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A Cauchy type inequality for Möbius operations

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Abstract

In this article, we show two fundamental features of the restriction of Möbius operations to the real numbers, that is, a Cauchy type inequality and a criterion for convergence of series.

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1 Introduction and preliminaries

Möbius addition is defined on the complex open unit disk $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ by

$$a \oplus b = \frac{a + b}{1 + \bar{a}b} \quad (a, b \in \mathbb{D}),$$

which appears in a wide variety of fields of mathematics. In particular, although Möbius addition is known in the literature as a *hyperbolic translation*, its group-like structure had gone unnoticed until it was uncovered by A.A. Ungar in 1988 [1], in the context of Einstein's special theory of relativity. Furthermore, Ungar extended Möbius addition, introduced Möbius scalar multiplication to open balls of arbitrary real inner product spaces and established the concept of gyrovector spaces, which have a vector space-like structure (see [2–4]).

In this article, we show two fundamental features of the restriction of Möbius operations to the real numbers, that is, a Cauchy type inequality and a criterion for convergence of series.

Definition 1.1 ([5]) Let $s > 0$ be a fixed number. The addition \oplus_s and scalar multiplication \otimes_s on the open interval $(-s, s)$ in the real line are defined by the equations

$$a \oplus_s b = \frac{a + b}{1 + \frac{1}{s^2}ab},$$
$$r \otimes_s a = s \tanh\left(r \tanh^{-1} \frac{a}{s}\right)$$

for any $-s < a, b < s, r \in \mathbb{R}$. These operations appear in one of the axioms (VV) of the real inner product gyrovector spaces defined by Ungar (see [2, Definition 6.2, Definition 6.83]).

Note that, on the interval $(-s, s)$, \oplus_s is commutative, associative, and the operations \oplus_s, \otimes_s together with the ordinary addition and multiplication have the following properties:

$$\begin{aligned} (r_1 r_2) \otimes_s a &= r_1 \otimes_s (r_2 \otimes_s a), \\ (r_1 + r_2) \otimes_s a &= r_1 \otimes_s a \oplus_s r_2 \otimes_s a, \\ r \otimes_s (a \oplus_s b) &= r \otimes_s a \oplus_s r \otimes_s b \end{aligned}$$

for any $-s < a, b < s, r_1, r_2, r \in \mathbb{R}$.

Proposition 1.2 (see [2, after Remark 3.41], [3, p. 1054]) *The Möbius addition (resp. Möbius scalar multiplication) reduces to the ordinary addition (resp. scalar multiplication) as $s \rightarrow \infty$, that is,*

$$\begin{aligned} a \oplus_s b &\rightarrow a + b \quad (s \rightarrow \infty), \\ r \otimes_s a &\rightarrow ra \quad (s \rightarrow \infty) \end{aligned}$$

for any $a, b, r \in \mathbb{R}$.

We simply denote \oplus_1, \otimes_1 by \oplus, \otimes , respectively. It is easy to see the following lemma, and we omit the proof.

Lemma 1.3 ([3, Theorem 4.2])

- (i) *If $0 < a, b < 1$, then $a \oplus b < a + b$.*
- (ii) *If $-1 < a < a' < 1$ and $0 < b < 1$, then $a \oplus b < a' \oplus b$.*

2 Results and discussion

2.1 A Cauchy type inequality

The classical Cauchy inequality for real numbers

$$x_1 y_1 + \dots + x_n y_n \leq (x_1^2 + \dots + x_n^2)^{\frac{1}{2}} (y_1^2 + \dots + y_n^2)^{\frac{1}{2}}$$

is one of the most fundamental inequalities in mathematics. In this section, we show an inequality of Cauchy type for Möbius operations.

Although some of the following lemmas might be well known, we present each proof for the sake of completeness and the convenience of readers.

Lemma 2.1 *If a, b are real numbers and $c = (a^2 + b^2)^{\frac{1}{2}}$, then we have*

$$\cosh c \leq \cosh a \cdot \cosh b.$$

The equality holds if and only if $a = 0$ or $b = 0$.

