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# Variation and oscillation for the multilinear singular integrals satisfying Hörmander type conditions

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**Abstract**

Suppose that the kernel $K$ satisfies a certain Hörmander type condition. Let $b$ be a function satisfying $D^\alpha b \in BMO(\mathbb{R}^n)$ for $|\alpha| = m$, and let $T^b = \{T^b_\epsilon\}_{\epsilon>0}$ be a family of multilinear singular integral operators, i.e.,

$$T^b_\epsilon f(x) = \int_{|x-y|>\epsilon} \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x,y) f(y)\,dy.$$

The main purpose of this paper is to establish the weighted $L^p$-boundedness of the variation operator and the oscillation operator for $T^b$.

**MSC:** 42B25; 42B20; 47G10

**Keywords:** Variation operator; Oscillation operator; Multilinear operator; Hörmander condition; Singular integral; BMO

## 1 Introduction and results

Let $K$ be a singular kernel in $\mathbb{R}^n$ and satisfy

$$\left|K(x)\right| \leq \frac{C}{|x|^n} \quad \text{for } |x| > 0, \tag{1}$$

where $C$ is a fixed constant. Consider the family of operators $T = \{T_\epsilon\}_{\epsilon>0}$, where

$$T_\epsilon f(x) = \int_{|x-y|>\epsilon} K(x-y) f(y)\,dy. \tag{2}$$

The $\rho$-variation operator is defined by

$$\mathcal{V}_\rho(T)(f)(x) = \sup_{\epsilon_i \searrow 0} \left( \sum_{i=1}^{\infty} \left| T_{\epsilon_{i+1}} f(x) - T_{\epsilon_i} f(x) \right|^\rho \right)^{1/\rho},$$

where the supremum is taken over all sequences of real numbers $\{\epsilon_i\}$ decreasing to zero. The oscillation operator is defined by

$$\mathcal{O}(T)(f)(x) = \left( \sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \epsilon_{i+1} < \epsilon_i \leq t_i} \left| T_{\epsilon_{i+1}} f(x) - T_{\epsilon_i} f(x) \right|^2 \right)^{1/2},$$

where $\{t_i\}$ is a fixed sequence which decreases to zero.

Variation and oscillation can be used to measure the speed of convergence of certain convergent families of operators. The operators of variation and oscillation have attracted many researchers' attention in probability, ergodic theory, and harmonic analysis. Bourgain [1] obtained variation inequality for the ergodic averages of a dynamic system, his work has launched a new research direction in harmonic analysis. Campbell, Jones, Reinhold, and Wierdl in [2] established the $L^p$-boundedness of variation operator and oscillation operator for the Hilbert transform. Recently, many other publications have enriched this research direction [3–7].

Let $1 \leq r \leq \infty, m \in \mathbb{N} \cup \{0\}$. We say that the kernel $K$ satisfies the $H_{r,m}$-Hörmander condition if there exist the constants $c \geq 1$ and $C_{r,m} > 0$ such that, for all $y \in \mathbb{R}^n$ and $R > c|x|$,

$$\sum_{k=1}^{\infty} k^m \left(2^k R\right)^n \left( \frac{1}{(2^k R)^n} \int_{2^k R < |y| \leq 2^{k+1} R} \left| K(x-y) - K(-y) \right|^r dy \right)^{1/r} \leq C_{r,m},$$

if $r < \infty$, and

$$\sum_{k=1}^{\infty} k^m \left(2^k R\right)^n \sup_{2^k R < |y| \leq 2^{k+1} R} \left| K(x-y) - K(-y) \right| \leq C_{\infty}$$

in the case $r = \infty$.

We notice that $H_{r,0}$ is $L^r$-Hörmander condition, which was studied in depth in [8].

Let $K$ satisfy (1) and $H_{r,1}$-Hörmander condition, and let $T_b = \{T_{\epsilon,b}\}_{\epsilon>0}$, where $T_{\epsilon,b}$ is the commutator of $T_\epsilon$ and $b$,

$$T_{\epsilon,b} f(x) = \int_{|x-y|>\epsilon} \big(b(x) - b(y)\big) K(x-y) f(y)\, dy. \tag{3}$$

Suppose $r > 1, \rho > 2$, and $b \in BMO(\mathbb{R}^n)$. Zhang and Wu proved in [9] that, if $V_\rho(T)$ and $\mathcal{O}(T)$ are bounded on $L^{p_0}(\mathbb{R}^n)$ for some $p_0 > 1$, then $V_\rho(T_b)$ and $\mathcal{O}(T_b)$ are bounded on $L^p(\mathbb{R}^n, \omega)$ for any $\max\{r', p_0\} < p < \infty, \omega \in A_{p/\max\{r',p_0\}}(\mathbb{R}^n)$.

Given $m$ is a positive integer, and $b$ is a function on $\mathbb{R}^n$. Let $R_{m+1}(b;x,y)$ be the $m+1$th Taylor series remainder of $b$ at $x$ expander about $y$, *i.e.*,

$$R_{m+1}(b;x,y) = b(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha b(y)(x-y)^\alpha.$$

We consider the family of operators $T^b = \{T^b_\epsilon\}_{\epsilon>0}$, where $T^b_\epsilon$ are the multilinear singular integral operators of $T_\epsilon$ as follows:

$$T^b_\epsilon f(x) = \int_{|x-y|>\epsilon} \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x-y) f(y)\, dy. \tag{4}$$

Note that when $m = 0$, $T_\epsilon^b$ is just the commutator of $T_\epsilon$ and $b$. However, when $m > 0$, $T_\epsilon^b$ is a non-trivial generation of the commutator. It is well known that multilinear operators have been widely studied by many authors (see [10–14]).

