$$

holds for any $x_1, x_2 \in \Omega$, where $0 < \beta < 1$ and M_2 is independent of x_i $(i = 1, 2)$.

Remark 2.2 The hypergenic function vector $F(x) = (f_1(x), \dots, f_q(x))$ is Hölder continuous on $\partial\Omega$ if and only if each component $f_i(x)$ $(i = 1, \dots, q)$ of $F(x)$ is Hölder continuous on $\partial\Omega$.

The set of all Hölder continuous function vectors which are defined on $\partial\Omega$ and valued in $Cl_{n+1,0}(\mathbb{R})$ is denoted by $H_q(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$. For any $F \in H_q(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$, the norm of F is defined as follows: $\|F\|_\beta = C_q(F, \partial\Omega) + H_q(F, \partial\Omega, \beta)$, where

$$C_q(F,\partial\Omega) = \max_{t\in\partial\Omega}\left|F(t)\right|, \qquad H_q(F,\partial\Omega,\beta) = \sup_{\substack{t_1\neq t_2\\ t_1,t_2\in\partial\Omega}} \frac{|f(t_1)-f(t_2)|}{|t_1-t_2|^{\beta}}.$$

It is easy to prove that $H_q(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$ forms a Banach space.

For any $F, K \in H_q(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$, we have

$$\|F+K\|_\beta \le \|F\|_\beta + \|K\|_\beta, \qquad \|F\otimes K\|_\beta \le J_3\|F\|_\beta\|K\|_\beta. \tag{7}$$

3 Some properties of hypergenic quasi-Cauchy type integrals

Theorem 3.1 *If $y \notin \partial\Omega$, $F(x) = (f_1(x), \dots, f_q(x)) \in H_q(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$,*

$$\Psi_F(y) = \frac{2^{n-1}y_0^{n-1}}{w_{n+1}}\left[\int_{\partial\Omega}E_1(x,y)\,d\sigma(x)F(x) - \int_{\partial\Omega}E_2(x,y)\widehat{d\sigma(x)}\widehat{F(x)}\right] \tag{8}$$

is a hypergenic function vector on $\mathbb{R}^{n+1}_+\backslash\partial\Omega$, $\Psi_F(\infty) = 0$, and $\Psi_F(y)$ is Hölder continuous on $\Omega^\pm \cup \partial\Omega$.

Proof

$$\begin{aligned}\Psi_F(y) &= \left(\frac{2^{n-1}y_0^{n-1}}{w_{n+1}}\left[\int_{\partial\Omega}E_1(x,y)\,d\sigma(x)f_1(x) - \int_{\partial\Omega}E_2(x,y)\widehat{d\sigma(x)}\widehat{f_1(x)}\right],\right.\\ &\qquad \left.\dots, \frac{2^{n-1}y_0^{n-1}}{w_{n+1}}\left[\int_{\partial\Omega}E_1(x,y)\,d\sigma(x)f_q(x) - \int_{\partial\Omega}E_2(x,y)\widehat{d\sigma(x)}\widehat{f_q(x)}\right]\right)\\ &= \left(\Psi_{f_1}(y), \dots, \Psi_{f_q}(y)\right).\end{aligned}$$

It follows from $F(x) = (f_1(x), \dots, f_q(x)) \in H_q(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$ that $f_i \in H_q(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$ $(i = 1, \dots, q)$. From Lemma 2.1, $\Psi_{f_i}(y)$ $(i = 1, \dots, q)$ is a hypergenic function on $R^{n+1}_+ \setminus \partial\Omega$. Hence $\Psi_F(y)$ is a hypergenic function vector on $R^{n+1}_+ \setminus \partial\Omega$. By Lemma 2.2 and Remark 2.2 $\Psi_F(y)$ is Hölder continuous on $\Omega^\pm \cup \partial\Omega$. By Remark 2.1 we conclude $\Psi_{f_i}(\infty) = 0$ $(i = 1, \dots, q)$, thus $\Psi_F(\infty) = 0$. □

Theorem 3.2 *If* $y \in \partial\Omega, F \in H_q(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$, *then*

$$\Phi_F(y) = \frac{2^{n-1}y_0^{n-1}}{w_{n+1}}\left[\int_{\partial\Omega} E_1(x,y)\,d\sigma(x)\big(F(x) - F(y)\big) - \int_{\partial\Omega} E_2(x,y)\widehat{d\sigma(x)}\big(\widehat{F(x)} - F(y)\big)\right] + \frac{1}{2}F(y). \tag{9}$$

From Lemma 2.4 we conclude to the following theorem.

Theorem 3.3

$$\begin{cases} \Psi_F^+(y) = \Phi_F(y) + \frac{1}{2}F(y), \\ \Psi_F^-(y) = \Phi_F(y) - \frac{1}{2}F(y). \end{cases} \tag{10}$$

Lemma 3.1 ([15]) *If* $x, y \in \mathbb{R}^{n+1}$ $(n \geq 2)$, m (≥ 0) *is an integer, then*

$$\left|\frac{x}{|x|^{m+2}} - \frac{y}{|y|^{m+2}}\right| \leq \frac{P_m(x,y)}{|x|^{m+1}|y|^{m+1}}|x-y|,$$

where

$$P_m(x,y) = \begin{cases} \sum_{l=0}^{m} |x|^{m-l}|y|^l, & m \neq 0; \\ 1, & m = 0. \end{cases}$$

Theorem 3.4 *If* $y \in \partial\Omega$, $F \in H_q(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$, $Q(y) = \frac{1}{2}F(y) - \Phi_F(y)$, *then*

$$\|Q(y)\|_\beta \leq J_{31}\|F\|_\beta.$$

Proof Similar to Ref. [15], we have

$$|d\sigma(x)| \leq M_3\rho^{n-1}\,d\rho,$$

where M_3 is a positive constant.

