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# Sharp bounds for the Sándor–Yang means in terms of arithmetic and contra-harmonic means

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**Abstract**

In the article, we provide several sharp upper and lower bounds for two Sándor–Yang means in terms of combinations of arithmetic and contra-harmonic means.

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**Keywords:** Schwab–Borchardt mean; Sándor–Yang mean; Arithmetic mean; Contra-harmonic mean; Quadratic mean

## 1 Preliminaries

Let $a, b > 0$ with $a \neq b$. Then the arithmetic mean $A(a,b)$ [1–4], the quadratic mean $Q(a,b)$ [5], the contra-harmonic mean $C(a,b)$ [6–9], the Neuman–Sándor mean $NS(a,b)$ [10–12], the second Seiffert mean $T(a,b)$ [13, 14], and the Schwab–Borchardt mean $SB(a,b)$ [15, 16] of $a$ and $b$ are defined by

$$A(a,b) = \frac{a+b}{2}, \qquad Q(a,b) = \sqrt{\frac{a^2+b^2}{2}}, \qquad C(a,b) = \frac{a^2+b^2}{a+b}, \tag{1.1}$$

$$NS(a,b) = \frac{a-b}{2\sinh^{-1}(\frac{a-b}{a+b})}, \tag{1.2}$$

$$T(a,b) = \frac{a-b}{2\arctan(\frac{a-b}{a+b})}, \tag{1.3}$$

$$SB(a,b) = \begin{cases} \frac{\sqrt{b^2-a^2}}{\arccos(a/b)}, & a < b, \\ \frac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & a > b, \end{cases}$$

respectively, where $\sinh^{-1}(x) = \log(x + \sqrt{x^2+1})$ and $\cosh^{-1}(x) = \log(x + \sqrt{x^2-1})$ are respectively the inverse hyperbolic sine and cosine functions. The Schwab–Borchardt mean $SB(a,b)$ is strictly increasing, non-symmetric and homogeneous of degree one with respect to its variables. It can be expressed by the degenerated completely symmetric elliptic integral of the first kind [17]. Recently, the Schwab–Borchardt mean has attracted the attention of many researchers. In particular, many remarkable inequalities for the Schwab–Borchardt mean and its generated means can be found in the literature [18–38].

Let $X(a,b)$ and $Y(a,b)$ denote symmetric bivariate means of $a$ and $b$. Then Yang [39] introduced the Sándor–Yang mean

$$R_{XY}(a,b) = Y(a,b)e^{\frac{X(a,b)}{SB[X(a,b),Y(a,b)]}-1}$$

and presented the explicit formulas for $R_{QA}(a,b)$ and $R_{AQ}(a,b)$ as follows:

$$R_{QA}(a,b) = A(a,b)e^{\frac{Q(a,b)}{NS(a,b)}-1}, \tag{1.4}$$

$$R_{AQ}(a,b) = Q(a,b)e^{\frac{A(a,b)}{T(a,b)}-1}. \tag{1.5}$$

Very recently, the bounds involving the Sándor–Yang means have been the subject of intensive research. Numerous interesting results and inequalities for $R_{QA}(a,b)$ and $R_{AQ}(a,b)$ can be found in the literature [40–42].

Neuman [43] established the inequality

$$R_{AQ}(a,b) < R_{QA}(a,b) \tag{1.6}$$

for $a,b > 0$ with $a \neq b$.

In [44], Xu proved that the double inequalities

$$\begin{aligned} &\alpha_1 C(a,b) + (1-\alpha_1)A(a,b) < R_{QA}(a,b) < \beta_1 C(a,b) + (1-\beta_1)A(a,b), \\ &\alpha_2 C(a,b) + (1-\alpha_2)A(a,b) < R_{AQ}(a,b) < \beta_2 C(a,b) + (1-\beta_2)A(a,b) \end{aligned} \tag{1.7}$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq (1+\sqrt{2})^{\sqrt{2}}/e - 1 = 0.2794\ldots$, $\beta_1 \geq 1/3$, $\alpha_2 \leq \sqrt{2}e^{\pi/4-1} - 1 = 0.1410\ldots$ and $\beta_2 \geq 1/6$.

From (1.6) and (1.7), together the well-known inequalities

$$C(a,b) > Q(a,b) > A(a,b), \qquad Q(a,b) > \frac{1}{3}C(a,b) + \frac{2}{3}A(a,b),$$

we clearly see that

$$A(a,b) < R_{AQ}(a,b) < R_{QA}(a,b) < Q(a,b) < C(a,b) \tag{1.8}$$

for all $a,b > 0$ with $a \neq b$.

