Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

CrossMark

Triple Diamond-Alpha integral and Hölder-type inequalities

Jing-Feng Tian[1*]

*Correspondence:
tianjf@ncepu.edu.cn
[1]College of Science and
Technology, North China Electric
Power University, Baoding, P.R. China

Abstract

In this paper, we first introduce the definition of triple Diamond-Alpha integral for functions of three variables. Therefore, we present the Hölder and reverse Hölder inequalities for the triple Diamond-Alpha integral on time scales, and then we obtain some new generalizations of the Hölder and reverse Hölder inequalities for the triple Diamond-Alpha integral. Moreover, using the obtained results, we give a new generalization of the Minkowski inequality for the triple Diamond-Alpha integral on time scales.

MSC: 26D15

Keywords: Hölder's inequality; Minkowski's inequality; Triple Diamond-Alpha integral; Time scales

1 Introduction

To unify and generalize discrete and continuous analysis, in 1998, Hilger [1] introduced the theory of time scales. Since then, many researchers have studied various aspects of the theory and obtained a lot of interesting results on time scales [1–10]. The first purpose of this paper is to give the definition of the triple Diamond-Alpha integral (triple diamond-α integral or triple $\diamond_\alpha$-integral) for functions of three variables on time scales.

Let $u(x)$ and $v(x)$ be continuous real-valued functions on $[a, b]$, and let $\frac{1}{p} + \frac{1}{q} = 1$.

(I) if $p > 1$ and if $u(x) \geq 0$, $v(x) \geq 0$, then the classical Hölder inequality holds (see [11]):

$$\int_a^b u(x)v(x)\,dx \leq \left(\int_a^b u^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_a^b v^q(x)\,dx\right)^{\frac{1}{q}}. \tag{1}$$

(II) if $0 < p < 1$ and if $u(x) > 0$, $v(x) > 0$, then the following reverse Hölder inequality (e.g., see [12]) holds:

$$\int_a^b u(x)v(x)\,dx \geq \left(\int_a^b u^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_a^b v^q(x)\,dx\right)^{\frac{1}{q}}. \tag{2}$$

The classical Hölder and reverse Hölder inequalities play a very important role and have wide applications in different branches of modern mathematics. A large number of papers dealing with refinements, generalizations, and applications of the Hölder and reverse Hölder inequalities and their series analogues in different ares of mathematics have

appeared. For example, Agahi et al. [13] gave generalizations of the Hölder and reverse Hölder inequalities for the pseudo-integral. Zhao et al. [14] found that the Hölder inequality for the pan-integral holds if the monotone measurer is subadditive. Tian [15–18] gave some new properties and refinements of the Hölder and reverse Hölder inequalities. For more detail, the reader may consult [19–25].

Among various extensions of (1) and (2), Agarwal, Bohner, and Peterson first presented the time scale versions of (1) via the Delta-integral (Δ-integral).

Theorem A *Assume that $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$, and $a < b$. If $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and if $u, v \in C_{rd}([a, b], \mathbb{R})$, then*

$$\int_a^b |u(x)v(x)| \Delta x \le \left(\int_a^b |u(x)|^p \Delta x \right)^{\frac{1}{p}} \left(\int_a^b |v(x)|^q \Delta x \right)^{\frac{1}{q}}. \tag{3}$$

Later, in 2005, Wong et al. [26] gave the following Hölder-type inequalities via the Delta-integral.

Theorem B *Assume that $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$, and $a < b$. Let $\frac{1}{p} + \frac{1}{q} = 1$ and $u, v, \omega \in C_{rd}([a, b], \mathbb{R})$. If $p > 1$, then*

$$\int_a^b |\omega(x)||u(x)v(x)| \Delta x \le \left(\int_a^b |\omega(x)||u(x)|^p \Delta x \right)^{\frac{1}{p}} \left(\int_a^b |\omega(x)||v(x)|^q \Delta x \right)^{\frac{1}{q}}. \tag{4}$$

If $p < 0$ or $q < 0$, then inequality (4) is reversed.

In 2008, Özkan et al. [27] presented the following time scale versions of inequalities (1) and (2) via the Nabla-integral (∇-integral) and Diamond-Alpha integral ($\diamond_\alpha$-integral).

Theorem C *Assume that $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$, and $a < b$. Let $\frac{1}{p} + \frac{1}{q} = 1$ and $u, v, \omega \in C_{ld}([a, b], \mathbb{R})$. If $p > 1$, then*

$$\int_a^b |\omega(x)||u(x)v(x)| \nabla x \le \left(\int_a^b |\omega(x)||u(x)|^p \nabla x \right)^{\frac{1}{p}} \left(\int_a^b |\omega(x)||v(x)|^q \nabla x \right)^{\frac{1}{q}}; \tag{5}$$

If $p < 0$ or $q < 0$, then inequality (5) is reversed.

Theorem D *Assume that $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$, and $a < b$. Let $\frac{1}{p} + \frac{1}{q} = 1$, and let $u, v, \omega : [a, b] \to \mathbb{R}$ be $\diamond_\alpha$-integrable functions. If $p > 1$, then*

$$\int_a^b |\omega(x)||u(x)v(x)| \diamond_\alpha x \le \left(\int_a^b |\omega(x)||u(x)|^p \diamond_\alpha x \right)^{\frac{1}{p}} \left(\int_a^b |\omega(x)||v(x)|^q \diamond_\alpha x \right)^{\frac{1}{q}}. \tag{6}$$

If $p < 0$ or $q < 0$, then inequality (6) is reversed.