Proof It is straightforward to see

$$\begin{aligned} \cosh a \cdot \cosh b &= \left(1 + \frac{a^2}{2!} + \frac{a^4}{4!} + \dots + \frac{a^{2k}}{(2k)!} + \dots \right) \\ &\quad \times \left(1 + \frac{b^2}{2!} + \frac{b^4}{4!} + \dots + \frac{b^{2k}}{(2k)!} + \dots \right) \\ &= 1 + \frac{a^2 + b^2}{2!} + \frac{a^4 + 6a^2b^2 + b^4}{4!} + \dots \\ &\quad + \frac{a^{2k} + \dots + c_j a^{2(k-j)} b^{2j} + \dots + b^{2k}}{(2k)!} + \dots, \end{aligned}$$

where the coefficient of the term $c_j a^{2(k-j)} b^{2j}$ in the numerator is

$$c_j = \frac{(2k)!}{(2k-2j)!(2j)!} \geq \frac{k!}{(k-j)!j!} = \binom{k}{j}.$$

So we obtain

$$\begin{aligned} \cosh a \cdot \cosh b &\geq 1 + \frac{a^2 + b^2}{2!} + \frac{a^4 + 2a^2b^2 + b^4}{4!} + \dots \\ &\quad + \frac{a^{2k} + \dots + \binom{k}{j} a^{2(k-j)} b^{2j} + \dots + b^{2k}}{(2k)!} + \dots \\ &= 1 + \frac{c^2}{2!} + \frac{c^4}{4!} + \dots + \frac{c^{2k}}{(2k)!} + \dots \\ &= \cosh c. \end{aligned}$$

It is obvious that $a = 0$ or $b = 0$ implies the equality in the formula above. Conversely, if the equality holds, then we must have

$$\frac{a^4 + 6a^2b^2 + b^4}{4!} = \frac{a^4 + 2a^2b^2 + b^4}{4!},$$

which implies that $a = 0$ or $b = 0$. This completes the proof. □

Lemma 2.2 *If a, b are real numbers and $c = (a^2 + b^2)^{\frac{1}{2}}$, then we have*

$$(\tanh c)^2 \leq (\tanh a)^2 \oplus (\tanh b)^2.$$

The equality holds if and only if $a = 0$ or $b = 0$.

Proof We have to show

$$\left(\frac{e^c - e^{-c}}{e^c + e^{-c}} \right)^2 \leq \frac{\left(\frac{e^a - e^{-a}}{e^a + e^{-a}} \right)^2 + \left(\frac{e^b - e^{-b}}{e^b + e^{-b}} \right)^2}{1 + \left(\frac{e^a - e^{-a}}{e^a + e^{-a}} \right)^2 \left(\frac{e^b - e^{-b}}{e^b + e^{-b}} \right)^2},$$

which is equivalent to

$$\begin{aligned} \frac{e^{2c} - 2 + e^{-2c}}{e^{2c} + 2 + e^{-2c}} &\leq \frac{(e^a - e^{-a})^2(e^b + e^{-b})^2 + (e^b - e^{-b})^2(e^a + e^{-a})^2}{(e^a + e^{-a})^2(e^b + e^{-b})^2 + (e^a - e^{-a})^2(e^b - e^{-b})^2} \\ &= \frac{(e^{2a} - 2 + e^{-2a})(e^{2b} + 2 + e^{-2b}) + (e^{2b} - 2 + e^{-2b})(e^{2a} + 2 + e^{-2a})}{(e^{2a} + 2 + e^{-2a})(e^{2b} + 2 + e^{-2b}) + (e^{2a} - 2 + e^{-2a})(e^{2b} - 2 + e^{-2b})}. \end{aligned} \tag{1}$$

It is straightforward to see that the numerator and denominator of the right-hand side in the formula (1) are respectively

$$2(e^{2a+2b} + e^{2a-2b} - 4 + e^{-2a+2b} + e^{-2a-2b})$$

and

$$2(e^{2a+2b} + e^{2a-2b} + 4 + e^{-2a+2b} + e^{-2a-2b}).$$

Hence, to prove inequality (1), it is sufficient to show the following inequality:

$$\frac{e^{2c} - 2 + e^{-2c}}{e^{2c} + 2 + e^{-2c}} \leq \frac{e^{2a+2b} + e^{2a-2b} - 4 + e^{-2a+2b} + e^{-2a-2b}}{e^{2a+2b} + e^{2a-2b} + 4 + e^{-2a+2b} + e^{-2a-2b}},$$

which is equivalent to

$$\begin{aligned} (e^{2c} - 2 + e^{-2c})(e^{2a+2b} + e^{2a-2b} + 4 + e^{-2a+2b} + e^{-2a-2b}) \\ \leq (e^{2c} + 2 + e^{-2c})(e^{2a+2b} + e^{2a-2b} - 4 + e^{-2a+2b} + e^{-2a-2b}). \end{aligned} \tag{2}$$