The main purpose of this paper is to find the best possible parameters $\alpha_i, \beta_i \in (0,1)$ $(i = 1,2,3,4)$ such that the double inequalities

$$\begin{aligned} &C^{\alpha_1}(a,b)A^{1-\alpha_1}(a,b) < R_{QA}(a,b) < C^{\beta_1}(a,b)A^{1-\beta_1}(a,b), \\ &C^{\alpha_2}(a,b)A^{1-\alpha_2}(a,b) < R_{AQ}(a,b) < C^{\beta_2}(a,b)A^{1-\beta_2}(a,b), \\ &\alpha_3\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right] + (1-\alpha_3)C^{1/3}(a,b)A^{2/3}(a,b) \\ &\quad < R_{QA}(a,b) < \beta_3\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right] + (1-\beta_3)C^{1/3}(a,b)A^{2/3}(a,b), \end{aligned}$$

$$\alpha_4\left[\frac{1}{6}C(a,b)+\frac{5}{6}A(a,b)\right]+(1-\alpha_4)C^{1/6}(a,b)A^{5/6}(a,b)$$
$$< R_{AQ}(a,b) < \beta_4\left[\frac{1}{6}C(a,b)+\frac{5}{6}A(a,b)\right]+(1-\beta_4)C^{1/6}(a,b)A^{5/6}(a,b)$$

hold for all $a,b>0$ with $a\neq b$.

## 2 Lemmas

In order to prove our main results, we need several lemmas, which we present in this section.

**Lemma 2.1** (see [45]) *Let $a,b\in\mathbb{R}$ with $a<b$, $f,g:[a,b]\mapsto\mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$, and $g'(x)\neq 0$ on $(a,b)$. If $f'(x)/g'(x)$ is increasing (decreasing) on $(a,b)$, then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)},\qquad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

*If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.*

**Lemma 2.2** (see [46]) *Let $A(t)=\sum_{k=0}^{\infty}a_kt^k$ and $B(t)=\sum_{k=0}^{\infty}b_kt^k$ be two real power series converging on $(-r,r)$ $(r>0)$ with $b_k>0$ for all $k$. If the non-constant sequence $\{a_k/b_k\}_{k=0}^{\infty}$ is increasing (decreasing) for all $k$, then the function $t\mapsto A(t)/B(t)$ is strictly increasing (decreasing) on $(0,r)$.*

**Lemma 2.3** *The function*

$$\phi(x)=\frac{x\coth(x)-1}{2\log[\cosh(x)]}$$

*is strictly increasing from $(0,\log(1+\sqrt{2})$ onto $(1/3,[\sqrt{2}\log(1+\sqrt{2})-1]/\log 2)$.*

*Proof* Let $\phi_1(x)=x\coth(x)-1$, $\phi_2(x)=2\log[\cosh(x)]$. Then elaborate computations lead to

$$\phi(x)=\frac{\phi_1(x)}{\phi_2(x)}=\frac{\phi_1(x)-\phi_1(0^+)}{\phi_2(x)-\phi_2(0)},\tag{2.1}$$

$$\begin{aligned}\frac{\phi_1'(x)}{\phi_2'(x)}&=\frac{\sinh(x)\cosh^2(x)-x\cosh(x)}{2\sinh^3(x)}\\&=\frac{\sinh(3x)+\sinh(x)-4x\cosh(x)}{2\sinh(3x)-6\sinh(x)}=\frac{\sum_{n=0}^{\infty}\frac{3^{2n+1}-8n-3}{(2n+1)!}x^{2n+1}}{\sum_{n=0}^{\infty}\frac{6(3^{2n}-1)}{(2n+1)!}x^{2n+1}}\\&=\frac{\sum_{n=1}^{\infty}\frac{3^{2n+1}-8n-3}{(2n+1)!}x^{2n+1}}{\sum_{n=1}^{\infty}\frac{6(3^{2n}-1)}{(2n+1)!}x^{2n+1}}=\frac{\sum_{n=0}^{\infty}\frac{3^{2n+3}-8n-11}{(2n+3)!}x^{2n+3}}{\sum_{n=0}^{\infty}\frac{6(3^{2n+2}-1)}{(2n+3)!}x^{2n+3}}.\end{aligned}\tag{2.2}$$

Let

$$a_n=\frac{3^{2n+3}-8n-11}{(2n+3)!},\qquad b_n=\frac{6(3^{2n+2}-1)}{(2n+3)!}.\tag{2.3}$$

Then

$$b_n > 0 \tag{2.4}$$

and

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = \frac{4[(72n+63)3^{2n}+1]}{3(3^{2n+4}-1)(3^{2n+2}-1)} > 0 \tag{2.5}$$

for all $n \geq 0$.