Remark 1.1 If $\alpha = 0$ in Theorem C, then inequality (6) reduces to inequality (5). If $\alpha = 1$ in Theorem C, then inequality (6) reduces to inequality (4).

The second purpose of this paper is to give the time scale versions of the Hölder and reverse Hölder inequalities for the triple Diamond-Alpha integral. Then we obtain some

new generalizations of the Hölder and reverse Hölder inequalities for the triple Diamond-Alpha integral on time scales. Moreover, using the obtained results, we present a new generalization of the Minkowski inequality for the triple Diamond-Alpha integral on time scales.

2 Main results

For details on time scales theory, the readers may consult [1, 3–9] and the references therein. Now we give the definition of triple Diamond-Alpha integral for functions of three variables.

The triple Diamond-Alpha integral is defined as an iterated integral. Suppose that $\mathbb{T}$ is a time scale and $a_i, b_i \in \mathbb{T}$ with $a_i < b_i$ ($i = 1, 2, 3$). Let $f(x_1, x_2, x_3)$ be a real-valued function on $\mathbb{T} \times \mathbb{T} \times \mathbb{T}$. Because we need the notation of partial derivatives with respect to variables x_i, we denote the time scale partial derivatives of $f(x_1, x_2, x_3)$ with respect to x_i by $f^{\diamond_\alpha^i}(x_1, x_2, x_3)$, $i = 1, 2, 3$. We now give the definition of these partial derivatives. Fixing $x_2, x_3 \in \mathbb{T}$, the diamond-α derivative of a function

$$\mathbb{T} \to \mathbb{R},$$
$$x_1 \to f(x_1, x_2, x_3)$$

is denoted by $f^{\diamond_\alpha^1}$. Next, fixing $x_1, x_3 \in \mathbb{T}$, the diamond-α derivative of a function

$$\mathbb{T} \to \mathbb{R},$$
$$x_2 \to f(x_1, x_2, x_3)$$

is denoted by $f^{\diamond_\alpha^2}$. Finally, fixing $x_1, x_2 \in \mathbb{T}$, the diamond-α derivative of a function

$$\mathbb{T} \to \mathbb{R},$$
$$x_3 \to f(x_1, x_2, x_3)$$

is denoted by $f^{\diamond_\alpha^3}$.

If a function f has a $\diamond_\alpha^1$ antiderivative F_1, F_1 has a $\diamond_\alpha^2$ antiderivative F_2, and F_2 has a $\diamond_\alpha^3$ antiderivative F_3, that is, $F_1^{\diamond_\alpha^1} = f$, $F_2^{\diamond_\alpha^2} = F_1$, and $F_3^{\diamond_\alpha^3} = F_2$, then

$$
\begin{aligned}
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} & f(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \\
:= & \int_{a_2}^{b_2} \int_{a_3}^{b_3} \big(F_1(b_1, x_2, x_3) - F_1(a_1, x_2, x_3) \big) \diamond_\alpha x_2 \diamond_\alpha x_3 \\
= & \int_{a_3}^{b_3} \big[F_2(b_1, b_2, x_3) - F_2(b_1, a_2, x_3) - \big(F_2(a_1, b_2, x_3) - F_2(a_1, a_2, x_3) \big) \big] \diamond_\alpha x_3 \\
= & F_3(b_1, b_2, b_3) - F_3(b_1, b_2, a_3) - F_3(b_1, a_2, b_3) + F_3(b_1, a_2, a_3) \\
& - F_3(a_1, b_2, b_3) + F_3(a_1, b_2, a_3) + F_3(a_1, a_2, b_3) - F_3(a_1, a_2, a_3).
\end{aligned}
\tag{7}
$$

By this definition it is easy to obtain the following property for the triple Diamond-Alpha integral.

Proposition 2.1 *Suppose that $\mathbb{T}$ is a time scale, $a_i, b_i \in \mathbb{T}$ with $a_i < b_i$ $(i = 1, 2, 3)$, and $f(x_1, x_2, x_3)$ and $g(x_1, x_2, x_3)$ are $\diamond_\alpha$-integrable functions on $[a_i, b_i]_{\mathbb{T}}^3$ $(i = 1, 2, 3)$.*

(i) *If $f(x_1, x_2, x_3) \geq 0$ for all $x_i \in [a_i, b_i]_{\mathbb{T}}$ $(i = 1, 2, 3)$, then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \geq 0;$$

(ii) *If $f(x_1, x_2, x_3) \leq g(x_1, x_2, x_3)$ for all $x_i \in [a_i, b_i]_{\mathbb{T}}$ $(i = 1, 2, 3)$, then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3$$

$$\leq \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} g(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3;$$

(iii) *If $f(x_1, x_2, x_3) \geq 0$ for all $x_i \in [a_i, b_i]_{\mathbb{T}}$ $(i = 1, 2, 3)$, then $f(x_1, x_2, x_3) = 0$ if and only if*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 = 0.$$

To prove the main results, we need the following lemmas.