From Theorem 3.2, Lemma 2.3 and Lemma 2.4, we get

$$\begin{aligned}
&|Q(y)| \\
&= \left|\frac{2^{n-1}y_0^{n-1}}{w_{n+1}}\left[\int_{\partial\Omega} E_1(x,y)\,d\sigma(x)F(y) - \int_{\partial\Omega} E_2(x,y)\widehat{d\sigma(x)}F(y)\right]\right. \\
&\quad \left. - \frac{2^{n-1}y_0^{n-1}}{w_{n+1}}\left[\int_{\partial\Omega} E_1(x,y)\,d\sigma(x)F(x) - \int_{\partial\Omega} E_2(x,y)\widehat{d\sigma(x)}\widehat{F(x)}\right]\right| \\
&\leq \left|\frac{2^{n-1}y_0^{n-1}}{w_{n+1}}\right|\left[\left|\int_{\partial\Omega} E_1(x,y)\,d\sigma(x)\big(F(y) - F(x)\big)\right| + \left|\int_{\partial\Omega} E_2(x,y)\widehat{d\sigma(x)}\big(\widehat{F(x)} - F(y)\big)\right|\right] \\
&\leq J_4 H_q(F, \partial\Omega, \beta)\int_{\partial\Omega} |E_1(x,y)||d\sigma(x)||y-x|^\beta + 2\max_{x\in\partial\Omega}|F(x)|\int_{\partial\Omega}|E_2(x,y)||\widehat{d\sigma(x)}| \\
&\leq J_5 H_q(F, \partial\Omega, \beta)\int_{\partial\Omega} \frac{1}{|x-y|^n}|d\sigma(x)||y-x|^\beta + 2\max_{x\in\partial\Omega}|F(x)|\int_{\partial\Omega}\frac{1}{|x-y|^{n-1}}|\widehat{d\sigma(x)}|
\end{aligned}$$

$$\leq J_6 H_q(F, \partial\Omega, \beta) + J_7 C_q(F, \partial\Omega)$$
$$\leq J_8 \|F\|_\beta.$$

So

$$C_q(Q, \partial\Omega) \leq J_8 \|F\|_\beta. \tag{11}$$

Next we consider $H_q(Q, \partial\Omega, \beta)$.

There is a ball with radius 3δ, centered at y^1 when $y^1, y^2 \in \partial\Omega$ and $6\delta < d$, $\delta = |y^1 - y^2|$. Remark that $\partial\Omega_1$ is located inside the ball and the rest of $\partial\Omega$ is $\partial\Omega_2$.

From equality (9) and (1), we have

$$\begin{aligned}
&\left|Q(y^1) - Q(y^2)\right| \\
&= \left|\frac{1}{2}F(y^1) - \Phi_F(y^1) - \left(\frac{1}{2}F(y^2) - \Phi_F(y^2)\right)\right| \\
&= \left|\frac{2^{n-1}(y_0^1)^{n-1}}{w_{n+1}}\left[\int_{\partial\Omega} E_1(x, y^1)\, d\sigma(x)\big(F(y^1) - F(x)\big)\right.\right. \\
&\quad \left. + \int_{\partial\Omega} E_2(x, y^1)\widehat{d\sigma(x)}\big(\widehat{F(x)} - F(y^1)\big)\right] \\
&\quad - \frac{2^{n-1}(y_0^2)^{n-1}}{w_{n+1}}\left[\int_{\partial\Omega} E_1(x, y^2)\, d\sigma(x)\big(F(y^2) - F(x)\big)\right. \\
&\quad \left.\left. + \int_{\partial\Omega} E_2(x, y^2)\widehat{d\sigma(x)}\big(\widehat{F(x)} - F(y^2)\big)\right]\right| \\
&= \left|\frac{2^{n-1}(y_0^1)^{n-1}}{w_{n+1}}\left[\int_{\partial\Omega_1} E_1(x, y^1)\, d\sigma(x)\big(F(y^1) - F(x)\big)\right.\right. \\
&\quad \left. + \int_{\partial\Omega_1} E_2(x, y^1)\widehat{d\sigma(x)}\big(\widehat{F(x)} - F(y^1)\big)\right] \\
&\quad + \frac{2^{n-1}(y_0^1)^{n-1}}{w_{n+1}}\left[\int_{\partial\Omega_2} E_1(x, y^1)\, d\sigma(x)\big(F(y^1) - F(x)\big)\right. \\
&\quad \left. + \int_{\partial\Omega_2} E_2(x, y^1)\widehat{d\sigma(x)}\big(\widehat{F(x)} - F(y^1)\big)\right] \\
&\quad - \frac{2^{n-1}(y_0^2)^{n-1}}{w_{n+1}}\left[\int_{\partial\Omega_1} E_1(x, y^2)\, d\sigma(x)\big(F(y^2) - F(x)\big)\right. \\
&\quad \left. + \int_{\partial\Omega_1} E_2(x, y^2)\widehat{d\sigma(x)}\big(\widehat{F(x)} - F(y^2)\big)\right] \\
&\quad - \frac{2^{n-1}(y_0^2)^{n-1}}{w_{n+1}}\left[\int_{\partial\Omega_2} E_1(x, y^2)\, d\sigma(x)\big(F(y^2) - F(x)\big)\right. \\
&\quad \left.\left. + \int_{\partial\Omega_2} E_2(x, y^2)\widehat{d\sigma(x)}\big(\widehat{F(x)} - F(y^2)\big)\right]\right| \\
&\leq \left|\frac{2^{n-1}(y_0^1)^{n-1}}{w_{n+1}}\right|\left|\int_{\partial\Omega_1} E_1(x, y^1)\, d\sigma(x)\big(F(y^1) - F(x)\big)\right. \\
&\quad \left. + \int_{\partial\Omega_1} E_2(x, y^1)\widehat{d\sigma(x)}\big(\widehat{F(x)} - F(y^1)\big)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{2^{n-1}(y_0^2)^{n-1}}{w_{n+1}}\right|\left|\int_{\partial\Omega_1} E_1(x,y^2)\,d\sigma(x)\big(F(y^2)-F(x)\big)\right.\\
&\left.+\int_{\partial\Omega_1} E_2(x,y^2)\widehat{d\sigma(x)}\big(\widehat{F(x)}-F(y^2)\big)\right|\\
&+\left|\frac{2^{n-1}(y_0^1)^{n-1}}{w_{n+1}}\left[\int_{\partial\Omega_2} E_1(x,y^1)\,d\sigma(x)\big(F(y^1)-F(x)\big)\right.\right.\\
&\left.+\int_{\partial\Omega_2} E_2(x,y^1)\widehat{d\sigma(x)}\big(\widehat{F(x)}-F(y^1)\big)\right]\\
&-\frac{2^{n-1}(y_0^2)^{n-1}}{w_{n+1}}\left[\int_{\partial\Omega_2} E_1(x,y^2)\,d\sigma(x)\big(F(y^2)-F(x)\big)\right.\\
&\left.\left.+\int_{\partial\Omega_2} E_2(x,y^2)\widehat{d\sigma(x)}\big(\widehat{F(x)}-F(y^2)\big)\right]\right|\\
&=I_1+I_2+I_3;
\end{aligned}
$$