In [15], Hu and Wang established the weighted $(L^p, L^q)$ inequalities of the variation and oscillation operators for the multilinear Calderón–Zygmund singular integral with a Lipschitz function in $\mathbb{R}$. In this paper, if $K$ satisfies (1) and $H_{r,1}$-Hörmander condition, we will study the bounded behaviors of variation and oscillation operators for the family of the multilinear singular integrals defined by (4) in $L^p(\mathbb{R}^n, \omega(x)\,dx)$ when $D^\alpha b \in BMO(\mathbb{R}^n)$ for $|\alpha| = m$. Our main results can be formulated as follows.

**Theorem 1** *Let $K$ satisfy* (1), $\rho > 2$, *and let $T = \{T_\epsilon\}_{\epsilon>0}$ and $T^b = \{T_\epsilon^b\}_{\epsilon>0}$ be given by* (2) *and* (4), *respectively. Suppose that $K \in H_{r,1}$, $V_\rho(T)$, and $\mathcal{O}(T)$ are bounded on $L^{p_0}(\mathbb{R}^n)$ for some $p_0 > 1$, and $D^\alpha b \in BMO(\mathbb{R}^n)$ for $|\alpha| = m$, then $V_\rho(T^b)$ and $\mathcal{O}(T^b)$ are bounded on $L^p(\mathbb{R}^n, \omega)$ for any* $\max\{r', p_0\} < p < \infty$, $\omega \in A_{p/\max\{r',p_0\}}(\mathbb{R}^n)$.

**Corollary 1** ([9]) *Let $K$ satisfy* (1), $\rho > 2$, *and let $T = \{T_\epsilon\}_{\epsilon>0}$ and $T_b = \{T_{\epsilon,b}\}_{\epsilon>0}$ be given by* (2) *and* (3), *respectively. Suppose that $K \in H_{r,1}$, $V_\rho(T)$ and $\mathcal{O}(T)$ are bounded on $L^{p_0}(\mathbb{R}^n)$ for some $p_0 > 1$, and $b \in BMO(\mathbb{R}^n)$, then $V_\rho(T_b)$ and $\mathcal{O}(T_b)$ are bounded on $L^p(\mathbb{R}^n, \omega)$ for any* $\max\{r', p_0\} < p < \infty$, $\omega \in A_{p/\max\{r',p_0\}}(\mathbb{R}^n)$.

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant $C$, independent of all important parameters, such that $A \le CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

## 2 Some preliminaries

### 2.1 $A_p(\mathbb{R}^n)$ weight

A weight $\omega$ belongs to $A_p(\mathbb{R}^n)$ for $1 < p < \infty$ if there exists a constant $C$ such that, for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{1-p'}\,dx\right)^{p-1} \le C,$$

where $p'$ is the dual of $p$ such that $1/p + 1/p' = 1$. A weight $\omega$ belongs to $A_1(\mathbb{R}^n)$ if

$$\frac{1}{|B|}\int_B w(y)\,dy \le C \cdot \operatorname*{ess\,inf}_{x\in B} w(x) \quad \text{for every ball } B \subset \mathbb{R}^n.$$

### 2.2 Function of $BMO(\mathbb{R}^n)$

Following [16], a locally integrable function $b$ is said to be in $BMO(\mathbb{R}^n)$ if

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B \left|b(x) - b_B\right|\,dx = \|b\|_* < \infty,$$

where

$$b_B = \frac{1}{|B|}\int_B b(y)\,dy.$$

The function of $BMO(\mathbb{R}^n)$ has the following property.

**Lemma 1** *Suppose $b \in BMO(\mathbb{R}^n)$, $B_k = 2^k B$, $k \in \mathbb{N} \cup \{0\}$. Then, for $1 \le p < \infty$,*

$$\left(\frac{1}{|B|}\int_B \left|b(x) - b_{B_k}\right|^p dx\right)^{1/p} \lesssim k\|b\|_*.$$

## 2.3 Maximal function

The Hardy–Littlewood maximal operator is defined by

$$M(f)(x) = \sup_{B \ni x} \frac{1}{|B|}\int_B \left|f(y)\right| dy.$$

We also define the maximal function

$$M_r(f)(x) = \sup_{B \ni x}\left(\frac{1}{|B|}\int_B \left|f(y)\right|^r dy\right)^{1/r}$$

for $1 \le r < \infty$. It is well known that $M_r$ is bounded on $L^p(\mathbb{R}^n, \omega)$ for $r < p < \infty$ and $\omega \in A_p(\mathbb{R}^n)$ (see [17]).

## 2.4 Taylor series remainder

By definition, it is obvious that

$$R_{m+1}(b; x, y) = R_m(b; x, y) - \sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha b(y)(x-y)^\alpha.$$

The following lemma gives an estimate on Taylor series remainder.