It is easy to see that (2) is equivalent to

$$\frac{e^{2c} + e^{-2c}}{2} \leq \frac{e^{2a} + e^{-2a}}{2} \cdot \frac{e^{2b} + e^{-2b}}{2},$$

which actually holds by *Lemma 2.1*, because if $c = (a^2 + b^2)^{\frac{1}{2}}$, then we have

$$\{(2a)^2 + (2b)^2\}^{\frac{1}{2}} = (4a^2 + 4b^2)^{\frac{1}{2}} = 2(a^2 + b^2)^{\frac{1}{2}} = 2c.$$

The equality condition also follows from the previous lemma. This completes the proof. \square

Lemma 2.3 *If a_1, \dots, a_n are real numbers, then we have*

$$\tanh\{(a_1^2 + \dots + a_n^2)^{\frac{1}{2}}\} \leq \{(\tanh a_1)^2 \oplus \dots \oplus (\tanh a_n)^2\}^{\frac{1}{2}}. \tag{3}$$

The equality holds if and only if $a_j = 0$ except at most one j .

Proof For $n = 2$, the lemma is just *Lemma 2.2*. Suppose that the lemma is valid up to n . Put $a = (a_1^2 + \dots + a_n^2)^{\frac{1}{2}}$. Then *Lemma 2.2* implies that

$$\tanh\{(a^2 + a_{n+1}^2)^{\frac{1}{2}}\} \leq \{(\tanh a)^2 \oplus (\tanh a_{n+1})^2\}^{\frac{1}{2}}. \tag{4}$$

By the assumption of our induction, we also have

$$\tanh a = \tanh\{(a_1^2 + \dots + a_n^2)^{\frac{1}{2}}\} \leq \{(\tanh a_1)^2 \oplus \dots \oplus (\tanh a_n)^2\}^{\frac{1}{2}}. \tag{5}$$

As an immediate consequence of (4) and (5), we obtain (3) for $n + 1$.

The equality condition follows from *Lemma 2.2* easily. This completes the proof. \square

The following lemma is an immediate consequence of the additional formula of \tanh , and we omit the proof.

Lemma 2.4 *If a_1, \dots, a_n are real numbers, then we have*

$$\tanh a_1 \oplus \dots \oplus \tanh a_n = \tanh(a_1 + \dots + a_n).$$

For Möbius operations restricted to the open interval $(-1, 1)$, the following inequality is a counterpart to the celebrated Cauchy inequality.

Theorem 2.5 *If $r_1, \dots, r_n \geq 0$ and $0 \leq x_1, \dots, x_n < 1$, then we have*

$$r_1 \otimes x_1 \oplus \dots \oplus r_n \otimes x_n \leq (r_1^2 + \dots + r_n^2)^{\frac{1}{2}} \otimes (x_1^2 \oplus \dots \oplus x_n^2)^{\frac{1}{2}}. \tag{6}$$

The equality holds if and only if one of the following conditions is satisfied:

- (i) $r_j = 0$ ($j = 1, \dots, n$);
- (ii) $x_j = 0$ ($j = 1, \dots, n$);
- (iii) $r_j = x_j = 0$ except for precisely one j .