$$
\begin{aligned}
I_1 &\le J_9\left[\int_{\partial\Omega_1}\big|E_1(x,y^1)\big|\big|d\sigma(x)\big|\big|F(y^1)-F(x)\big| + \int_{\partial\Omega_1}\big|E_2(x,y^1)\big|\big|\widehat{d\sigma(x)}\big|\big|\widehat{F(x)}-F(y^1)\big|\right]\\
&\le J_{10}\int_0^{3\delta}\frac{1}{|x-y^1|^n|x-\widehat{y^1}|^{n-1}}\rho^{n-1}H_q(F,\partial\Omega,\beta)\big|y^1-x\big|^\beta d\rho\\
&\quad+2J_{10}\int_0^{3\delta}\frac{1}{|x-y^1|^{n-1}|x-\widehat{y^1}|^n}\rho^{n-1}C_q(F,\partial\Omega)d\rho\\
&\le J_{11}H_q(F,\partial\Omega,\beta)\int_0^{3\delta}\frac{1}{|x-y^1|^{n-\beta}}\rho^{n-1}\,d\rho + J_{12}C_q(F,\partial\Omega)\int_0^{3\delta}\frac{1}{|x-y^1|^{n-1}}\rho^{n-1}\,d\rho\\
&\le J_{11}H_q(F,\partial\Omega,\beta)\int_0^{3\delta}\rho^{\beta-1}\,d\rho + J_{12}C_q(F,\partial\Omega)\int_0^{3\delta}d\rho\\
&\le J_{13}\big(H_q(F,\partial\Omega,\beta)+C_q(F,\partial\Omega)\big)\big|y^1-y^2\big|^\beta\\
&\le J_{13}\|F\|_\beta\big|y^1-y^2\big|^\beta,
\end{aligned}
$$

that is,

$$
I_1 \le J_{13}\|F\|_\beta\big|y^1-y^2\big|^\beta. \tag{12}
$$

In a similar way, we have

$$
I_2 \le J_{14}\|F\|_\beta\big|y^1-y^2\big|^\beta, \tag{13}
$$

$$
\begin{aligned}
I_3 = &\left|\frac{2^{n-1}(y_0^1)^{n-1}}{w_{n+1}}\int_{\partial\Omega_2}\big[E_1(x,y^1)-E_1(x,y^2)\big]\,d\sigma(x)\big(F(y^1)-F(x)\big)\right.\\
&+\frac{2^{n-1}(y_0^1)^{n-1}}{w_{n+1}}\int_{\partial\Omega_2}E_1(x,y^2)\,d\sigma(x)\big(F(y^1)-F(x)\big)\\
&+\frac{2^{n-1}(y_0^1)^{n-1}}{w_{n+1}}\int_{\partial\Omega_2}\big[E_2(x,y^1)-E_2(x,y^2)\big]\widehat{d\sigma(x)}\big(\widehat{F(x)}-F(y^1)\big)\\
&+\frac{2^{n-1}(y_0^1)^{n-1}}{w_{n+1}}\int_{\partial\Omega_2}E_2(x,y^2)\widehat{d\sigma(x)}\big(\widehat{F(x)}-F(y^1)\big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2^{n-1}(y_0^2)^{n-1}}{w_{n+1}} \int_{\partial\Omega_2} E_1(x,y^2)\,d\sigma(x)\big(F(y^2)-F(x)\big) \\
&- \frac{2^{n-1}(y_0^2)^{n-1}}{w_{n+1}} \int_{\partial\Omega_2} E_2(x,y^2)\widehat{d\sigma(x)}\big(\widehat{F(x)}-F(y^2)\big)\bigg| \\
&\le J_{15}\bigg|\int_{\partial\Omega_2}\big[E_1(x,y^1)-E_1(x,y^2)\big]\,d\sigma(x)\big(F(y^1)-F(x)\big)\bigg| \\
&+ J_{15}\bigg|\int_{\partial\Omega_2}\big[E_2(x,y^1)-E_2(x,y^2)\big]\widehat{d\sigma(x)}\big(\widehat{F(x)}-F(y^1)\big)\bigg| \\
&+ \bigg|\frac{2^{n-1}(y_0^1)^{n-1}}{w_{n+1}} \int_{\partial\Omega_2} E_1(x,y^2)\,d\sigma(x)\big(F(y^1)-F(x)\big) \\
&- \frac{2^{n-1}(y_0^2)^{n-1}}{w_{n+1}} \int_{\partial\Omega_2} E_1(x,y^2)\,d\sigma(x)\big(F(y^2)-F(x)\big)\bigg| \\
&+ \bigg|\frac{2^{n-1}(y_0^1)^{n-1}}{w_{n+1}} \int_{\partial\Omega_2} E_2(x,y^2)\widehat{d\sigma(x)}\big(\widehat{F(x)}-F(y^1)\big) \\
&- \frac{2^{n-1}(y_0^2)^{n-1}}{w_{n+1}} \int_{\partial\Omega_2} E_2(x,y^2)\widehat{d\sigma(x)}\big(\widehat{F(x)}-F(y^2)\big)\bigg| \\
&= J_{15}(I_4+I_5)+I_6+I_7,
\end{aligned}
$$