Lemma 2.2 (Bernoulli's inequality; see [28]) *If $x > 0$ and $p > 1$, then*

$$x^p \geq px + 1 - p. \tag{8}$$

Lemma 2.3 (Young inequality; see [11]) *Let $a, b > 0$.*

(i) *If $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q. \tag{9}$$

(ii) *If $p > 0$, $q < 0$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \geq \frac{1}{p}a^p + \frac{1}{q}b^q. \tag{10}$$

Lemma 2.4 (AG inequality; see [11]) *Let $\alpha_i > 0$ $(i = 1, 2, 3)$.*

(i) *If $1 < \lambda_1, \lambda_2, \ldots, \lambda_n < \infty$ with $\sum_{i=1}^{n} \frac{1}{\lambda_i} = 1$, then*

$$\prod_{i=1}^{n} \alpha_i \leq \sum_{i=1}^{n} \frac{\alpha_i^{\lambda_i}}{\lambda_i}. \tag{11}$$

(ii) *If $\lambda_1 > 0, \lambda_2, \ldots, \lambda_m < 0$ with $\sum_{i=1}^{m} \frac{1}{\lambda_i} = 1$, then*

$$\prod_{i=1}^{m} \alpha_i \geq \sum_{i=1}^{m} \frac{\alpha_i^{\lambda_i}}{\lambda_i}. \tag{12}$$

Lemma 2.5 (Schlömilch's inequality for triple Diamond-Alpha integral) *Suppose that* $\mathbb{T}$ *is a time scale,* $a_i, b_i \in \mathbb{T}$ *with* $a_i < b_i$ $(i = 1, 2, 3)$, *and* $f(x_1, x_2, x_3), \omega(x_1, x_2, x_3) : [a_i, b_i]_{\mathbb{T}}^3 \to [0, +\infty)$ *are* $\diamond_\alpha$-*integrable functions with* $\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 > 0$. *Then, for* $s > r > 0$, *we have*

$$
\left(\frac{\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1, x_2, x_3) f^s(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3}{\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3} \right)^{\frac{1}{s}}
$$

$$
\geq \left(\frac{\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1, x_2, x_3) f^r(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3}{\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3} \right)^{\frac{1}{r}}.
$$

Proof Without loss of generality, we may suppose that $\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} h(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 = 1$. If $s > r > 0$, then $\frac{s}{r} > 1$. Therefore, by inequality (8) we have

$$
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left[\left(\frac{f(x_1, x_2, x_3)}{\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1, x_2, x_3) f(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3} \right)^{\frac{s}{r}} \right.
$$

$$
\left. \times \omega(x_1, x_2, x_3) \right] \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3
$$

$$
\geq \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left[\left(\frac{\frac{s}{r} f(x_1, x_2, x_3)}{\int_a^b \int_a^b \cdots \int_a^b \omega(x_1, x_2, x_3) f(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3} + 1 - \frac{s}{r} \right) \right.
$$

$$
\left. \times \omega(x_1, x_2, x_3) \right] \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3
$$

$$
= 1,
$$

which implies

$$
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1, x_2, x_3) f^{\frac{s}{r}}(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3
$$

$$
\geq \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1, x_2, x_3) f(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{s}{r}}. \tag{13}
$$

Replacing f by f^r in (13), we find

$$
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1, x_2, x_3) f^s(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3
$$

$$
\geq \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1, x_2, x_3) f^r(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{s}{r}}.
$$

Then

$$
\left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1, x_2, x_3) f^s(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{s}}
$$

$$
\geq \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1, x_2, x_3) f^r(x_1, x_2, x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{r}}.
$$

Thus, the proof of Lemma 2.5 is completed. $\qquad\square$

Now, we give the following Hölder and reverse Hölder inequalities for triple Diamond-Alpha integral.

Theorem 2.6 *Suppose that* $\mathbb{T}$ *is a time scale,* $a_i, b_i \in \mathbb{T}$ *with* $a_i < b_i$ *(*$i = 1, 2, 3$*), and* $f(x_1, x_2, x_3), g(x_1, x_2, x_3), \omega(x_1, x_2, x_3) : [a_i, b_i]_{\mathbb{T}}^3 \to \mathbb{R}$ *are* $\diamond_\alpha$*-integrable functions.*

(i) *If* $p, q > 0$ *are such that* $\frac{1}{p} + \frac{1}{q} = 1$*, then*

$$
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) f(x_1, x_2, x_3) g(x_1, x_2, x_3) \right| \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3
$$

$$
\leq \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) \right| \left| f(x_1, x_2, x_3) \right|^p \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{p}}
$$

$$
\times \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) \right| \left| g(x_1, x_2, x_3) \right|^q \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{q}}. \qquad (14)
$$

(ii) *If* $q < 0$ *is such that* $\frac{1}{p} + \frac{1}{q} = 1$*, then*

$$
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) f(x_1, x_2, x_3) g(x_1, x_2, x_3) \right| \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3
$$

$$
\geq \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) \right| \left| f(x_1, x_2, x_3) \right|^p \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{p}}
$$

$$
\times \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) \right| \left| g(x_1, x_2, x_3) \right|^q \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{q}}. \qquad (15)
$$