**Lemma 2** ([11]) *Let $b$ be a function on $\mathbb{R}^n$ with $m$th order derivatives in $L^q(\mathbb{R})$ for some $q > n$. Then*

$$\left|R_m(b; x, y)\right| \lesssim |x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{|Q(x,y)|}\int_{Q(x,y)} \left|D^\alpha b(z)\right|^q dz\right)^{1/q},$$

*where $Q(x, y)$ is the cube centered at $x$ and having diameter $5\sqrt{n}|x-y|$.*

## 2.5 Variation and oscillation operators

Following [2], let $\Theta = \{\beta : \beta = \{\epsilon_i\}, \epsilon_i \in \mathbb{R}, \epsilon_i \searrow 0\}$. We denote by $F_\rho$ the mixed norm space of two variable functions $g(i, \beta)$ such that

$$\|g\|_{F_\rho} \equiv \sup_\beta \left(\sum_i \left|g(i, \beta)\right|^\rho\right)^{1/\rho}.$$

We also consider the $F_\rho$-valued operator $\mathcal{V}(T) : f \to \mathcal{V}(T)f$ by

$$\mathcal{V}(T)f(x) = \left\{T_{\epsilon_{i+1}} f(x) - T_{\epsilon_i} f(x)\right\}_{\beta=\{\epsilon_i\}\in\Theta}.$$

This implies that

$$\mathcal{V}_\rho(T)f(x) = \left\|\mathcal{V}(T)f(x)\right\|_{F_\rho}.$$

On the other hand, we consider the operator

$$\mathcal{O}'(Tf)(x) = \left(\sum_{i=1}^{\infty} \sup_{t_{i+1}<\delta_i<t_i} \left|T_{t_{i+1}}f(x) - T_{\delta_i}f(x)\right|^2\right)^{1/2}.$$

It is easy to check that

$$\mathcal{O}'(Tf) \approx \mathcal{O}(Tf).$$

We denote by $E$ the mixed norm Banach space of two-variable function $h$ defined on $\mathbb{R} \times \mathbb{N}$ such that

$$\|h\|_E \equiv \left(\sum_i \left(\sup_s \left|h(s,i)\right|\right)^2\right)^{1/2} < \infty.$$

For a fixed decreasing sequence $\{t_i\}$ with $t_i \searrow 0$, let $J_i = (t_{i+1}, t_i]$ and define the $E$-valued operator $\mathcal{U}(T) : f \to \mathcal{U}(T)f$ given by

$$\mathcal{U}(T)f(x) = \left\{T_{t_{i+1}}f(x) - T_s f(x)\right\}_{s\in J_i, i\in\mathbb{N}} = \left\{\int_{\{t_{i+1}<|x-y|<s\}} K(x,y)f(y)\,dy\right\}_{s\in J_i, i\in\mathbb{N}}.$$

Then

$$\begin{aligned}\mathcal{O}'(Tf)(x) &= \left\|\mathcal{U}(T)f(x)\right\|_E = \left\|\left\{T_{t_{i+1}}f(x) - T_s f(x)\right\}_{s\in J_i, i\in\mathbb{N}}\right\|_E \\ &= \left\|\left\{\int_{\{t_{i+1}<|x-y|<s\}} K(x,y)f(y)\,dy\right\}_{s\in J_i, i\in\mathbb{N}}\right\|_E.\end{aligned}$$

Let $B$ be a Banach space and $\varphi$ be a $B$-valued function, we define the sharp maximal operator as follows:

$$\varphi^\sharp(x) = \sup_{x\in I} \frac{1}{|I|} \int_I \left\|\varphi(y) - \frac{1}{|I|}\int_I \varphi(z)\,dz\right\|_B dy.$$

Then

$$M^\sharp\left(\mathcal{V}_\rho(Tf)\right) \le 2\left(\mathcal{V}(T)f\right)^\sharp(x)$$

and

$$M^\sharp\left(\mathcal{O}'(Tf)\right) \le 2\left(\mathcal{U}(T)f\right)^\sharp(x).$$

Finally, let us recall some results about the boundedness of $V_\rho(T)$ and $\mathcal{O}(T)$.

**Lemma 3** ([9]) *Let $K$ satisfy* (1), $\rho > 2$, $T = \{T_\epsilon\}_{\epsilon>0}$ *be given by* (2). *Suppose that $K$ satisfies* (1) *and $K \in H_{r,0}$-Hörmander condition, $V_\rho(T)$ and $\mathcal{O}(T)$ are bounded on $L^{p_0}(\mathbb{R}^n)$ for some* $1 < p_0 < \infty$. *Then $V_\rho(T)$ and $\mathcal{O}(T)$ are bounded on $L^p(\mathbb{R}^n)$ for any* $\max\{r', p_0\} < p < \infty$.

## 3 Proof of Theorem 1

By the fact that $M_s$ is bounded on $L^p(\mathbb{R}^n,\omega)$ for $1 \le s < p < \infty$ and $\omega \in A_p(\mathbb{R}^n)$, we need to prove

$$M^\sharp(\mathcal{V}_\rho(T^b)f(x) \lesssim \sum_{|\alpha|=m} \|D^\alpha b\|_* M_s(f)(x) \tag{5}$$

and

$$M^\sharp(\mathcal{O}'(T^b)f(x) \lesssim \sum_{|\alpha|=m} \|D^\alpha b\|_* M_s(f)(x) \tag{6}$$

for any $s > \max\{r', p_0\}$.

We will only need to prove (5), since (6) can be obtained by a similar argument. To prove (5), it suffices to prove that for any fixed $x_0 \in \mathbb{R}^n$ and some $F_\rho$-valued constant $c_1$, such that for every ball $B = B(x_0, l)$ with radius $l$, centered at $x_0$, the following inequality

$$\frac{1}{|B|}\int_B \|V(T^b)f(x) - c_1\|_E \, dx \lesssim \sum_{|\alpha|=m} \|D^\alpha b\|_* M_s(f)(x_0)$$

holds.