that is,

$$
I_3 \le J_{15}(I_4+I_5)+I_6+I_7. \tag{14}
$$

Because $x \in \partial\Omega_2 \setminus \lambda_{3\delta}$, and $y^1, y^2 \in \partial\Omega_1$, $|\frac{x-y^2}{x-y^1}|^{l+1}$ and $|\frac{x-y^1}{x-y^2}|^{l+1}$ $(l = 0,1,\ldots,n)$ are continuous on $\partial\Omega_2$, there is a positive constant J_{16}, such that

$$
\left|\frac{x-y^2}{x-y^1}\right|^{l+1} \le J_{16}, \qquad \left|\frac{x-y^1}{x-y^2}\right|^{l+1} \le J_{16} \quad (l = 0,1,\ldots,n). \tag{15}
$$

From inequality (1), the Hile lemma and inequality (15), we get

$$
\begin{aligned}
I_4 &= \bigg|\int_{\partial\Omega_2}\big[E_1(x,y^1)-E_1(x,y^2)\big]\,d\sigma(x)\big(F(y^1)-F(x)\big)\bigg| \\
&= \bigg|\int_{\partial\Omega_2}\bigg(\frac{x-y^1}{|x-y^1|^{n+1}|x-\widehat{y^1}|^{n-1}} - \frac{x-y^2}{|x-y^2|^{n+1}|x-\widehat{y^2}|^{n-1}}\bigg)\,d\sigma(x)\big(F(y^1)-F(x)\big)\bigg| \\
&\le J_{17}\int_{\partial\Omega_2}\bigg|\frac{x-y^1}{|x-y^1|^{n+1}|x-\widehat{y^1}|^{n-1}} - \frac{x-y^2}{|x-y^2|^{n+1}|x-\widehat{y^2}|^{n-1}}\bigg|\big|d\sigma(x)\big|\big|F(y^1)-F(x)\big| \\
&\le J_{17}\int_{\partial\Omega_2}\frac{1}{|x-\widehat{y^1}|^{n-1}}\bigg|\frac{x-y^1}{|x-y^1|^{n+1}} - \frac{x-y^2}{|x-y^2|^{n+1}}\bigg|\big|d\sigma(x)\big|\big|F(y^1)-F(x)\big| \\
&\quad + J_{17}\int_{\partial\Omega_2}\frac{1}{|x-y^2|^{n+1}}\bigg|\frac{x-y^2}{|x-\widehat{y^1}|^{n-1}} - \frac{x-y^2}{|x-\widehat{y^2}|^{n-1}}\bigg|\big|d\sigma(x)\big|\big|F(y^1)-F(x)\big| \\
&\le \int_{\partial\Omega_2}\Bigg(J_{18}\sum_{l=0}^{n-1}\left|\frac{x-y^2}{x-y^1}\right|^{l+1}\frac{|y^1-y^2|}{|x-y^2|^{n+1}} + J_{19}\frac{|y^1-y^2|}{|x-y^2|^{n}}\Bigg)\big|d\sigma(x)\big|\big|F(y^1)-F(x)\big| \\
&\le J_{20}H_q(F,\partial\Omega,\beta)\int_{\partial\Omega_2}\bigg(\frac{1}{|x-y^2|^{n+1}} + \frac{1}{|x-y^2|^{n}}\bigg)\big|d\sigma(x)\big|\big|y^1-x\big|^{\beta}\big|y^1-y^2\big|
\end{aligned}
$$

$$\begin{aligned}
&\le H_q(Q,\partial\Omega,\beta)\left(J_{21}\int_{3\delta}^{L}\rho^{\beta-2}\,d\rho + J_{22}\int_{3\delta}^{L}\rho^{\beta-1}\,d\rho\right)\left|y^1-y^2\right| \\
&\le J_{23}H_q(F,\partial\Omega,\beta)\left|y^1-y^2\right|^{\beta} \\
&\le J_{24}\|F\|_{\beta}\left|y^1-y^2\right|^{\beta},
\end{aligned}$$