Proof Case (i): Let $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$. Without loss of generality, we may suppose that

$$
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) \right| \left| f(x_1, x_2, x_3) \right|^p \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \neq 0
$$

and

$$
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) \right| \left| g(x_1, x_2, x_3) \right|^q \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \neq 0.
$$

Let

$$
\mu = \frac{|\omega(x_1, x_2, x_3)|^{\frac{1}{p}} |f(x_1, x_2, x_3)|}{\left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |\omega(x_1, x_2, x_3)| |f(x_1, x_2, x_3)|^p \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{p}}}
$$

and

$$
\nu = \frac{|\omega(x_1, x_2, x_3)|^{\frac{1}{q}} |g(x_1, x_2, x_3)|}{\left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |\omega(x_1, x_2, x_3)| |g(x_1, x_2, x_3)|^q \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{q}}}.
$$

From the Young inequality (9) we get

$$
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \mu \nu \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3
$$

$$
\leq \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left(\frac{\mu^p}{p} + \frac{\nu^q}{q} \right) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3
$$

$$
= \frac{1}{p} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left(\frac{|\omega(x_1,x_2,x_3)|}{\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |\omega(x_1,x_2,x_3)||f(x_1,x_2,x_3)|^p \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3} \right.
$$

$$
\times \left. |f(x_1,x_2,x_3)|^p \right) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3
$$

$$
+ \frac{1}{q} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left(\frac{|\omega(x_1,x_2,x_3)|}{\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |\omega(x_1,x_2,x_3)||f(x_1,x_2,x_3)|^q \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3} \right.
$$

$$
\times \left. |f(x_1,x_2,x_3)|^q \right) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3
$$

$$
= \frac{1}{p} + \frac{1}{q} = 1.
$$

Therefore, we get the desired inequality (14).

Case (ii). Let $q < 0$ with $\frac{1}{p} + \frac{1}{q} = 1$. Write $\alpha = -\frac{p}{q}, \beta = \frac{1}{q}$. Then $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ with $\alpha, \beta > 0$. Setting $f(x_1,x_2,x_3) = F(x_1,x_2,x_3)$ and $g(x_1,x_2,x_3) = G(x_1,x_2,x_3)$ in (14), we get

$$
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1,x_2,x_3) F(x_1,x_2,x_3) G(x_1,x_2,x_3) \right| \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3
$$

$$
\leq \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1,x_2,x_3) \right| \left| F(x_1,x_2,x_3) \right|^\alpha \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{\alpha}}
$$

$$
\times \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1,x_2,x_3) \right| \left| G(x_1,x_2,x_3) \right|^\beta \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{\beta}}. \tag{16}
$$

Putting $F(x_1,x_2,x_3) = f^{-q}(x_1,x_2,x_3)$ and $G(x_1,x_2,x_3) = f^q(x_1,x_2,x_3) g^q(x_1,x_2,x_3)$ in equality (16), we immediately obtain the desired inequality (15). □

Next, we present the following generalizations of inequalities (14) and (15).

Theorem 2.7 *Suppose that $\mathbb{T}$ is a time scale, $a_i, b_i \in \mathbb{T}$ with $a_i < b_i$ ($i = 1,2,3$), and $f(x_1,x_2,x_3), g(x_1,x_2,x_3), \omega(x_1,x_2,x_3) : [a_i, b_i]_{\mathbb{T}}^3 \to \mathbb{R}$ are $\diamond_\alpha$-integrable functions.*

(i) *Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ with $p, q, r \in \mathbb{R} - \{0\}$, $p > 0$ and $q > 0$ or $p > 0$, $q < 0$, and $r < 0$. Then*

$$
\left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1,x_2,x_3) f^r(x_1,x_2,x_3) g^r(x_1,x_2,x_3) \right| \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{r}}
$$

$$
\leq \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1,x_2,x_3) \right| \left| f(x_1,x_2,x_3) \right|^p \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{p}}
$$

$$
\times \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1,x_2,x_3) \right| \left| g(x_1,x_2,x_3) \right|^q \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{q}}. \tag{17}
$$

(ii) *Let* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ *with* $p, q, r \in \mathbb{R} - \{0\}$, $p > 0$, $q < 0$, *and* $r > 0$ *or* $p < 0$ *and* $q < 0$. *Then*

$$\left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) f^r(x_1, x_2, x_3) g^r(x_1, x_2, x_3) \right| \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{r}}$$

$$\geq \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) \right| \left| f(x_1, x_2, x_3) \right|^p \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{p}}$$

$$\times \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) \right| \left| g(x_1, x_2, x_3) \right|^q \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{q}}. \qquad (18)$$

Proof (i) Case 1. When $p > 0$, $q > 0$, by $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ we get

$$\frac{p}{r} > 1, \qquad \frac{1}{p/r} + \frac{1}{q/r} = 1.$$

Then, by (14) we find that

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) f^r(x_1, x_2, x_3) g^r(x_1, x_2, x_3) \right| \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3$$

$$\leq \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) f^r(x_1, x_2, x_3) \right|^{\frac{p}{r}} \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{r}{p}}$$

$$\times \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) g^r(x_1, x_2, x_3) \right|^{\frac{q}{r}} \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{r}{q}}.$$

Therefore

$$\left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) f^r(x_1, x_2, x_3) g^r(x_1, x_2, x_3) \right| \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{r}}$$

$$\leq \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) f^r(x_1, x_2, x_3) \right|^{\frac{p}{r}} \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{p}}$$

$$\times \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) g^r(x_1, x_2, x_3) \right|^{\frac{q}{r}} \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{q}}.$$

Case 2. When $p > 0$, $q < 0$, and $r < 0$, by the same method as in Case 1, we can obtain inequality (17).