We write $f = f_1 + f_2 = f\chi_{10B} + f\chi_{\mathbb{R}^n\setminus 10B}$. Then

$$\begin{aligned} V(T^b)f(x) &= \left\{\int_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}} \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x-y)f(y)\,dy\right\}_{\beta=\{\epsilon_i\}\in\Theta} \\ &= V(T)\left(\frac{R_{m+1}(b;x,\cdot)}{|x-\cdot|^m} f_1\right)(x) + V(T^b)(f_2)(x). \end{aligned}$$

Let $x_1$ be a point at the boundary of $2B$, and let

$$c_1 = \left\{\int_{\{\epsilon_{i+1}<|x_1-y|<\epsilon_i\}} \frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m} K(x_1-y)f_2(y)\,dy\right\}_{\beta=\{\epsilon_i\}\in\Theta} = V(T)(f_2)(x_1).$$

Then

$$\begin{aligned} &\frac{1}{|B|}\int_B \|V(T^b)f(x) - c_1\|_{F_\rho}\,dx \\ &\quad\le \frac{1}{|B|}\int_B \left\|V(T)\left(\frac{R_{m+1}(b;x,\cdot)}{|x-\cdot|^m} f_1\right)(x)\right\|_{F_\rho} dx \\ &\qquad + \frac{1}{|B|}\int_B \|V(T^b)(f_2)(x) - V(T^b)(f_2)(x_1)\|_{F_\rho}\,dx \\ &\quad= M_1 + M_2. \end{aligned}$$

For any $x \in B, k \in \mathbb{Z}$, let $E_k = \{y : 2^k \cdot 3l \le |y-x| < 2^{k+1} \cdot 3l\}$, let $F_k = \{y : |y-x| < 2^{k+1} \cdot 3l\}$, and let

$$b_k(z) = b(z) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha b)_{F_k} z^m.$$

By [11] we have $R_{m+1}(b;x,y) = R_{m+1}(b_k;x,y)$ for any $y \in E_k$. From Lemma 3, we know $V_\rho(T)$ is bounded on $L^u(\mathbb{R}^n)$ for $\max\{r', p_0\} < u < s$. Then, using Hölder's inequality, we deduce

$$\begin{aligned}
M_1 &\lesssim \left(\frac{1}{|B|}\int_B \left\| V(T)\left(\frac{R_{m+1}(b;x,\cdot)}{|x-\cdot|^m} f_1\right)\right\|_{F_\rho}^u dx\right)^{1/u} \\
&\lesssim \left(\frac{1}{|B|}\int_{\{y:|y-x|<11l\}} \left|\frac{R_{m+1}(b;\cdot,y)}{|y-\cdot|^m} f(y)\right|^u dy\right)^{1/u} \\
&= \left(\frac{1}{|B|}\int_{\{y:|y-x|<11l\}} \left|\sum_{k=-\infty}^{1}\left(\frac{R_{m+1}(b_k;\cdot,y)}{|y-\cdot|^m}\chi_{E_k}(y)\right) f(y)\right|^u dy\right)^{1/u} \\
&\lesssim \left(\frac{1}{|B|}\int_{\{y:|y-x|<11l\}} \left|\left(\sum_{k=-\infty}^{1}\frac{R_m(b_k;\cdot,y)}{|y-\cdot|^m}\chi_{E_k}(y)\right.\right.\right. \\
&\quad \left.\left.\left. - \frac{1}{|y-\cdot|^m}\sum_{|\alpha|=m}\frac{1}{\alpha!}\sum_{k=-\infty}^{1} D^\alpha b_k(y)\chi_{E_k}(y)(y-\cdot)^\alpha\right) f(y)\right|^u dy\right)^{1/u} \\
&\lesssim \left(\frac{1}{|B|}\sum_{k=-\infty}^{1}\int_{E_k}\left|\frac{R_m(b_k;\cdot,y)}{|y-\cdot|^m} f(y)\right|^u dy\right)^{1/u} \\
&\quad + \sum_{|\alpha|=m}\left(\frac{1}{|B|}\int_{\{y:|y-x|<11l\}}\left(\left(\sum_{k=-\infty}^{1}\left|D^\alpha b_k(y)\right|\chi_{E_k}(y)\right)\left|f(y)\right|\right)^u dy\right)^{1/u} \\
&= M_{11} + M_{12}.
\end{aligned}$$

For any $y \in E_k$, by Lemma 2 and Lemma 1,

$$\begin{aligned}
\left|R_m(b_k;x,y)\right| &\lesssim |x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{|Q(x,y)|}\int_{Q(x,y)}\left|D^\alpha b(z)\right|^q dz\right)^{1/q} \\
&\lesssim |x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{(\sqrt{n}2^k\cdot 30l)^n}\int_{|z-x|<\sqrt{n}2^k\cdot 30l}\left|D^\alpha b(z) - \left(D^\alpha b\right)_{F_k}\right|^q dz\right)^{1/q} \\
&\lesssim |x-y|^m \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_*.
\end{aligned}$$

Then we have

$$\begin{aligned}
M_{11} &\lesssim \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_*\left(\frac{1}{|B|}\sum_{k=-\infty}^{1}\int_{E_k}\left|f(y)\right|^u dy\right)^{1/u} \\
&= \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_*\left(\frac{1}{|B|}\int_{12B}\left|f(y)\right|^u dy\right)^{1/u} \\
&\lesssim \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_* M_s(f)(x_0).
\end{aligned}$$