Proof Put $a_j = \tanh^{-1} x_j$ ($j = 1, \dots, n$).

$$\begin{aligned} & r_1 \otimes x_1 \oplus \dots \oplus r_n \otimes x_n \\ &= \tanh(r_1 \tanh^{-1} x_1) \oplus \dots \oplus \tanh(r_n \tanh^{-1} x_n) \\ &= \tanh(r_1 \tanh^{-1} x_1 + \dots + r_n \tanh^{-1} x_n) \\ &\leq \tanh\{(r_1^2 + \dots + r_n^2)^{\frac{1}{2}} (a_1^2 + \dots + a_n^2)^{\frac{1}{2}}\} \\ &\leq \tanh\left[(r_1^2 + \dots + r_n^2)^{\frac{1}{2}} \tanh^{-1}\{(x_1^2 \oplus \dots \oplus x_n^2)^{\frac{1}{2}}\}\right] \\ &= (r_1^2 + \dots + r_n^2)^{\frac{1}{2}} \otimes (x_1^2 \oplus \dots \oplus x_n^2)^{\frac{1}{2}}, \end{aligned}$$

because of the definition of the operation \otimes , *Lemma 2.4*, the ordinary Cauchy inequality, *Lemma 2.3*, and the definition of \otimes again.

Finally, it is obvious that the equality holds in (6) if one of the conditions (i)–(iii) is satisfied. Conversely, if the equality holds in (6), then, from the equality conditions of the ordinary Cauchy inequality and *Lemma 2.3*, one has (r_1, \dots, r_n) and (a_1, \dots, a_n) are linearly dependent, and $a_j = 0$ except at most one j . It yields the condition stated above. This completes the proof. \square

Remark 2.6 Let $s > 0$. It is immediate to see that

$$\frac{a}{s} \oplus \frac{b}{s} = \frac{a \oplus_s b}{s} \quad \text{and} \quad r \otimes \frac{a}{s} = \frac{r \otimes_s a}{s}$$

for any $-s < a, b < s, r \in \mathbb{R}$.

If $r_1, \dots, r_n \geq 0$ and $x_1, \dots, x_n \geq 0$, then, for sufficiently large $s > 0$, we have

$$r_1 \otimes_s x_1 \oplus_s \dots \oplus_s r_n \otimes_s x_n \leq (r_1^2 + \dots + r_n^2)^{\frac{1}{2}} \otimes_s (x_1^2 \oplus_{s^2} \dots \oplus_{s^2} x_n^2)^{\frac{1}{2}}. \tag{7}$$

Moreover, by letting $s \rightarrow \infty$ in inequality (7), we can get the classical Cauchy inequality.

2.2 A criterion for convergence of series

In this section, we show a criterion for convergence of series in Möbius addition.

Definition 2.7

$$\begin{aligned} T_e(2m-1) &= \sum_{(0)} x_j + \sum_{(2)} x_{j_1} x_{j_2} + \dots + \sum_{(2l)} \prod_{k=1}^{2l} x_{j_k} + \dots + \sum_{(2m-2)} \prod_{k=1}^{2m-2} x_{j_k}, \\ T_o(2m-1) &= \sum_{(1)} x_j + \sum_{(3)} x_{j_1} x_{j_2} x_{j_3} + \dots + \sum_{(2l+1)} \prod_{k=1}^{2l+1} x_{j_k} + \dots + \sum_{(2m-1)} \prod_{k=1}^{2m-1} x_{j_k}, \\ T_e(2m) &= \sum_{(0)} x_j + \sum_{(2)} x_{j_1} x_{j_2} + \dots + \sum_{(2l)} \prod_{k=1}^{2l} x_{j_k} + \dots + \sum_{(2m)} \prod_{k=1}^{2m} x_{j_k}, \\ T_o(2m) &= \sum_{(1)} x_j + \sum_{(3)} x_{j_1} x_{j_2} x_{j_3} + \dots + \sum_{(2l+1)} \prod_{k=1}^{2l+1} x_{j_k} + \dots + \sum_{(2m-1)} \prod_{k=1}^{2m-1} x_{j_k}. \end{aligned}$$

Here, each $\sum_{(q)}$ appearing in $T_e(p)$ or $T_o(p)$ indicates the sum of all combinations which choose q items from x_1, \dots, x_p . Exceptionally, we define $\sum_{(0)} x_j = 1$. For example, $m = 2$

$$\begin{aligned} T_e(3) &= \sum_{(0)} x_j + \sum_{(2)} x_{j_1} x_{j_2} = 1 + x_1 x_2 + x_1 x_3 + x_2 x_3, \\ T_o(3) &= \sum_{(1)} x_j + \sum_{(3)} x_{j_1} x_{j_2} x_{j_3} = x_1 + x_2 + x_3 + x_1 x_2 x_3, \\ T_e(4) &= \sum_{(0)} x_j + \sum_{(2)} x_{j_1} x_{j_2} + \sum_{(4)} \prod_{k=1}^4 x_{j_k} \\ &= 1 + x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4 + x_1 x_2 x_3 x_4, \\ T_o(4) &= \sum_{(1)} x_j + \sum_{(3)} x_{j_1} x_{j_2} x_{j_3} \\ &= x_1 + x_2 + x_3 + x_4 + x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4. \end{aligned}$$