It follows from Lemma 2.2 and (2.2)–(2.5) that $\phi_1'(x)/\phi_2'(x)$ is strictly increasing on $(0, \log(1+\sqrt{2}))$.

Note that

$$\phi(0^+) = \frac{a_0}{b_0} = \frac{1}{3}, \qquad \phi(\log(1+\sqrt{2})) = \frac{\sqrt{2}\log(1+\sqrt{2})-1}{\log 2} = 0.3555\ldots. \tag{2.6}$$

Therefore, Lemma 2.3 follows from Lemma 2.1, (2.1), and (2.6) together with the monotonicity of $\phi_1'(x)/\phi_2'(x)$. □

**Lemma 2.4** *The function*

$$\varphi(x) = \frac{\log\sec(x) + x\cot(x) - 1}{2\log\sec(x)}$$

*is strictly increasing from* $(0, \pi/4)$ *onto* $(1/6, 1/2 - (4-\pi)(4\log 2))$.

*Proof* Let $\varphi_1(x) = \log\sec(x) + x\cot(x) - 1$, $\varphi_2(x) = 2\log[\sec(x)]$, $\varphi_3(x) = \sin(x) - x\cos(x)$, and $\varphi_4(x) = 2\sin^3(x)$. Then elaborate computations lead to

$$\varphi(x) = \frac{\varphi_1(x)}{\varphi_2(x)} = \frac{\varphi_1(x) - \varphi_1(0^+)}{\varphi_2(x) - \varphi_2(0)}, \tag{2.7}$$

$$\frac{\varphi_1'(x)}{\varphi_2'(x)} = \frac{\varphi_3(x)}{\varphi_4(x)} = \frac{\varphi_3(x) - \varphi_3(0)}{\varphi_4(x) - \varphi_4(0)} \tag{2.8}$$

and

$$\frac{\varphi_3'(x)}{\varphi_4'(x)} = \frac{x}{3\sin(2x)} = \frac{1}{6} \times \frac{1}{\sin(2x)/(2x)}. \tag{2.9}$$

It is well known that the function $x \to \sin(x)/x$ is strictly decreasing on $(0, \pi/2)$, hence equation (2.9) leads to the conclusion that the function $\varphi_3'(x)/\varphi_4'(x)$ is strictly increasing on $(0, \pi/4)$.

Note that

$$\begin{aligned} \varphi(0^+) &= \lim_{x\to 0^+} \frac{\varphi_3'(x)}{\varphi_4'(x)} = \frac{1}{6}, \\ \varphi\left(\frac{\pi}{4}\right) &= \frac{1}{2} - \frac{4-\pi}{4\log 2} = 0.1903\ldots. \end{aligned} \tag{2.10}$$

Therefore, Lemma 2.4 follows from Lemma 2.1 and (2.7)–(2.9) together with the monotonicity of $\varphi_3'(x)/\varphi_4'(x)$. □

**Lemma 2.5** *Let* $p \in (0,1)$ *and*

$$f(x) = 3p^2x^{10} + 14p(1-p)x^6 + 18p^2x^4 - 9(1-p)^2x^2 - 2p(1-p).$$

*Then the following statements are true*:

(1) *If* $p = 3/10$, *then* $f(x) > 0$ *for all* $x \in (1, \sqrt[6]{2})$;

(2) *If* $p = 3[(1+\sqrt{2})^{\sqrt{2}}/e - \sqrt[3]{2}]/(4 - 3\sqrt[3]{2}) = 0.2663\ldots$, *then there exists* $\lambda_0(= 1.0808\ldots) \in (1, \sqrt[6]{2})$ *such that* $f(x) < 0$ *for* $x \in (1, \lambda_0)$ *and* $f(x) > 0$ *for* $x \in (\lambda_0, \sqrt[6]{2})$.

*Proof* Part (1) follows easily from

$$f(x) = \frac{3}{100}\left(x^2 - 1\right)\left(9x^8 + 9x^6 + 107x^4 + 161x^2 + 14\right) > 0$$

for all $x \in (1, \sqrt[6]{2})$ if $p = 3/10$.