By $D^\alpha b_k(y) = D^\alpha b(y) - (D^\alpha b)_{F_k}$, applying Hölder's inequality and Lemma 2, we get

$$\begin{aligned} M_{12} &\lesssim \left( \frac{1}{|B|} \left( \int_{12B} |f(y)|^s \, dy \right)^{u/s} \left( \sum_{|\alpha|=m} \sum_{k=-\infty}^{1} \int_{F_k} |D^\alpha b(y) - (D^\alpha b)_{F_k}|^{\frac{us}{s-u}} \, dy \right)^{1-u/s} \right)^{1/u} \\ &\lesssim \sum_{|\alpha|=m} \|D^\alpha b\|_* \left( \frac{1}{|B|} \left( \int_{12B} |f(y)|^s \, dy \right)^{u/s} \sum_{k=-\infty}^{1} |F_k|^{1-u/s} \right)^{1/u} \\ &\lesssim \sum_{|\alpha|=m} \|D^\alpha b\|_* \left( \frac{1}{|B|} \int_{12B} |f(y)|^s \, dy \right)^{1/s} \lesssim \sum_{|\alpha|=m} \|D^\alpha b\|_* M_s(f)(x_0). \end{aligned}$$

We now estimate $M_2$. We write

$$\begin{aligned} &\|V(T^b)f_2(x) - V(T^b)f_2(x_1)\|_{F_\rho} \\ &= \left\| \left\{ \int_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}} \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x-y) f_2(y) \, dy \right.\right. \\ &\quad \left.\left. - \int_{\{\epsilon_{i+1}<|x_1-y|<\epsilon_i\}} \frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m} K(x_1-y) f_2(y) \, dy \right\}_{\beta=\{\epsilon_i\}\in\Theta} \right\|_{F_\rho} \\ &\leq \left\| \left\{ \int_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}} \left( \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x-y) \right.\right.\right. \\ &\quad \left.\left.\left. - \frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m} K(x_1-y) \right) f_2(y) \, dy \right\}_{\beta=\{\epsilon_i\}\in\Theta} \right\|_{F_\rho} \\ &\quad + \left\| \left\{ \int_{\mathbb{R}^n} \left( \chi_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}}(y) - \chi_{\{\epsilon_{i+1}<|x_1-y|<\epsilon_i\}}(y) \right) \right.\right. \\ &\quad \left.\left. \times \frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m} K(x_1-y) f_2(y) \, dy \right\}_{\beta=\{\epsilon_i\}\in\Theta} \right\|_{F_\rho} \\ &= N_1 + N_2. \end{aligned}$$

By Minkowski's inequality and $\|\{\epsilon_{i+1} < |x-y| < \epsilon_i\}_{\beta=\{\epsilon_i\}\in\Theta}\|_{F_\rho} \leq 1$, we obtain

$$\begin{aligned} N_1 &\leq \int_{\mathbb{R}^n} \left\| \{\chi_{\{t_{i+1}<|x-y|<s\}}\}_{\beta=\{\epsilon_i\}\in\Theta} \right\|_{F_\rho} \\ &\quad \times \left| \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x-y) - \frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m} K(x_1-y) \right| |f_2(y)| \, dy \\ &\leq \int_{\mathbb{R}^n} \left| \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x-y) - \frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m} K(x_1-y) \right| |f_2(y)| \, dy \\ &\leq \sum_{k=1}^{\infty} \int_{E_k} \left| \left( \frac{R_{m+1}(b_k;x,y)}{|x-y|^m} - \frac{R_{m+1}(b_k;x_1,y)}{|x_1-y|^m} \right) K(x-y) \right| |f_2(y)| \, dy \\ &\quad + \int_{\mathbb{R}^n} \left| \frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m} \right| \left| K(x-y) - K(x_1-y) \right| |f_2(y)| \, dy \\ &\leq \sum_{k=1}^{\infty} \int_{E_k} \frac{1}{|x-y|^m} |R_m(b_k;x,y) - R_m(b_k;x_1,y)| |K(x-y)| |f(y)| \, dy \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{\infty}\int_{E_k}\left|R_m(b_k;x_1,y)\right|\left|\frac{1}{|x-y|^m}-\frac{1}{|x_1-y|^m}\right|\left|K(x-y)\right|\left|f(y)\right|\,dy\\
&+\sum_{k=1}^{\infty}\int_{E_k}\sum_{|\alpha|=m}\frac{1}{\alpha!}\left|D^\alpha b_k(y)\right|\left|\frac{(x-y)^\alpha}{|x-y|^m}-\frac{(x_1-y)^\alpha}{|x_1-y|^m}\right|\left|K(x-y)\right|\left|f(y)\right|\,dy\\
&+\int_{(10B)^c}\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right|\left|K(x-y)-K(x_1-y)\right|\left|f(y)\right|\,dy\\
&=N_{11}+N_{12}+N_{13}+N_{14}.
\end{aligned}
$$

Applying the formula ([11] p. 448), we have

$$
R_m(b_k;x,y)-R_m(b_k;x_1,y)=R_m(b_k;x,x_1)+\sum_{0<|\beta|<m}\frac{(x-x_1)^\beta}{\beta!}R_{m-\beta}\big(D^\beta b_k;x_1,y\big).
$$