that is,

$$I_4 \le J_{24}\|F\|_{\beta}\left|y^1-y^2\right|^{\beta}. \tag{16}$$

In a similar way, we have

$$I_5 \le J_{25}\|F\|_{\beta}\left|y^1-y^2\right|^{\beta}, \tag{17}$$

$$\begin{aligned}
I_6 \le{}& \left|\frac{2^{n-1}[(y_0^2)^{n-1}-(y_0^1)^{n-1}]}{w_{n+1}}\int_{\partial\Omega_2}\frac{x-y^2}{|x-y^2|^{n+1}|x-\widehat{y^2}|^{n-1}}\,d\sigma(x)F(x)\right| \\
&+\left|\frac{2^{n-1}(y_0^1)^{n-1}}{w_{n+1}}\int_{\partial\Omega_2}\frac{x-y^2}{|x-y^2|^{n+1}|x-\widehat{y^2}|^{n-1}}\,d\sigma(x)\bigl(F(y^1)-F(y^2)\bigr)\right| \\
&+\left|\frac{2^{n-1}[(y_0^1)^{n-1}-(y_0^2)^{n-1}]}{w_{n+1}}\int_{\partial\Omega_2}\frac{x-y^2}{|x-y^2|^{n+1}|x-\widehat{y^2}|^{n-1}}\,d\sigma(x)F(y^2)\right|.
\end{aligned}$$

Because $\lim_{\delta\to 0}\frac{x-y^2}{|x-y^2|^{n+1}|x-\widehat{y^2}|^{n-1}}$ exists, there is a constant $\delta_2>0$, when $0<\delta<\delta_2$, such that

$$\left|\int_{\partial\Omega_2}\frac{(x-y^2)}{|x-y^2|^{n+1}|x-\widehat{y^2}|^{n-1}}\,d\sigma(x)\right| \le J_{26}.$$

Hence

$$\begin{aligned}
I_6 &\le J_{27}\bigl[2C_q(F,\partial\Omega)\bigl|\bigl(y_0^1\bigr)^{n-1}-\bigl(y_0^2\bigr)^{n-1}\bigr|+\bigl|F(y^1)-F(y^2)\bigr|\bigr] \\
&\le J_{28}\bigl[C_q(F,\partial\Omega)+H_q(F,\partial\Omega,\beta)\bigr]\left|y^1-y^2\right|^{\beta} \\
&= J_{28}\|F\|_{\beta}\left|y^1-y^2\right|^{\beta},
\end{aligned}$$

that is,

$$I_6 \le J_{28}\|F\|_{\beta}\left|y^1-y^2\right|^{\beta}. \tag{18}$$

In a similar way, we have

$$I_7 \le J_{29}\|F\|_{\beta}\left|y^1-y^2\right|^{\beta}. \tag{19}$$

From inequalities (14), (16), (17), (18) and (19), we have

$$|I_3| \le \bigl[J_{15}(J_{24}+J_{25})+J_{28}+J_{29}\bigr]\|F\|_{\beta}\left|y^1-y^2\right|^{\beta}. \tag{20}$$

From inequalities (12), (13) and (20), we have

$$\frac{|Q(y^1)-Q(y^2)|}{|y^1-y^2|^{\beta}} \le \bigl[J_{13}+J_{14}+J_{15}(J_{24}+J_{25})+J_{28}+J_{29}\bigr]\|F\|_{\beta} = J_{30}\|F\|_{\beta}.$$

So

$$H_q\big(Q(y),\partial\Omega,\beta\big)\le J_{30}\|F\|_\beta. \tag{21}$$

From inequalities (11) and (21), we have $\|Q(y)\|_\beta \le (J_8+J_{30})\|F\|_\beta = J_{31}\|F\|_\beta$. □

Remark 3.1 If $y\in\partial\Omega$, $F\in H_q(\beta,\partial\Omega,Cl_{n+1,0}(\mathbb{R}))$, then

$$\big\|\Phi_F(y)\big\|_\beta \le J_{32}\|F\|_\beta.$$

Remark 3.2 If $y\in\partial\Omega$, $F\in H_q(\beta,\partial\Omega,Cl_{n+1,0}(\mathbb{R}))$, then

$$\begin{cases}\|\Psi_F^+(y)\|_\beta \le J_{33}\|F\|_\beta,\\ \|\Psi_F^-(y)\|_\beta \le J_{33}\|F\|_\beta.\end{cases} \tag{22}$$

4 The existence of the solution to the nonlinear boundary value problem for the hypergenic function vector

Let $A(y),B(y),G(y)\in H_q(\beta,\partial\Omega,Cl_{n+1,0}(\mathbb{R}))$ be Hölder continuous function vectors on $\partial\Omega$, we find a function vector $\Psi_F^*(y)$, which is hypergenic on $\Omega^+\cup\Omega^-$, and continuous on $\Omega^+\cup\partial\Omega$ and $\Omega^-\cup\partial\Omega$, satisfying $\Psi_F^*(\infty)=0$, and the nonlinear boundary value condition:

$$A(y)\otimes\Psi_F^{*+}(y)+B(y)\otimes\Psi_F^{*-}(y)=G(y)\otimes P\big(\Psi_F^{*+}(y),\Psi_F^{*-}(y)\big), \tag{23}$$

where $P(\Psi_F^{*+}(y),\Psi_F^{*-}(y))$ is a Hölder continuous function vector on $\partial\Omega$ which is related to $\Psi_F^{*+}(y)$, $\Psi_F^{*-}(y)$.

The above problem is called the nonlinear boundary value problem *SR*. If $P=1$, then the above problem is called the linear boundary value problem *SR*.

By Theorem 3.1, $\Psi_F(y)$ is hypergenic on $\Omega^+\cup\Omega^-$, and $\Psi_F(y)$ is continuous on $\Omega^+\cup\partial\Omega$ and $\Omega^-\cup\partial\Omega$, and $\Psi_F(\infty)=0$. If $P(\Psi_F^+(y),\Psi_F^-(y))$ satisfies equality (23) under certain conditions, then $\Psi_F(y)$ is a solution to the nonlinear boundary value problem *SR*.