Lemma 2.8 *The following identities hold:*

$$T_o(2m) + T_e(2m)x_{2m+1} = T_o(2m+1), \tag{8}$$

$$T_e(2m) + T_o(2m)x_{2m+1} = T_e(2m + 1), \tag{9}$$

$$T_o(2m + 1) + T_e(2m + 1)x_{2m+2} = T_o(2m + 2), \tag{10}$$

$$T_e(2m + 1) + T_o(2m + 1)x_{2m+2} = T_e(2m + 2). \tag{11}$$

Proof For the first identity (8), it is immediate to see that

$$\begin{aligned} & T_o(2m) + T_e(2m)x_{2m+1} \\ &= \left\{ \sum_{(1)} x_j + \sum_{(3)} x_{j_1}x_{j_2}x_{j_3} + \cdots + \sum_{(2l+1)} \prod_{k=1}^{2l+1} x_{j_k} + \cdots + \sum_{(2m-1)} \prod_{k=1}^{2m-1} x_{j_k} \right\} \\ &+ \left\{ \sum_{(0)} x_j + \sum_{(2)} x_{j_1}x_{j_2} + \cdots + \sum_{(2l)} \prod_{k=1}^{2l} x_{j_k} + \cdots + \sum_{(2m)} \prod_{k=1}^{2m} x_{j_k} \right\} x_{2m+1} \\ &= T_o(2m + 1). \end{aligned}$$

Other three identities (9), (10), and (11) can be obtained in the same manner. This completes the proof. □

Definition 2.9 Let $s > 0$ be a fixed number. For any sequence $\{x_n\}_n$ in the open interval $(-s, s)$, we say that a series

$$x_1 \oplus_s x_2 \oplus_s \cdots \oplus_s x_n \oplus_s \cdots$$

converges if there exists $S \in (-s, s)$ such that $S_n \rightarrow S$, where the sequence $\{S_n\}_n$ is defined recursively by $S_1 = x_1$ and $S_n = S_{n-1} \oplus_s x_n$. In this case, we say the series converges to S and denote

$$S = x_1 \oplus_s x_2 \oplus_s \cdots \oplus_s x_n \oplus_s \cdots .$$

Lemma 2.10 *If $s = 1$, then the equality $\frac{T_o(n)}{T_e(n)} = S_n$ holds for all $n = 1, 2, \dots$*

Proof Recall the definition $S_n = x_1 \oplus \cdots \oplus x_n$. For $n = 1, 2$, we trivially have

$$\begin{aligned} S_1 &= x_1 = \frac{T_o(1)}{T_e(1)}, \\ S_2 &= x_1 \oplus x_2 = \frac{x_1 + x_2}{1 + x_1x_2} = \frac{T_o(2)}{T_e(2)}. \end{aligned}$$

Suppose that the lemma is valid up to $n \geq 2$. Then, if $n = 2m$,

$$\begin{aligned} S_{2m+1} &= S_{2m} \oplus x_{2m+1} = \frac{S_{2m} + x_{2m+1}}{1 + S_{2m}x_{2m+1}} = \frac{\frac{T_o(2m)}{T_e(2m)} + x_{2m+1}}{1 + \frac{T_o(2m)}{T_e(2m)}x_{2m+1}} = \frac{T_o(2m) + T_e(2m)x_{2m+1}}{T_e(2m) + T_o(2m)x_{2m+1}} \\ &= \frac{T_o(2m + 1)}{T_e(2m + 1)} \tag{12} \end{aligned}$$

by identities (8) and (9). Moreover, if $n = 2m + 1$, then identity (12), which has just been established, identities (10) and (11) yield that

$$\begin{aligned} S_{2m+2} &= S_{2m+1} \oplus x_{2m+2} = \frac{S_{2m+1} + x_{2m+2}}{1 + S_{2m+1}x_{2m+2}} = \frac{\frac{T_o(2m+1)}{T_e(2m+1)} + x_{2m+2}}{1 + \frac{T_o(2m+1)}{T_e(2m+1)}x_{2m+2}} \\ &= \frac{T_o(2m + 1) + T_e(2m + 1)x_{2m+2}}{T_e(2m + 1) + T_o(2m + 1)x_{2m+2}} \\ &= \frac{T_o(2m + 2)}{T_e(2m + 2)}. \end{aligned}$$

This completes the proof. □

The following theorem is a criterion for convergence of series in Möbius addition, which subsumes [6, Example 34].