For part (2), if $p = 3[(1+\sqrt{2})^{\sqrt{2}}/e - \sqrt[3]{2}]/(4 - 3\sqrt[3]{2})$, then numerical computations lead to

$$20p - 3 = 2.3273\cdots > 0, \tag{2.11}$$

$$f(1) = 3(10p - 3) = -1.008\cdots < 0, \tag{2.12}$$

$$f\left(\sqrt[6]{2}\right) = 1.6809\cdots > 0, \tag{2.13}$$

$$f'(x) = 30p^2x^9 + 84p(1-p)x^5 + 72p^2x^3 - 18(1-p)^2x. \tag{2.14}$$

It follows from (2.11) and (2.14) that

$$f'(x) > \left[30p^2 + 84p(1-p) + 72p^2 - 18(1-p)^2\right]x = 6(20p-3)x > 0 \tag{2.15}$$

for all $x \in (1, \sqrt[6]{2})$.

Therefore, part (2) follows easily from (2.12), (2.13), (2.15), and the numerical results $f(1.0808) < 0$ and $f(1.0809) > 0$. □

**Lemma 2.6** *Let* $p \in (0,1)$ *and*

$$g(x) = 3p^2x^{11} + 56p(1-p)x^6 + 75p^2x^5 - 72(1-p)^2x - 50p(1-p).$$

*Then the following statements are true*:

(1) *If* $p = 12/25$, *then* $g(x) > 0$ *for all* $x \in (1, \sqrt[6]{2})$;

(2) *If* $p = 6[\sqrt{2}e^{\pi/4-1} - \sqrt[6]{2}]/(7 - 6\sqrt[6]{2}) = 0.4210\ldots$, *then there exists* $\mu_0(= 1.0577\ldots) \in (1, \sqrt[6]{2})$ *such that* $g(x) < 0$ *for* $x \in (1, \mu_0)$ *and* $g(x) > 0$ *for* $x \in (\mu_0, \sqrt[6]{2})$.

*Proof* Part (1) follows easily from

$$\begin{aligned} g(x) = {} & \frac{24}{625}(x-1)\bigl(18x^{10} + 18x^9 + 18x^8 + 18x^7 + 18x^6 + 382x^5 + 832x^4 \\ & + 832x^3 + 832x^2 + 832x + 325\bigr) > 0 \end{aligned}$$

for all $x \in (1, \sqrt[6]{2})$ if $p = 12/25$.

For part (2), if $p = 6[\sqrt{2}e^{\pi/4-1} - \sqrt[6]{2}]/(7-6\sqrt[6]{2}) = 0.4210\ldots$, then numerical computations lead to

$$20p - 3 = 5.4217\cdots > 0, \tag{2.16}$$

$$g(1) = 6(25p-12) = -8.8367\cdots < 0, \tag{2.17}$$

$$g\bigl(\sqrt[6]{2}\bigr) = 13.6200\cdots > 0, \tag{2.18}$$

$$g'(x) = 3\bigl[11p^2x^{10} + 112p(1-p)x^5 + 125p^2x^4 - 24(1-p)^2\bigr]. \tag{2.19}$$

It follows from (2.16) and (2.19) that

$$\begin{aligned} g'(x) & > 11p^2 + 112p(1-p) + 125p^2 - 24(1-p)^2 \\ & = 24(20p-3) > 0 \end{aligned} \tag{2.20}$$

for $x \in (1, \sqrt[6]{2})$.

Therefore, part (2) follows easily from (2.17), (2.18), and (2.20) together with the numerical results $g(1.0577) < 0$ and $g(1.0578) > 0$. □

## 3 Main results

We are now in a position to state and prove our main results.

**Theorem 3.1** *The double inequality*

$$C^{\alpha_1}(a,b)A^{1-\alpha_1}(a,b) < R_{QA}(a,b) < C^{\beta_1}(a,b)A^{1-\beta_1}(a,b) \tag{3.1}$$

*holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/3$ and $\beta_1 \geq [\sqrt{2}\log(1+\sqrt{2}) - 1]/\log 2$.*

*Proof* Clearly, inequality (3.1) can be rewritten as

$$\left[\frac{C(a,b)}{A(a,b)}\right]^{\alpha_1} < \frac{R_{QA}(a,b)}{A(a,b)} < \left[\frac{C(a,b)}{A(a,b)}\right]^{\beta_1}. \tag{3.2}$$