(ii) Case I. When $p > 0$, $q < 0$, and $r > 0$, we find from $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ that

$$\frac{r}{p} > 1, \qquad \frac{1}{r/p} + \frac{1}{-q/p} = \frac{p}{r} - \frac{p}{q} = p\left(\frac{1}{r} - \frac{1}{q} \right) = 1.$$

Then, by inequality (14) we get

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) f^p(x_1, x_2, x_3) \right| \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3$$

$$= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left(\left| \omega(x_1, x_2, x_3) f^p(x_1, x_2, x_3) \right| \right.$$

$$\times \big|g^p(x_1,x_2,x_3)g^{-p}(x_1,x_2,x_3)\big|\bigg) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3$$

$$\leq \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\big|\omega(x_1,x_2,x_3)\big|\big|f^p(x_1,x_2,x_3)g^p(x_1,x_2,x_3)\big|^{\frac{r}{p}}\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{\frac{p}{r}}$$

$$\times \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\big|\omega(x_1,x_2,x_3)\big|\big|g^{-p}(x_1,x_2,x_3)\big|^{-\frac{q}{p}}\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{-\frac{p}{q}}$$

$$= \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\big|\omega(x_1,x_2,x_3)\big|\big|f^r(x_1,x_2,x_3)g^r(x_1,x_2,x_3)\big|\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{\frac{p}{r}}$$

$$\times \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\big|\omega(x_1,x_2,x_3)\big|\big|g^q(x_1,x_2,x_3)\big|\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{-\frac{p}{q}},$$

which implies

$$\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\big|\omega(x_1,x_2,x_3)f^p(x_1,x_2,x_3)\big|\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{\frac{1}{p}}$$

$$\leq \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\big|\omega(x_1,x_2,x_3)\big|\big|f^r(x_1,x_2,x_3)g^r(x_1,x_2,x_3)\big|\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{\frac{1}{r}}$$

$$\times \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\big|\omega(x_1,x_2,x_3)\big|\big|g^q(x_1,x_2,x_3)\big|\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{-\frac{1}{q}}. \qquad (19)$$

Thus from inequality (19) we obtain

$$\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\big|\omega(x_1,x_2,x_3)f^r(x_1,x_2,x_3)g^r(x_1,x_2,x_3)\big|\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{\frac{1}{r}}$$

$$\geq \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\big|\omega(x_1,x_2,x_3)\big|\big|f(x_1,x_2,x_3)\big|^p\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{\frac{1}{p}}$$

$$\times \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\big|\omega(x_1,x_2,x_3)\big|\big|g(x_1,x_2,x_3)\big|^q\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{\frac{1}{q}}.$$

Case II. When $p < 0$ and $q < 0$, by the same method as in Case I, we can obtain the desired inequality (18). The proof of Theorem 2.7 is completed. $\qquad\square$

We present another generalization of inequality (14).

Theorem 2.8 *Suppose that $\mathbb{T}$ is a time scale, $a_i, b_i \in \mathbb{T}$ with $a_i < b_i$ ($i = 1, 2, 3$), and that $f(x_1,x_2,x_3)$, $g(x_1,x_2,x_3)$, $\omega(x_1,x_2,x_3) : [a_i, b_i]_\mathbb{T}^3 \to \mathbb{R}$ are $\diamond_\alpha$-integrable functions. If $p > 0$, $q > 0$ are such that $0 < \frac{1}{p} + \frac{1}{q} < 1$, then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\big|\omega(x_1,x_2,x_3)f(x_1,x_2,x_3)g(x_1,x_2,x_3)\big|\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3$$

$$\leq \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\big|\omega(x_1,x_2,x_3)\big|\big|f(x_1,x_2,x_3)\big|^p\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{\frac{1}{p}}$$

$$\times \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\big|\omega(x_1,x_2,x_3)\big|\big|g(x_1,x_2,x_3)\big|^q\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{\frac{1}{q}}. \qquad (20)$$

Proof Denote $\gamma := \frac{1}{p} + \frac{1}{q}$, $\zeta = \gamma p$, $\delta = q\gamma$. Then $\zeta > 1$, $\delta > 1$, and $\frac{1}{\zeta} + \frac{1}{\delta} = 1$. Hence from Lemma 2.5 with $\zeta < p$, $\delta < q$ and inequality (14) we find that

$$
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1,x_2,x_3) f(x_1,x_2,x_3) g(x_1,x_2,x_3) \right| \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3
$$

$$
\leq \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1,x_2,x_3) \right| \left| f(x_1,x_2,x_3) \right|^\zeta \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{\zeta}}
$$

$$
\times \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1,x_2,x_3) \right| \left| g(x_1,x_2,x_3) \right|^\delta \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{\delta}}
$$

$$
\leq \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1,x_2,x_3) \right| \left| f(x_1,x_2,x_3) \right|^p \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{p}}
$$

$$
\times \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1,x_2,x_3) \right| \left| g(x_1,x_2,x_3) \right|^q \diamond_\alpha x_1 \diamond_\alpha x_2 \cdots \diamond_\alpha x_n \right)^{\frac{1}{q}}.
$$