Putting (10) into (23), we have

$$A(y)\otimes\left(\Phi_F(y)+\frac{1}{2}F(y)\right)+B(y)\otimes\left(\Phi_F(y)-\frac{1}{2}F(y)\right)=G(y)\otimes P\big(\Psi_F^+(y),\Psi_F^-(y)\big). \tag{24}$$

Let

$$\begin{aligned}NF(y)={}&\big(A(y)+B(y)\big)\otimes\left(-\frac{1}{2}F(y)+\Phi_F(y)\right)+\big(1+A(y)\big)\otimes F(y)\\&-G(y)\otimes P\big(\Psi_F^+(y),\Psi_F^-(y)\big),\end{aligned} \tag{25}$$

and equality (23) is transformed into the following singular integral equation:

$$NF=F. \tag{26}$$

Theorem 4.1 *If $A(y), B(y), G(y) \in H_q(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$, for any $y^1, y^2 \in \partial\Omega$, $P(\Psi_F^+(y), \Psi_F^-(y))$ satisfies*

$$\begin{aligned} &\left|P\left(\Psi_F^+\left(y^1\right), \Psi_F^-\left(y^1\right)\right) - P\left(\Psi_F^+\left(y^2\right), \Psi_F^-\left(y^2\right)\right)\right| \\ &\quad \leq J_{34}\left|\Psi_F^+\left(y^1\right) - \Psi_F^+\left(y^2\right)\right| + J_{35}\left|\Psi_F^-\left(y^1\right) - \Psi_F^-\left(y^2\right)\right|, \end{aligned} \tag{27}$$

where J_{34} and J_{35} are positive constants independent of y^i $(i = 1,2)$ and F. If $P(0,0) = 0$, $0 < \gamma = J_{36}(\|A + B\|_\beta + \|1 + A\|_\beta) < 1$, $\|G(y)\|_\beta < \delta$, when $0 < \delta \leq \frac{1-\gamma}{J_3 J_{41}}$, Problem SR has at least one solution and the integral expression of the solution is (8).

Proof Let $T = \{F \mid \|F\|_\beta \leq M_4$ and F is uniformly Hölder continuous on $\partial\Omega$, that is, to say, there is a positive constant M_2, for any $x_1, x_2 \in \partial\Omega$, $F \in H_q(\beta, \partial\Omega, Cl_{n+1,0}(\mathbb{R}))$, we have $|F(x_1) - F(x_2)| \leq M_2|x_1 - x_2|^\beta\}$. Obviously T is a convex subset of the continuous function vector space $C_q(\partial\Omega)$.

(1) We prove that N maps the set T to itself.

From inequality (7), Theorem 3.1 and Remark 3.2, it follows that

$$\begin{aligned} &\|NF\| \\ &\quad \leq J_3\left\|A(y) + B(y)\right\|_\beta \left\|-\frac{1}{2}F(y) + \Phi_F(y)\right\|_\beta + J_3\left\|1 + A(y)\right\|_\beta \|F\|_\beta + J_3\|G\|_\beta\|P\|_\beta \\ &\quad \leq J_3\left\|A(y) + B(y)\right\|_\beta J_{31}\|F\|_\beta + J_3\left\|1 + A(y)\right\|_\beta \|F\|_\beta + J_3\|G\|_\beta\|P\|_\beta \\ &\quad \leq J_{36}\left(\|A + B\|_\beta + \|1 + A\|_\beta\right)\|F\|_\beta + J_3\|G\|_\beta\|P\|_\beta \\ &\quad \leq \gamma\|F\|_\beta + J_3\delta\|P\|_\beta. \end{aligned}$$

By inequality (27) and Remark 3.2, we have

$$\begin{aligned} &\left|P\left(\Psi_F^+(y), \Psi_F^-(y)\right)\right| \\ &\quad = \left|P\left(\Psi_F^+(y), \Psi_F^-(y)\right) - P(0,0)\right| \\ &\quad \leq J_{34}\left|\Psi_F^+(y)\right| + J_{35}\left|\Psi_F^-(y)\right| \\ &\quad \leq J_{34}J_{33}\|F\|_\beta + J_{35}J_{33}\|F\|_\beta \\ &\quad = J_{37}\|F\|_\beta. \end{aligned}$$

So

$$C_q(P, \partial\Omega, \beta) = \max_{y \in \partial\Omega}|P| \leq J_{37}\|F\|_\beta. \tag{28}$$

By inequality (27) and Remark 3.2, we have

$$\begin{aligned} &\left|P\left(\Psi_F^+\left(y^1\right), \Psi_F^-\left(y^1\right)\right) - P\left(\Psi_F^+\left(y^2\right), \Psi_F^-\left(y^2\right)\right)\right| \\ &\quad \leq J_{34}\left|\Psi_F^+\left(y^1\right) - \Psi_F^+\left(y^2\right)\right| + J_{35}\left|\Psi_F^-\left(y^1\right) - \Psi_F^-\left(y^2\right)\right| \\ &\quad \leq J_{34}H_q\left(\Psi_F^+(y), \partial\Omega, \beta\right)\left|y^1 - y^2\right|^\beta + J_{35}H_q\left(\Psi_F^+(y), \partial\Omega, \beta\right)\left|y^1 - y^2\right|^\beta \end{aligned}$$

$$
\begin{aligned}
&\le \left[J_{34}\left\|\Psi_F^+(y)\right\|_\beta + J_{35}\left\|\Psi_F^+(y)\right\|_\beta\right]\left|y^1 - y^2\right|^\beta \\
&\le \left(J_{38}\|F\|_\beta + J_{39}\|F\|_\beta\right)\left|y^1 - y^2\right|^\beta \\
&\le J_{40}\|F\|_\beta\left|y^1 - y^2\right|^\beta, \qquad (29)
\end{aligned}
$$

then

$$H_q(P, \partial\Omega, \beta) \le J_{40}\|F\|_\beta.$$

So

$$
\begin{aligned}
\|P\|_\beta &= C_q(P, \partial\Omega, \beta) + H_q(P, \partial\Omega, \beta) \\
&\le J_{37}\|F\|_\beta + J_{40}\|F\|_\beta \\
&\le J_{41}\|F\|_\beta. \qquad (30)
\end{aligned}
$$