Since $A(a,b)$, $R_{QA}(a,b)$, and $C(a,b)$ are symmetric and homogenous of degree one, we assume that $a > b > 0$. Let $v = (a-b)/(a+b) \in (0,1)$. Then from (1.1), (1.2), and (1.4) we know that inequality (3.2) is equivalent to

$$\alpha_1 < \frac{[\sqrt{1+v^2}\sinh^{-1}(v)]/v - 1}{\log(1+v^2)} < \beta_1. \tag{3.3}$$

Let $x = \sinh^{-1}(v)$. Then $x \in (0, \log(1+\sqrt{2}))$ and

$$\frac{[\sqrt{1+v^2}\sinh^{-1}(v)]/v - 1}{\log(1+v^2)} = \frac{x\coth(x) - 1}{2\log[\cosh(x)]} := \phi(x). \tag{3.4}$$

Therefore, inequality (3.1) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/3$ and $\beta_1 \geq [\sqrt{2}\log(1+\sqrt{2}) - 1]/\log 2$ follows from (3.2)–(3.4) and Lemma 2.3. □

**Theorem 3.2** *The double inequality*

$$C^{\alpha_2}(a,b)A^{1-\alpha_2}(a,b) < R_{AQ}(a,b) < C^{\beta_2}(a,b)A^{1-\beta_2}(a,b) \tag{3.5}$$

*holds for all* $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_2 \leq 1/6$ *and* $\beta_2 \geq 1/2 - (4-\pi)/(4\log 2) = 0.1903\ldots$ .

*Proof* Clearly, inequality (3.5) can be rewritten as

$$\left[\frac{C(a,b)}{A(a,b)}\right]^{\alpha_2} < \frac{R_{AQ}(a,b)}{A(a,b)} < \left[\frac{C(a,b)}{A(a,b)}\right]^{\beta_2}. \tag{3.6}$$

Since $A(a,b)$, $R_{AQ}(a,b)$, and $C(a,b)$ are symmetric and homogenous of degree one, we assume that $a > b > 0$. Let $v = (a-b)/(a+b) \in (0,1)$. Then from (1.1), (1.3), and (1.5) we see that inequality (3.6) is equivalent to

$$\alpha_2 < \frac{\log\sqrt{1+v^2} + [\arctan(v)]/v - 1}{\log(1+v^2)} < \beta_2. \tag{3.7}$$

Let $x = \arctan(v)$. Then $x \in (0, \pi/4)$ and

$$\begin{aligned} &\frac{\log\sqrt{1+v^2} + [\arctan(v)]/v - 1}{\log(1+v^2)} \\ &\quad = \frac{\log\sec(x) + x\cot(x) - 1}{2\log\sec(x)} := \varphi(x). \end{aligned} \tag{3.8}$$

Therefore, inequality (3.5) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq 1/6$ and $\beta_2 \geq 1/2 - (4-\pi)/(4\log 2) = 0.1903\ldots$ follows from (3.6)–(3.8) and Lemma 2.4. □

**Theorem 3.3** *The double inequality*

$$\begin{aligned} &\alpha_3\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right] + (1-\alpha_3)C^{1/3}(a,b)A^{2/3}(a,b) \\ &\quad < R_{QA}(a,b) < \beta_3\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right] + (1-\beta_3)C^{1/3}(a,b)A^{2/3}(a,b) \end{aligned}$$

*holds for all* $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_3 \leq 3[(1+\sqrt{2})^{\sqrt{2}}/e - \sqrt[3]{2}]/(4 - 3\sqrt[3]{2}) = 0.2663\ldots$ *and* $\beta_3 \geq 3/10$.

*Proof* Since $R_{QA}(a,b)$, $A(a,b)$, and $C(a,b)$ are symmetric and homogenous of degree one, without loss generality, we assume that $a > b > 0$. Let $v = (a-b)/(a+b)$, $x = \sqrt[6]{1+v^2}$, and $p \in (0,1)$. Then $v \in (0,1)$, $x \in (1, \sqrt[6]{2})$, and (1.1), (1.2), and (1.4) lead to

$$\begin{aligned}
&\log \frac{R_{QA}(a,b)}{p[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)]+(1-p)C^{1/3}(a,b)A^{2/3}(a,b)} \\
&\quad = \frac{\sqrt{1+v^2}\sinh^{-1}(v)}{v} - \log\left[p\left(\frac{1}{3}v^2+1\right)+(1-p)\sqrt[3]{1+v^2}\right]-1 \\
&\quad = \frac{x^3\sinh^{-1}(\sqrt{x^6-1})}{\sqrt{x^6-1}} - \log\left[p\left(\frac{1}{3}x^6+\frac{2}{3}\right)+(1-p)x^2\right]-1. \qquad (3.9)
\end{aligned}$$