The proof of Theorem 2.8 is completed. □

Theorem 2.9 *Suppose that $\mathbb{T}$ is a time scale, $a_i, b_i \in \mathbb{T}$ with $a_i < b_i$ ($i = 1, 2, 3$), and that $f_i(x_1, x_2, x_3)$ ($i = 1, 2, \ldots, m$), $\omega(x_1, x_2, x_3) : [a_i, b_i]_\mathbb{T}^3 \to \mathbb{R}$ are $\diamond_\alpha$-integrable functions.*

(I) *Let $1 < \lambda_1, \lambda_2, \ldots, \lambda_m < \infty$ such that $\sum_{i=1}^m \frac{1}{\lambda_i} = 1$. Then*

$$
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1,x_2,x_3) \left(\prod_{i=1}^m f_i(x_1,x_2,x_3) \right) \right| \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3
$$

$$
\leq \prod_{i=1}^m \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1,x_2,x_3) \right| \left| f_i(x_1,x_2,x_3) \right|^{\lambda_i} \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{\lambda_i}}. \quad (21)
$$

(II) *Let $\lambda_1 > 0$, $\lambda_2, \ldots, \lambda_m < 0$ be such that $\sum_{i=1}^m \frac{1}{\lambda_i} = 1$. Then inequality (21) is reversed.*

Proof Case (I). Without loss of generality, we may suppose that

$$
\prod_{i=1}^m \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1,x_2,x_3) \right| \left| f_i(x_1,x_2,x_3) \right|^{\lambda_i} \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{\lambda_i}} \neq 0.
$$

Write

$$
\alpha_i = \frac{|\omega(x_1,x_2,x_3)|^{\frac{1}{\lambda_i}} |f_i(x_1,x_2,x_3)|}{\left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |\omega(x_1,x_2,x_3)| |f_i(x_1,x_2,x_3)|^{\lambda_i} \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{\lambda_i}}}
$$

for $i = 1, 2, \ldots, m$.

From *AG* inequality (11) we have

$$
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \prod_{i=1}^m \alpha_i \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3
$$

$$
\leq \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left(\sum_{i=1}^m \frac{\alpha_i^{\lambda_i}}{\lambda_i} \right) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3
$$

$$= \sum_{i=1}^{m} \frac{1}{\lambda_i} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left(\frac{|\omega(x_1,x_2,x_3)|}{\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |\omega(x_1,x_2,x_3)||f_i(x_1,x_2,x_3)|^{\lambda_i} \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3} \right.$$

$$\left. \times |f_i(x_1,x_2,x_3)|^{\lambda_i} \right) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3$$

$$= \sum_{i=1}^{m} \frac{1}{\lambda_i} = 1.$$

Therefore, we get the desired inequality (21).

Case (II). By the same method as in Case (I) and using the reversed inequality (11), we can obtain the desired result. □

Theorem 2.10 *Suppose that $\mathbb{T}$ is a time scale, $a_i, b_i \in \mathbb{T}$ with $a_i < b_i$ ($i = 1, 2, 3$), and that $f_i(x_1,x_2,x_3)$, $\omega(x_1,x_2,x_3) : [a_i,b_i]_{\mathbb{T}}^3 \to [0,+\infty)$ ($i = 1,2,\dots,m$) are $\diamond_\alpha$-integrable functions with $\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1,x_2,x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 = 1$. Let $0 < \lambda_1, \lambda_2, \dots, \lambda_m < 1$ be such that $\lambda_1 + \lambda_2 + \cdots + \lambda_m = k < 1$. Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1,x_2,x_3) \left(\prod_{i=1}^{m} f_i^{\lambda_i}(x_1,x_2,x_3) \right) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3$$

$$\leq \prod_{i=1}^{m} \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1,x_2,x_3) f_i(x_1,x_2,x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\lambda_i}. \tag{22}$$

Proof Denote $\xi_i = \frac{\lambda_i}{k}$ ($i = 1, 2, \dots, m$). Then

$$\xi_1 + \xi_2 + \cdots + \xi_m = 1.$$

Write

$$\psi_i(x_1,x_2,x_3) = f_i^k(x_1,x_2,x_3) \quad (i = 1, 2, \dots, m).$$

From Theorem 2.9 and Lemma 2.5 we have

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1,x_2,x_3) \prod_{i=1}^{m} f_i^{\lambda_i}(x_1,x_2,x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3$$

$$= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1,x_2,x_3) \prod_{i=1}^{m} \psi_i^{\xi_i}(x_1,x_2,x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3$$

$$\leq \prod_{i=1}^{m} \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1,x_2,x_3) \psi_i(x_1,x_2,x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\xi_i}$$

$$= \prod_{i=1}^{m} \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1,x_2,x_3) f_i^k(x_1,x_2,x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{\lambda_i}{k}}$$

$$\leq \prod_{i=1}^{m} \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \omega(x_1,x_2,x_3) f_i(x_1,x_2,x_3) \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\lambda_i} \tag{23}$$

for $k < 1$. Therefore the proof of Theorem 2.10 is completed. □

Let $f_i^{\lambda_i}(x_1,x_2,x_3)=\mu_i(x_1,x_2,x_3)$, that is, $f_i(x_1,x_2,x_3)=\mu_i^{\frac{1}{\lambda_i}}(x_1,x_2,x_3)$ for $i=1,2,\ldots,m$. Then from Theorem 2.10 we get the following Hölder-type inequality.