As $\gamma = J_{35}(\|A + B\|_\beta + \|1 + A\|_\beta) < 1$,

$$
\begin{aligned}
\|NF\|_\beta &\le \gamma\|F\|_\beta + J_3\delta\|P\|_\beta \\
&\le \gamma M_4 + J_3\frac{1-\gamma}{J_3 J_{41}}J_{41}M_4 = M_4. \qquad (31)
\end{aligned}
$$

If F is uniformly Hölder continuous on $\partial\Omega$, then $\Phi_F(y)$, Ψ_F^+, Ψ_F^- are uniformly Hölder continuous on $\partial\Omega$. So NF is uniformly Hölder continuous on $\partial\Omega$.

Hence N maps the set T to itself.

(2) We prove that N is a continuous mapping.

Any $F_n \in T$, $\{F_n\}$ uniformly converges to F on $\partial\Omega$. As for $\varepsilon > 0$, when n is fully large and $|F_n - F|$ is sufficiently small. There is a ball with radius 3δ, centered at y when $6\delta\langle d, \delta\rangle 0$, and remark that $\partial\Omega_1$ is located inside the ball and the rest of $\partial\Omega$ is $\partial\Omega_2$

By inequality (27), Theorem 3.3, we have

$$
\begin{aligned}
&\left|P\left(\Psi_{F_n}^+(y), \Psi_{F_n}^-(y)\right) - P\left(\Psi_F^+(y), \Psi_F^-(y)\right)\right| \\
&\quad\le J_{34}\left|\Psi_{F_n}^+(y) - \Psi_F^+(y)\right| + J_{35}\left|\Psi_{F_n}^-(y) - \Psi_F^-(y)\right| \\
&\quad= J_{34}\left|\Phi_{F_n}(y) - \Phi_F(y) + \frac{1}{2}\left(F_n(y) - F(y)\right)\right| + J_{35}\left|\Phi_{F_n}(y) - \Phi_F(y) + \frac{1}{2}\left(F(y) - F_n(y)\right)\right| \\
&\quad\le (J_{34} + J_{35})\left|\Phi_{F_n}(y) - \Phi_F(y)\right| + (J_{34} + J_{35})\left|\frac{1}{2}\left(F_n(y) - F(y)\right)\right| \\
&\quad\le (J_{34} + J_{35})\left|\frac{2^{n-1}y_0^{n-1}}{w_{n+1}}\int_{\partial\Omega} E_1(x, y)\,d\sigma(x)\left[\left(F_n(x) - F_n(y)\right) + \left(F(y) - F(x)\right)\right]\right| \\
&\qquad+ (J_{34} + J_{35})\left|\frac{2^{n-1}y_0^{n-1}}{w_{n+1}}\int_{\partial\Omega} E_2(x, y)\widehat{d\sigma(x)}\left[\left(\widehat{F(x)} - \widehat{F(y)}\right) + \left(\widehat{F_n(y)} - \widehat{F_n(x)}\right)\right]\right| \\
&\qquad+ (J_{34} + J_{35})|F_n(y) - F(y)| \\
&\quad\le (J_{34} + J_{35})\left(I_8 + I_9 + \|F_n - F\|_\beta\right),
\end{aligned}
$$

that is,

$$\begin{aligned}
&\bigl|P\bigl(\Psi^+_{F_n}(y),\Psi^-_{F_n}(y)\bigr)-P\bigl(\Psi^+_F(y),\Psi^-_F(y)\bigr)\bigr| \le (J_{34}+J_{35})\bigl(I_8+I_9+\|F_n-F\|_\beta\bigr), \qquad (32)\\
&I_8 \le \left|\frac{2^{n-1}y_0^{n-1}}{w_{n+1}}\int_{\partial\Omega}E_1(x,y)\,d\sigma(x)\bigl[\bigl(F_n(x)-F_n(y)\bigr)+\bigl(F(y)-F(x)\bigr)\bigr]\right|\\
&\quad\le \left|\frac{2^{n-1}y_0^{n-1}}{w_{n+1}}\int_{\partial\Omega_1}E_1(x,y)\,d\sigma(x)\bigl[\bigl(F_n(x)-F_n(y)\bigr)+\bigl(F(y)-F(x)\bigr)\bigr]\right|\\
&\qquad+\left|\frac{2^{n-1}y_0^{n-1}}{w_{n+1}}\int_{\partial\Omega_2}E_1(x,y)\,d\sigma(x)\bigl[\bigl(F_n(x)-F_n(y)\bigr)+\bigl(F(y)-F(x)\bigr)\bigr]\right|\\
&\quad= I_{10}+I_{11};\\
&I_{10} \le J_{42}\int_{\partial\Omega_1}\bigl|E_1(x,y)\bigr|\bigl|d\sigma(x)\bigr||x-y|^\beta\\
&\quad\le J_{43}\int_{\partial\Omega_1}\frac{1}{|x-y|^n}|x-y|^\beta\bigl|d\sigma(x)\bigr|\\
&\quad\le J_{44}\int_0^{3\delta}\rho^{\beta-1}\,d\rho\\
&\quad= J_{45}\delta^\beta,
\end{aligned}$$