Then, by Lemmas 1 and 2, we have

$$
\begin{aligned}
\left|R_m(b_k;x,x_1)\right| &\lesssim |x-x_1|^m\sum_{|\alpha|=m}\left(\frac{1}{Q(x,x_1)}\int_{Q(x,x_1)}\left|D^\alpha b_k-\big(D^\alpha b_k\big)_{F_k}\right|^q\right)^{1/q}\\
&\lesssim kl^m\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_*
\end{aligned}
$$

and

$$
\left|R_{m-\beta}\big(D^\beta b_k;x_1,y\big)\right|\lesssim|x-y|^{m-\beta}\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_*.
$$

So

$$
\begin{aligned}
\left|R_m(b_k;x,y)-R_m(b_k;x_1,y)\right| &\lesssim \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_*\left(kl^m+\sum_{0<|\beta|<m}l^\beta|x-y|^{m-\beta}\right)\\
&\lesssim kl|x-y|^{m-1}\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_*.
\end{aligned}
$$

By (1) we have

$$
\left|K(x-y)\right|\lesssim|x-y|^{-n}.
$$

Then

$$
\begin{aligned}
N_{11} &= \sum_{k=1}^{\infty}\int_{E_k}\frac{1}{|x-y|^m}\left|R_m(b_k;x,y)-R_m(b_k;x_1,y)\right|K(x-y)|\left|f(y)\right|\,dy\\
&\lesssim \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_*\sum_{k=1}^{\infty}\int_{E_k}\frac{kl}{|x-y|^{n+1}}\left|f(y)\right|\,dy\\
&\lesssim \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_*\sum_{k=1}^{\infty}\frac{k}{2^k}\frac{1}{|F_k|}\int_{F_k}\left|f(y)\right|\,dy\\
&\lesssim \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_*M_s(f)(x_0).
\end{aligned}
$$

For $y \in E_k, x \in B$ and $x_1$ being a point at the boundary of $2B$, we get $|x_1 - y| \approx |x - y|$. Thus

$$\left|\frac{1}{|x-y|^m} - \frac{1}{|x_1-y|^m}\right| \lesssim \frac{|x-x_1|}{|x-y|^{m+1}} \lesssim \frac{l}{|x-y|^{m+1}}$$

and

$$\left|R_m(b_k; x_1, y)\right| \lesssim |x-y|^m \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_*.$$

Then

$$N_{12} \lesssim \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_* \sum_{k=1}^{\infty} \int_{E_k} \frac{l|f(y)|}{|x-y|^{n+1}}\, dy \lesssim \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_* M_s(f)(x_0).$$

As for $N_{13}$, due to

$$\left|\frac{(x-y)^\alpha}{|x-y|^m} - \frac{(x_1-y)^\alpha}{|x_1-y|^m}\right| \lesssim \frac{|x-x_1|}{|x-y|}$$

for $|\alpha| = m$, and $D^\alpha b_k(y) = D^\alpha b(y) - (D^\alpha b)_{F_k}$, we have

$$\begin{aligned}
N_{13} &\lesssim \sum_{k=1}^{\infty} \int_{E_k} \frac{l|f(y)|}{|x-y|^{n+1}} \sum_{|\alpha|=m} \left|D^\alpha b(y) - \left(D^\alpha b\right)_{F_k}\right| dy \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{1}{|F_k|} \int_{F_k} \left|f(y)\right| \sum_{|\alpha|=m} \left|D^\alpha b(y) - \left(D^\alpha b\right)_{F_k}\right| dy \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{2^k} \left(\frac{1}{|F_k|} \int_{F_k} \left|f(y)\right|^s dy\right)^{1/s} \sum_{|\alpha|=m} \left(\frac{1}{|F_k|} \int_{F_k} \left|D^\alpha b(y) - \left(D^\alpha b\right)_{F_k}\right|^{s'} dy\right)^{1/s'} \\
&\lesssim \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_* M_s(f)(x_0) \sum_{k=1}^{\infty} \frac{1}{2^k} \\
&\lesssim \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_* M_s(f)(x_0).
\end{aligned}$$

Let us estimate $N_{14}$ now. For $y \in (10B)^c$, we have $|y - x_1| \geq |y - x_0| - |x_0 - x_1| > 8l$. For $k = 1, 2, \dots$, let $\widetilde{E}_k = \{y : 2^k \cdot 3l \leq |y - x_1| < 2^{k+1} \cdot 3l\}$, let $\widetilde{F}_k = \{y : |y - x_1| < 2^{k+1} \cdot 3l\}$, and let

$$\widetilde{b}_k(z) = b(z) - \sum_{|\alpha|=m} \frac{1}{\alpha!} \left(D^\alpha b\right)_{\widetilde{F}_k} z^m.$$

Note that $Q(x, y) \subset 2\sqrt{n}Q(x_1, y)$, then for $x \in B, y \in \widetilde{E}_k$ we have

$$\begin{aligned}
\left|R_m(\widetilde{b}_k; x, y)\right| &\lesssim |x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|2\sqrt{n}Q(x_1, y)|} \int_{2\sqrt{n}Q(x_1,y)} \left|D^\alpha \widetilde{b}_k(z)\right|^q dz\right)^{1/q} \\
&\lesssim |x_1-y|^m \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_*.
\end{aligned}$$

So

$$\left|R_{m+1}(\widetilde{b}_k;x,y)\right| \lesssim |x_1-y|^m\left(\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_* + \sum_{|\alpha|=m}\left|D^\alpha b(y) - \left(D^\alpha b\right)_{\widetilde{F}_k}\right|\right).$$