Theorem 2.11 *Let $\{x_n\}_{n=1}^\infty$ be a sequence satisfying $0 \leq x_n < 1$ ($n = 1, 2, \dots$). Then the series $x_1 \oplus x_2 \oplus \dots \oplus x_n \oplus \dots$ converges in the open interval $(-1, 1)$ if and only if the series $\sum_{n=1}^\infty x_n < \infty$ in the ordinary sense.*

Proof (\Rightarrow) Suppose that $x_1 \oplus x_2 \oplus \dots \oplus x_n \oplus \dots = S < 1$. It is trivial that $S_n \leq S$ and that the sequence of ordinary partial sums $\{x_1 + \dots + x_n\}$ is non-decreasing. Moreover, it follows from Lemma 2.10 that

$$\begin{aligned} x_1 + \dots + x_n &\leq \frac{x_1 + \dots + x_n}{\prod_{j=1}^n (1 - x_j)} \leq \frac{T_o(n)}{\prod_{j=1}^n (1 - x_j)} = \frac{T_o(n)}{T_e(n) - T_o(n)} = \frac{\frac{T_o(n)}{T_e(n)}}{1 - \frac{T_o(n)}{T_e(n)}} \\ &= \frac{S_n}{1 - S_n} \leq \frac{S}{1 - S}. \end{aligned}$$

Thus the sequence $\{x_1 + \dots + x_n\}$ is bounded above, so that $\sum_{n=1}^\infty x_n < \infty$ in the ordinary sense.

(\Leftarrow) Suppose that $\sum_{n=1}^\infty x_n < \infty$. Then

$$\begin{aligned} d_M(S_n, S_m) &= |S_n \oplus (-S_m)| \\ &= (x_1 \oplus \dots \oplus x_m \oplus x_{m+1} \oplus \dots \oplus x_n) \oplus \{-(x_1 \oplus \dots \oplus x_m)\} \\ &= x_{m+1} \oplus \dots \oplus x_n \\ &\leq x_{m+1} + \dots + x_n \rightarrow 0 \quad (n > m \rightarrow \infty), \end{aligned}$$

by the definition of the function d_M (see [2, (6.288), p. 216]), the definition of the partial sum S_n in the Möbius addition, commutativity and associativity of \oplus on the interval $(-1, 1)$, $x_n \geq 0$, and the assumption $\sum_{n=1}^\infty x_n < \infty$. Since the space $((-1, 1), d_M)$ is complete (see [6, Lemma 20 and Theorem 26]), there exists a value $0 \leq S < 1$ such that $d_M(S_n, S) \rightarrow 0$, which implies that $S = x_1 \oplus x_2 \oplus \dots \oplus x_n \oplus \dots$. This completes the proof. □

Remark 2.12 *Let $\{x_n\}_{n=1}^\infty$ be a sequence satisfying $0 \leq x_n < s$ ($n = 1, 2, \dots$). Then the series $x_1 \oplus_s x_2 \oplus_s \dots \oplus_s x_n \oplus_s \dots$ converges in the open interval $(-s, s)$ if and only if the series $\sum_{n=1}^\infty x_n < \infty$ in the ordinary sense.*

Corollary 2.13 *If $0 \leq x < 1$, then the series $x \oplus x^2 \oplus \cdots \oplus x^n \oplus \cdots$ converges in the interval $(-1, 1)$.*

3 Conclusions

We proved two fundamental theorems, that is, a Cauchy type inequality and a criterion for convergence of series for the restriction of Möbius operations to the open interval. They have certain importance such as the classical Cauchy inequality and convergence of series have importance for the ordinary addition and multiplication in the real and complex numbers.

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