Let

$$F(x) = \frac{x^3\sinh^{-1}(\sqrt{x^6-1})}{\sqrt{x^6-1}} - \log\left[p\left(\frac{1}{3}x^6+\frac{2}{3}\right)+(1-p)x^2\right]-1. \qquad (3.10)$$

Then simple computations lead to

$$F(1^+) = 0, \qquad F(\sqrt[6]{2}) = \sqrt{2}\log(1+\sqrt{2}) - \log\left[\frac{4}{3}p + \sqrt[3]{2}(1-p)\right]-1, \qquad (3.11)$$

$$F'(x) = \frac{3x^2}{(x^6-1)^{3/2}}F_1(x), \qquad (3.12)$$

where

$$\begin{aligned}
&F_1(x) = \frac{\sqrt{x^6-1}[-px^{10}+(1-p)x^6+4px^4+2(1-p)]}{x[p(x^6+2)+3(1-p)x^2]} - \sinh^{-1}(\sqrt{x^6-1}), \\
&F_1(1) = 0, \qquad F_1(\sqrt[6]{2}) = \frac{(\sqrt[6]{2048}-2\sqrt{2})p - \sqrt[6]{2048}}{3\sqrt[3]{2}p - 3\sqrt[3]{2} - 4p} - \log(1+\sqrt{2}), \qquad (3.13) \\
&F_1'(x) = -\frac{2(x^6-1)^{3/2}}{x^2[p(x^6+2)+3(1-p)x^2]^2}f(x), \qquad (3.14)
\end{aligned}$$

where $f(x)$ is defined as in Lemma 2.5.

We divide the proof into four cases.

*Case 1* $p = 3/10$. Then it follows from (3.9)–(3.14) and Lemma 2.5(1) that

$$R_{QA}(a,b) < \frac{3}{10}\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right] + \frac{7}{10}C^{1/3}(a,b)A^{2/3}(a,b).$$

*Case 2* $0 < p < 3/10$. Let $v > 0$ and $v \to 0^+$. Then power series expansion leads to

$$\begin{aligned}
&\frac{\sqrt{1+v^2}\sinh^{-1}(v)}{v} - \log\left[p\left(\frac{1}{3}v^2+1\right)+(1-p)\sqrt[3]{1+v^2}\right]-1 \\
&\quad = \left(\frac{1}{30}-\frac{1}{9}p\right)v^4 + O(v^6). \qquad (3.15)
\end{aligned}$$

Equations (3.9), (3.10), and (3.15) lead to the conclusion that there exists $0 < \delta_1 < 1$ such that

$$R_{QA}(a,b) > p\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right] + (1-p)C^{1/3}(a,b)A^{2/3}(a,b)$$

for all $a > b > 0$ with $(a-b)/(a+b) \in (0,\delta_1)$.

*Case 3* $p = 3[(1+\sqrt{2})^{\sqrt{2}}/e - \sqrt[3]{2}]/(4 - 3\sqrt[3]{2})$. Then (3.13) leads to

$$F_1\left(\sqrt[6]{2}\right) = -0.0039\cdots < 0. \tag{3.16}$$

Let $\lambda_0 = 1.0808\ldots$ be the number given in Lemma 2.5(2). Then we divide the discussion into two subcases.

*Subcase 1* $x \in (1,\lambda_0]$. Then $F_1(x) > 0$ for $x \in (1,\lambda_0]$ follows easily from (3.13) and (3.14) together with Lemma 2.5(2).

*Subcase 2* $x \in (\lambda_0, \sqrt[6]{2})$. Then Lemma 2.5(2) and (3.14) lead to the conclusion that $F_1(x)$ is strictly decreasing on the interval $[\lambda_0, \sqrt[6]{2})$. Then, from (3.16) and Subcase 1, we know that there exists $\lambda_1 \in (\lambda_0, \sqrt[6]{2})$ such that $F_1(x) > 0$ for $x \in [\lambda_0, \lambda_1)$ and $F_1(x) < 0$ for $x \in (\lambda_1, \sqrt[6]{2})$.