that is,

$$\begin{aligned}
&I_{10} \le J_{45}\delta^\beta, \qquad (33)\\
&I_{11} = \left|\frac{2^{n-1}y_0^{n-1}}{w_{n+1}}\int_{\partial\Omega_2}E_1(x,y)\,d\sigma(x)\bigl[\bigl(F_n(x)-F(x)\bigr)-\bigl(F_n(y)-F(y)\bigr)\bigr]\right|\\
&\quad\le J_{46}\left|\int_{\partial\Omega_2}E_1(x,y)\,d\sigma(x)\right|\bigl[\bigl|F_n(x)-F(x)\bigr|+\bigl|F_n(y)-F(y)\bigr|\bigr]\\
&\quad\le J_{47}\|F_n-F\|_\beta\left|\int_{\partial\Omega_2}E_1(x,y)\,d\sigma(x)\right|\\
&\quad\le J_{48}\|F_n-F\|_\beta,
\end{aligned}$$

that is,

$$I_{11} \le J_{48}\|F_n-F\|_\beta. \qquad (34)$$

From inequality (33) and (34), we have

$$I_8 \le J_{45}\delta^\beta + J_{48}\|F_n-F\|_\beta. \qquad (35)$$

In a similar way, we get

$$I_9 \le J_{49}\delta^\beta + J_{50}\|F_n-F\|_\beta. \qquad (36)$$

From inequality (32), (35) and (36), we get

$$\bigl|P\bigl(\Psi^+_{F_n}(y),\Psi^-_{F_n}(y)\bigr)-P\bigl(\Psi^+_F(y),\Psi^-_F(y)\bigr)\bigr|$$

$$\begin{aligned}&\leq (J_{34}+J_{35})\big[(J_{45}+J_{49})\delta^{\beta}+(J_{48}+J_{50})\|F_n-F\|_{\beta}\big]\\&=J_{51}\delta^{\beta}+J_{52}\|F_n-F\|_{\beta}.\end{aligned}$$

Select a sufficiently small positive number δ such that $J_{51}\delta^{\beta}<\frac{\varepsilon}{2}$; and let n be large enough such that $J_{52}\|F_n-F\|_{\beta}<\frac{\varepsilon}{2}$. So for any $y\in\partial\Omega$, we have $|P(\Psi^{+}_{F_n}(y),\Psi^{-}_{F_n}(y))-P(\Psi^{+}_{F}(y),\Psi^{-}_{F}(y))|<\varepsilon$, thus $|NF_n(y)-NF(y)|<\varepsilon$, then N is a continuous mapping which maps T to itself.

From the Arzela–Ascoli theorem we conclude that T is a compact set in $\mathbf{C_q}(\partial\Omega)$. As the continuous mapping N maps the closed convex set T to itself, $N(T)$ is compact in $\mathbf{C_q}(\partial\Omega)$. From the Schauder fixed point principle it follows that there is at least $F\in H_q(\beta,\partial\Omega,Cl_{n+1,0}(\mathbb{R}))$ that satisfies (26). Hence the nonlinear boundary value problem SR has at least one solution $\Psi_F(y)$, and the expression of the solution is (8). □

5 The existence and uniqueness of the solution to the linear boundary value problem for the hypergenic function vector

Theorem 5.1 *If $A(y),B(y),G(y)\in H_q(\beta,\partial\Omega,Cl_{n+1,0}(\mathbb{R}))$, when $0<\gamma=J_3(J_{32}+\frac{1}{2})\|A+B\|_{\beta}+J_3\|1+A\|_{\beta}<1$, the linear boundary value problem SR has a unique solution.*

Proof Let T be as in Theorem 4.1. N is a continuous mapping which maps T to itself from Theorem 4.1.

From inequalities (7), (25) and Remark 3.1, we get

$$\begin{aligned}&\|NF_1-NF_2\|_{\beta}\\&\leq J_3\|A+B\|_{\beta}\left\|\frac{1}{2}(F_2-F_1)+\Phi_{F_1}-\Phi_{F_2}\right\|_{\beta}+J_3\|1+A\|_{\beta}\|F_1-F_2\|_{\beta}\\&\leq J_3\|A+B\|_{\beta}\left[\left\|\frac{1}{2}(F_1-F_2)\right\|_{\beta}+\|\Phi_{F_1}-\Phi_{F_2}\|_{\beta}\right]+J_3\|1+A\|_{\beta}\|F_1-F_2\|_{\beta}\\&\leq J_3\|A+B\|_{\beta}\left(\frac{1}{2}+J_{31}\right)\|F_1-F_2\|_{\beta}+J_3\|1+A\|_{\beta}\|F_1-F_2\|_{\beta}\\&\leq\left(J_3\left(J_{32}+\frac{1}{2}\right)\|A+B\|_{\beta}+J_3\|1+A\|_{\beta}\right)\|F_1-F_2\|_{\beta}\\&=\gamma\|F_1-F_2\|_{\beta}.\end{aligned}$$
□

There is only one solution to the equation $NF=F$ by the compression mapping principle. So there is a unique solution to the linear boundary value problem SR, and the integral expression of the solution is (8).

6 Conclusions

In this paper, we prove the existence of the solution to the nonlinear boundary value problem for the hypergenic function vector by virtue of the Arzela–Ascoli theorem and prove the existence and uniqueness of the solution to the linear boundary value problem for the hypergenic function vector by the compression mapping principle.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YX has presented the main purpose of the article. All authors read and approved the final manuscript.

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