Corollary 2.11 *Suppose that $\mathbb{T}$ is a time scale, $a_i,b_i \in \mathbb{T}$ with $a_i < b_i$ ($i=1,2,3$), and that $g_i(x_1,x_2,x_3)$, $\omega(x_1,x_2,x_3):[a_i,b_i]_{\mathbb{T}}^3 \to [0,+\infty)$ ($i=1,2,\ldots,m$) are $\diamond_\alpha$-integrable functions with $\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\omega(x_1,x_2,x_3)\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 = 1$. Let $0 < \lambda_1,\lambda_2,\ldots,\lambda_m < 1$ be such that $\lambda_1 + \lambda_2 + \cdots + \lambda_m = k < 1$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\omega(x_1,x_2,x_3)\left(\prod_{i=1}^{m}g_i(x_1,x_2,x_3)\right)\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3$$

$$\leq \prod_{i=1}^{m}\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\omega(x_1,x_2,x_3)g_i^{\frac{1}{\lambda_i}}(x_1,x_2,x_3)\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{\lambda_i}. \tag{24}$$

3 Application

In this section, using the obtained results, we give the following generalization of the Minkowski inequality for the triple Diamond-Alpha integral on time scales.

Theorem 3.1 *Suppose that $\mathbb{T}$ is a time scale, $a_i,b_i \in \mathbb{T}$ with $a_i < b_i$ ($i=1,2,3$), and that $f_i(x_1,x_2,x_3)$ ($i=1,2,\ldots,m$) and $\omega(x_1,x_2,x_3):[a_i,b_i]_{\mathbb{T}}^3 \to \mathbb{R}$ are $\diamond_\alpha$-integrable functions.*

(I) *If $p > 1$, then*

$$\left[\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left|\omega(x_1,x_2,x_3)\left(\sum_{i=1}^{m}f_i(x_1,x_2,x_3)\right)^p\right|\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right]^{\frac{1}{p}}$$

$$\leq \sum_{i=1}^{m}\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}|\omega(x_1,x_2,x_3)|\,|f_i(x_1,x_2,x_3)|^p\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{\frac{1}{p}}. \tag{25}$$

(II) *If $0 < p < 1$, then inequality (25) is reversed.*

Proof We prove only case (I). Write $\Phi(x_1,x_2,x_3) = \sum_{i=1}^{m}f_i(x_1,x_2,x_3)$. Without loss of generality, we may assume that $\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\omega(x_1,x_2,x_3)\Phi^p(x_1,x_2,x_3)\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \neq 0$. From inequality (14) for $p,q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left|f_i(x_1,x_2,x_3)\omega(x_1,x_2,x_3)\Phi^{p-1}(x_1,x_2,x_3)\right|\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3$$

$$\leq \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left|\omega(x_1,x_2,x_3)\right|\left|f_i(x_1,x_2,x_3)\right|^p\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{\frac{1}{p}}$$

$$\times \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left|\omega(x_1,x_2,x_3)\right|\left|\Phi(x_1,x_2,x_3)\right|^{(p-1)q}\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{\frac{1}{q}}$$

$$= \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left|\omega(x_1,x_2,x_3)\right|\left|f_i(x_1,x_2,x_3)\right|^p\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{\frac{1}{p}}$$

$$\times \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left|\omega(x_1,x_2,x_3)\right|\left|\Phi(x_1,x_2,x_3)\right|^p\diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3\right)^{\frac{1}{q}}.$$

Therefore we get

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) \Phi^p(x_1, x_2, x_3) \right| \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3$$

$$= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) \Phi(x_1, x_2, x_3) \Phi^{p-1}(x_1, x_2, x_3) \right| \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3$$

$$= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) f_1(x_1, x_2, x_3) \Phi^{p-1}(x_1, x_2, x_3) \right| \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3$$

$$+ \cdots + \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) f_m(x_1, x_2, x_3) \Phi^{p-1}(x_1, x_2, x_3) \right| \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3$$

$$\leq \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) \right| \left| \Phi(x_1, x_2, x_3) \right|^p \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{q}}$$

$$\times \left[\sum_{i=1}^m \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| \omega(x_1, x_2, x_3) \right| \left| f_i(x_1, x_2, x_3) \right|^p \diamond_\alpha x_1 \diamond_\alpha x_2 \diamond_\alpha x_3 \right)^{\frac{1}{p}} \right].$$

Thus we obtain the desired inequality (25). □

4 Conclusions

As is well known, the Hölder inequality and its various extensions play a very important role in mathematical analysis. In this paper, based on the definition of the triple Diamond-Alpha integral for functions of three variables, we have presented the Hölder and reverse Hölder inequalities for the triple Diamond-Alpha integral on time scales. Moreover, we gave some new generalizations of the Hölder and reverse Hölder inequalities for the triple Diamond-Alpha integral. Finally, using the obtained results, we have obtained a new generalization of the Minkowski inequality for the triple Diamond-Alpha integral on time scales. In the future research, we will continue to explore other inequalities for the triple Diamond-Alpha integral on time scales.