It follows from Subcases 1 and 2 together with (3.12) that $F(x)$ is strictly increasing on $(1,\lambda_1]$ and strictly decreasing on $[\lambda_1, \sqrt[6]{2})$. Therefore,

$$R_{QA}(a,b) > p\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right] + (1-p)C^{1/3}(a,b)A^{2/3}(a,b)$$

follows from (3.9)–(3.11) and (3.16) together with the piecewise monotonicity of $F(x)$.

*Case 4* $3[(1+\sqrt{2})^{\sqrt{2}}/e - \sqrt[3]{2}]/(4 - 3\sqrt[3]{2}) < p < 1$. Then (3.11) leads to

$$F\left(\sqrt[6]{2}\right) = \sqrt{2}\log(1+\sqrt{2}) - \log\left[\frac{4}{3}p + \sqrt[3]{2}(1-p)\right] - 1 < 0. \tag{3.17}$$

Equations (3.9) and (3.10) together with inequality (3.17) imply that there exists $0 < \delta_1^* < 1$ such that

$$R_{QA}(a,b) < p\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right] + (1-p)C^{1/3}(a,b)A^{2/3}(a,b)$$

for all $a > b > 0$ with $(a-b)/(a+b) \in (1-\delta_1^*, 1)$. □

**Theorem 3.4** *The double inequality*

$$\begin{aligned}
&\alpha_4\left[\frac{1}{6}C(a,b) + \frac{5}{6}A(a,b)\right] + (1-\alpha_4)C^{1/6}(a,b)A^{5/6}(a,b) \\
&\quad < R_{AQ}(a,b) < \beta_4\left[\frac{1}{6}C(a,b) + \frac{5}{6}A(a,b)\right] + (1-\beta_4)C^{1/6}(a,b)A^{5/6}(a,b)
\end{aligned}$$

*holds for all* $a,b > 0$ *with* $a \neq b$ *if and only if* $\alpha_4 \leq 6[\sqrt{2}e^{(\pi/4-1)} - \sqrt[6]{2}]/(7 - 6\sqrt[6]{2}) = 0.4210\ldots$ *and* $\beta_4 \geq 12/25$.

*Proof* Since $R_{AQ}(a,b)$, $A(a,b)$, and $C(a,b)$ are symmetric and homogenous of degree one, without loss generality, we assume that $a > b > 0$. Let $v = (a-b)/(a+b)$, $x = \sqrt[6]{1+v^2}$, and $p \in (0,1)$. Then $v \in (0,1)$, $x \in (1, \sqrt[6]{2})$ and (1.1), (1.3), and (1.5) lead to

$$\begin{aligned}
&\log \frac{R_{AQ}(a,b)}{p[\frac{1}{6}C(a,b) + \frac{5}{6}A(a,b)] + (1-p)C^{1/6}(a,b)A^{5/6}(a,b)} \\
&\quad = \log\sqrt{1+v^2} + \frac{\arctan(v)}{v} - \log\left[p\left(\frac{1}{6}v^2+1\right) + (1-p)\sqrt[6]{1+v^2}\right] - 1 \\
&\quad = 3\log(x) + \frac{\arctan(\sqrt{x^6-1})}{\sqrt{x^6-1}} - \log\left[p\left(\frac{1}{6}x^6+\frac{5}{6}\right) + (1-p)x\right] - 1.
\end{aligned} \tag{3.18}$$

Let

$$G(x) = 3\log(x) + \frac{\arctan(\sqrt{x^6-1})}{\sqrt{x^6-1}} - \log\left[p\left(\frac{1}{6}x^6+\frac{5}{6}\right) + (1-p)x\right] - 1. \tag{3.19}$$

Then simple computations lead to

$$G(1^+) = 0, \qquad G(\sqrt[6]{2}) = \log(\sqrt{2}) + \frac{\pi}{4} - \log\left[\frac{7}{6}p + \sqrt[6]{2}(1-p)\right] - 1, \tag{3.20}$$

$$G'(x) = \frac{3x^5}{(x^6-1)^{3/2}} G_1(x), \tag{3.21}$$

where

$$G_1(x) = \frac{\sqrt{x^6-1}[-px^{11} + 4(1-p)x^6 + 7px^5 + 2(1-p)]}{x^5[p(x^6+5) + 6(1-p)x]} - \arctan(\sqrt{x^6-1}),$$

$$G_1(1) = 0, \qquad G_1(\sqrt[6]{2}) = \frac{5[\sqrt[6]{2}(1-p)+p]}{6\sqrt[6]{2}(1-p)+7p} - \frac{\pi}{4}, \tag{3.22}$$

$$G_1'(x) = -\frac{(x^6-1)^{3/2}}{x^6[p(x^6+5)+6(1-p)x]^2} g(x), \tag{3.23}$$

where $g(x)$ is defined as in Lemma 2.6.