Then

$$\begin{aligned} N_{14} &\lesssim \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_* \sum_{k=1}^{\infty}\int_{\widetilde{E}_k}\left|K(x-y)-K(x_1-y)\right|\left|f(y)\right|dy \\ &\quad + \sum_{k=1}^{\infty}\sum_{|\alpha|=m}\int_{\widetilde{E}_k}\left|D^\alpha b(y) - \left(D^\alpha b\right)_{\widetilde{F}_k}\right|\left|K(x-y)-K(x_1-y)\right|\left|f(y)\right|dy \\ &= N_{141} + N_{142}. \end{aligned}$$

Taking $R = 3l$, then $|x-x_1| < R$. By Hölder's inequality and $K \in H_{r,1} \subset H_{r,0}$, we have

$$\begin{aligned} N_{141} &\lesssim \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_* \sum_{k=1}^{\infty}\left(\int_{\widetilde{E}_k}\left|K(x-y)-K(x_1-y)\right|^r dy\right)^{1/r}\left(\int_{\widetilde{E}_k}\left|f(y)\right|^{r'}dy\right)^{1/r'} \\ &\lesssim \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_* \sum_{k=1}^{\infty}\left(2^k\cdot 3l\right)^n\left(\frac{1}{(2^k\cdot 3l)^n}\int_{\widetilde{E}_k}\left|K(x-y)-K(x_1-y)\right|^r dy\right)^{1/r} \\ &\quad\times\left(\frac{1}{(2^k\cdot 3l)^n}\int_{\widetilde{E}_k}\left|f(y)\right|^{r'}dy\right)^{1/r'} \\ &\lesssim \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_* M_{r'}(f)(x) \lesssim \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_* M_s(f)(x_0) \end{aligned}$$

and

$$\begin{aligned} N_{142} &\lesssim \sum_{k=1}^{\infty}\left(2^k\cdot 3l\right)^n\left(\frac{1}{(2^k\cdot 3l)^n}\int_{\widetilde{E}_k}\left|K(x-y)-K(x_1-y)\right|^r dy\right)^{1/r} \\ &\quad\times\sum_{|\alpha|=m}\left(\frac{1}{(2^k\cdot 3l)^n}\int_{\widetilde{E}_k}\left|\left(D^\alpha b(y) - \left(D^\alpha b\right)_{\widetilde{F}_k}\right)f(y)\right|^{r'}dy\right)^{1/r'} \\ &\lesssim \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_* M_s(f)(x_0). \end{aligned}$$

Finally, let us estimate $N_2$. Note that the integral

$$\int_{\mathbb{R}^n}\left(\chi_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}}(y) - \chi_{\{\epsilon_{i+1}<|x_1-y|<\epsilon_i\}}(y)\right)\frac{R_{m+1}(b;x_1-y)}{|x_1-y|^m}K(x_1-y)f_2(y)\,dy$$

will only be non-zero if either $\chi_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}}(y) = 1$ and $\chi_{\{\epsilon_{i+1}<|x_1-y|<\epsilon_i\}}(y) = 0$ or vice versa. That means the integral will only be non-zero in the following cases:

(i) $\epsilon_{i+1} < |x-y| < \epsilon_i$ and $|x_1-y| \le \epsilon_{i+1}$;
(ii) $\epsilon_{i+1} < |x-y| < \epsilon_i$ and $|x_1-y| \ge \epsilon_i$;
(iii) $\epsilon_{i+1} < |x_1-y| < \epsilon_i$ and $|x-y| \le \epsilon_{i+1}$;
(iv) $\epsilon_{i+1} < |x_1-y| < \epsilon_i$ and $|x-y| \ge \epsilon_i$.

In case (i) we observe that $\epsilon_{i+1} < |x-y| \le |x_1 - x| + |x_1 - y| < 3l + \epsilon_{i+1}$ as $|x - x_0| < l$. Similarly, in case (iii) we have $\epsilon_{i+1} < |x_1 - y| < 3l + \epsilon_{i+1}$ as $|x - x_0| < l$. In case (ii) we have $\epsilon_i < |x_1 - y| < 3l + \epsilon_i$, and in case (iv) we have $\epsilon_i < |x - y| < 3l + \epsilon_i$. By (1) we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \big(\chi_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}}(y) - \chi_{\{\epsilon_{i+1}<|x_1-y|<\epsilon_i\}}(y)\big) \frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m} K(x_1-y) f_2(y)\,dy\right| \\
&\quad\lesssim \int_{\mathbb{R}^n} \chi_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}}(y)\chi_{\{\epsilon_{i+1}<|x-y|<3l+\epsilon_{i+1}\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right| \frac{|f_2(y)|}{|x_1-y|^n}\,dy \\
&\qquad + \int_{\mathbb{R}^n} \chi_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}}(y)\chi_{\{\epsilon_i<|x_1-y|<3l+\epsilon_i\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right| \frac{|f_2(y)|}{|x_1-y|^n}\,dy \\
&\qquad + \int_{\mathbb{R}^n} \chi_{\{\epsilon_{i+1}<|x_1-y|<\epsilon_i\}}(y)\chi_{\{\epsilon_{i+1}<|x_1-y|<3l+\epsilon_{i+1}\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right| \frac{|f_2(y)|}{|x_1-y|^n}\,dy \\
&\qquad + \int_{\mathbb{R}^n} \chi_{\{\epsilon_{i+1}<|x_1-y|<\epsilon_i\}}(y)\chi_{\{\epsilon_i<|x-y|<3l+\epsilon_i\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right| \frac{|f_2(y)|}{|x_1-y|^n}\,dy \\
&\quad= P_1 + P_2 + P_3 + P_4.
\end{aligned}$$