Competing interests
The author declares that there is no conflict of interests regarding the publication of this paper.

Authors' contributions
The author read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References
1. Hilger, S.: Ein Maßkettenkalk ül mit Anwendung auf Zentrumsmannigfaltigkeiten. Ph.D. thesis, Universität Würzburg (1988)
2. Agarwal, R.P., Bohner, M., Peterson, A.: Inequalities on time scales: a survey. Math. Inequal. Appl. **4**(4), 535–557 (2001)
3. Bohner, M., Peterson, A.: Dynamic Equations on Time Scales. Birkhäuser, Basel (2001)
4. Barić, J., Bibi, R., Bohner, M., Nosheen, A., Pečarić, J.: Jensen Inequalities on Time Scales. Theory and Applications. Element, Zagreb (2015)
5. Bohner, M., Peterson, A.: Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston (2003)

6. Hilger, S.: Analysis on measure chains—a unified approach to continuous and discrete calculus. Results Math. **18**(1–2), 18–56 (1990)
7. Hilger, S.: Differential and difference calculus—unified!. Nonlinear Anal. **30**(5), 2683–2694 (1997)
8. Rogers, J.W.J., Sheng, Q.: Notes on the diamond-α dynamic derivative on time scales. J. Math. Anal. Appl. **326**(1), 228–241 (2007)
9. Sheng, Q., Fadag, M., Henderson, J., Davis, J.M.: An exploration of combined dynamic derivatives on time scales and their applications. Nonlinear Anal., Real World Appl. **7**(3), 395–413 (2006)
10. Wong, F.H., Yeh, C.C., Lian, W.C.: An extension of Jensen's inequality on time scales. Adv. Dyn. Syst. Appl. **1**(1), 113–120 (2006)
11. Mitrinović, D.S.: Analytic Inequalities. Springer, New York (1970)
12. Kuang, J.: Applied Inequalities. Shandong Science Press, Jinan (2003)
13. Agahi, H., Ouyang, Y., Mesiar, R., Pap, E., Štrboja, M.: Hölder and Minkowski type inequalities for pseudo-integral. Appl. Math. Comput. **217**(21), 8630–8639 (2011)
14. Zhao, Y., Yan, T., Ouyang, Y.: Several inequalities for the pan-integral. Inf. Sci. **372**, 625–633 (2016)
15. Tian, J., Yang, Z.-H.: Generalizations of Hu-type inequalities and their applications. J. Nonlinear Sci. Appl. **10**(4), 1971–1985 (2017)
16. Tian, J.-F., Ha, M.-H.: Extensions of Hölder-type inequalities on time scales and their applications. J. Nonlinear Sci. Appl. **10**(3), 937–953 (2017)
17. Tian, J.-F., Ha, M.-H.: Properties of generalized sharp Hölder's inequalities. J. Math. Inequal. **11**(2), 511–525 (2017)
18. Tian, J.-F., Ha, M.-H., Wang, C.: Improvements of generalized Hölder's inequalities and their applications. J. Math. Inequal. **12**(2), 459–471 (2018). https://doi.org/10.7153/jmi-2018-12-34
19. Tian, J., Ha, M.-H.: Properties and refinements of Aczél-type inequalities. J. Math. Inequal. **12**(1), 175–189 (2018)
20. Tian, J., Wang, W., Cheung, W.-S.: Periodic boundary value problems for first-order impulsive difference equations with time delay. Adv. Differ. Equ. **2018**, 79 (2018). https://doi.org/10.1186/s13662-018-1539-5
21. Tian, J.-F., Pedrycz, W.: New refinements of generalized Hölder's inequality and their applications. Math. Inequal. Appl. **19**(3), 805–822 (2016)
22. Yang, Z.-H., Tian, J.: Monotonicity and sharp inequalities related to gamma function. J. Math. Inequal. **12**(1), 1–22 (2018)
23. Yang, Z.-H., Tian, J.: Optimal inequalities involving power-exponential mean, arithmetic mean and geometric mean. J. Math. Inequal. **11**(4), 1169–1183 (2017)
24. Yang, Z.-H., Tian, J.: Monotonicity and inequalities for the gamma function. J. Inequal. Appl. **2017**, 317 (2017)
25. Zhao, C.-J., Cheung, W.-S.: Hölder's reverse inequality and its applications. Publ. Inst. Math. **99**(113), 211–216 (2016)
26. Wong, F.H., Yeh, C.C., Yu, S.L., Hong, C.H.: Young's inequality and related results on time scales. Appl. Math. Lett. **18**(9), 983–988 (2005)
27. Özkan, U.M., Sarikaya, M.Z., Yildirim, H.: Extensions of certain integral inequalities on time scales. Appl. Math. Lett. **21**(10), 993–1000 (2008)
28. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge University Press, Cambridge (1952)