We divide the proof into four cases.

*Case 1* $p = 12/25$. Then it follows from (3.18)–(3.23) and Lemma 2.6(1) that

$$R_{AQ}(a,b) < \frac{12}{25}\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right] + \frac{13}{25}C^{1/3}(a,b)A^{2/3}(a,b).$$

*Case 2* $0 < p < 12/25$. Let $v > 0$ and $v \to 0^+$, then power series expansion leads to

$$\begin{aligned}
&\log\sqrt{1+v^2} + \frac{\arctan(v)}{v} - \log\left[p\left(\frac{1}{6}v^2+1\right) + (1-p)\sqrt[6]{1+v^2}\right] - 1 \\
&\quad = \left(\frac{1}{30} - \frac{5}{72}p\right)v^4 + O(v^6).
\end{aligned} \tag{3.24}$$

Equations (3.18), (3.19), and (3.24) lead to the conclusion that there exists $0 < \delta_2 < 1$ such that

$$R_{AQ}(a,b) > p\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right] + (1-p)C^{1/3}(a,b)A^{2/3}(a,b)$$

for all $a > b > 0$ with $(a-b)/(a+b) \in (0, \delta_2)$.

*Case 3* $p = 6[\sqrt{2}e^{(\pi/4-1)} - \sqrt[6]{2}]/(7 - 6\sqrt[6]{2})$. Then, from (3.20) and (3.22) together with numerical computations, we get

$$G(\sqrt[6]{2}) = 0, \qquad G_1(\sqrt[6]{2}) = -0.0033\cdots < 0. \tag{3.25}$$

Let $\mu_0 = 1.0577\ldots$ be the number given in Lemma 2.6(2). Then we divide the discussion into two subcases.

*Subcase 1* $x \in (1, \mu_0]$. Then $G_1(x) > 0$ for $x \in (1, \mu_0]$ follows easily from (3.22) and (3.23) together with Lemma 2.6(2).

*Subcase 2* $x \in (\mu_0, \sqrt[6]{2})$. Then Lemma 2.6(2) and (3.23) lead to the conclusion that $G_1(x)$ is strictly decreasing on the interval $[\mu_0, \sqrt[6]{2})$. Then, from (3.25) and Subcase 1, we know that there exists $\mu_1 \in (\mu_0, \sqrt[6]{2})$ such that $G_1(x) > 0$ for $x \in [\mu_0, \mu_1)$ and $G_1(x) < 0$ for $x \in (\mu_1, \sqrt[6]{2})$.

It follows from Subcases 1 and 2 together with (3.21) that $G(x)$ is strictly increasing on $(1, \mu_1]$ and strictly decreasing on $[\mu_1, \sqrt[6]{2})$. Therefore,

$$R_{AQ}(a,b) > p\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right] + (1-p)C^{1/3}(a,b)A^{2/3}(a,b)$$

follows from (3.18)–(3.20) and (3.25) together with the piecewise monotonicity of $G(x)$.

*Case 4* $6[\sqrt{2}e^{(\pi/4-1)} - \sqrt[6]{2}]/(7 - 6\sqrt[6]{2}) < p < 1$. Then (3.21) leads to

$$G(\sqrt[6]{2}) = \log(\sqrt{2}) + \frac{\pi}{4} - \log\left[\frac{7}{6}p + \sqrt[6]{2}(1-p)\right] - 1 < 0. \tag{3.26}$$

Equations (3.18) and (3.19) together with inequality (3.26) imply that there exists $0 < \delta_2^* < 1$ such that

$$R_{AQ}(a,b) < p\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right] + (1-p)C^{1/3}(a,b)A^{2/3}(a,b)$$

for all $a > b > 0$ with $(a-b)/(a+b) \in (1 - \delta_2^*, 1)$. □

## 4 Results and discussion

In this paper, we provide the optimal upper and lower bounds for the Sándor–Yang means $R_{QA}(a,b)$ and $R_{AQ}(a,b)$ in terms of combinations of the arithmetic mean $A(a,b)$ and the contra-harmonic mean $C(a,b)$. Our approach may have further applications in the theory of bivariate means.

## 5 Conclusion

In the article, we find several best possible bounds for the Sándor–Yang means $R_{QA}(a,b)$ and $R_{AQ}(a,b)$. These results are improvements and refinements of the previous results.

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**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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