It is easy to check that $|x-y| \ge 9l$, $|x_1 - y| \ge 8l$, and $\frac{8}{11}|x-y| \le |x_1-y| \le \frac{4}{3}|x-y|$ for $x \in B, y \in (10B)^c$; moreover, if $3l \ge \epsilon_{i+1}, i \in \mathbb{N}$, we have

$$\big\{y \in (10B)^c : \epsilon_{i+1} < |x-y| < 3l + \epsilon_{i+1}\big\} \subset \big\{y \in (10B)^c : |x-y| < 6l\big\} = \phi$$

and

$$\big\{y \in (10B)^c : \epsilon_{i+1} < |x_1-y| < 3l + \epsilon_{i+1}\big\} \subset \big\{y \in (10B)^c : |x-y| < 6l\big\} = \phi;$$

this means $P_1 = P_3 = 0$. Similarly, $P_2 = P_4 = 0$ for $3l \ge \epsilon_i, i \in \mathbb{N}$. By Hölder's inequality with $t$ satisfying $1 < t < \sqrt{\min(r', \rho)}$, we get

$$\begin{aligned}
P_1 &\lesssim \left(\int_{\mathbb{R}^n} \chi_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right|^t \frac{|f_2(y)|^t}{|x_1-y|^{nt}}\,dy\right)^{1/t} \big[(3l+\epsilon_{i+1})^n - (\epsilon_{i+1})^n\big]^{1/t'}, \\
P_2 &\lesssim \left(\int_{\mathbb{R}^n} \chi_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right|^t \frac{|f_2(y)|^t}{|x_1-y|^{nt}}\,dy\right)^{1/t} \big[(3l+\epsilon_i)^n - (\epsilon_i)^n\big]^{1/t'}, \\
P_3 &\lesssim \left(\int_{\mathbb{R}^n} \chi_{\{\epsilon_{i+1}<|x_1-y|<\epsilon_i\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right|^t \frac{|f_2(y)|^t}{|x_1-y|^{nt}}\,dy\right)^{1/t} \big[(3l+\epsilon_{i+1})^n - (\epsilon_{i+1})^n\big]^{1/t'}
\end{aligned}$$

and

$$P_4 \lesssim \left(\int_{\mathbb{R}^n} \chi_{\{\epsilon_{i+1}<|x_1-y|<\epsilon_i\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right|^t \frac{|f_2(y)|^t}{|x_1-y|^{nt}}\,dy\right)^{1/t} \big[(3l+\epsilon_i)^n - (\epsilon_i)^n\big]^{1/t'}.$$

Note that for $3l < \epsilon_{i+1}$, we have $(3l+\epsilon_{i+1})^n - (\epsilon_{i+1})^n \lesssim (\epsilon_{i+1})^{n-1} l$. Then

$$\begin{aligned}
P_1 &\lesssim \frac{(3l+\epsilon_{i+1})^n - (\epsilon_{i+1})^n}{(\epsilon_{i+1})^{(n-1)/t'}} \left(\int_{\mathbb{R}^n} \chi_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right|^t \frac{|f_2(y)|^t}{|x_1-y|^{nt}}\,dy\right)^{1/t} \\
&\lesssim l^{1/t'} \left(\int_{\mathbb{R}^n} \chi_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right|^t \frac{|f_2(y)|^t}{|x_1-y|^{n+t-1}}\,dy\right)^{1/t}.
\end{aligned}$$

Similarly, we have

$$P_2 \lesssim l^{1/t'}\left(\int_{\mathbb{R}^n}\chi_{\{\max\{\epsilon_{i+1},\frac{3}{4}\epsilon_i\}<|x-y|<\epsilon_i\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right|^t\frac{|f_2(y)|^t}{|x_1-y|^{n+t-1}}\,dy\right)^{1/t},$$

$$P_3 \lesssim l^{1/t'}\left(\int_{\mathbb{R}^n}\chi_{\{\epsilon_{i+1}<|x_1-y|<\epsilon_i\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right|^t\frac{|f_2(y)|^t}{|x_1-y|^{n+t-1}}\,dy\right)^{1/t}$$

and

$$P_4 \lesssim l^{1/t'}\left(\int_{\mathbb{R}^n}\chi_{\{\max\{\epsilon_{i+1},\frac{8}{11}\epsilon_i\}<|x_1-y|<\epsilon_i\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right|^t\frac{|f_2(y)|^t}{|x_1-y|^{n+t-1}}\,dy\right)^{1/t}.$$

Then

$$\begin{aligned}
N_2 &= \Bigg\|\Bigg\{\int_{\mathbb{R}^n}\big(\chi_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}}(y)-\chi_{\{\epsilon_{i+1}<|x_1-y|<\epsilon_i\}}(y)\big)\\
&\quad\times\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}K(x_1-y)f_2(y)\,dy\Bigg\}_{\beta=\{\epsilon_i\}\in\Theta}\Bigg\|_{F_\rho}\\
&\lesssim l^{1/t'}\Bigg\|\Bigg\{\Bigg(\int_{\mathbb{R}^n}\chi_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right|^t\frac{|f_2(y)|^t}{|x_1-y|^{n+t-1}}\,dy\Bigg)^{1/t}\Bigg\}_{\beta=\{\epsilon_i\}\in\Theta}\Bigg\|_{F_\rho}\\
&\quad+l^{1/t'}\Bigg\|\Bigg\{\Bigg(\int_{\mathbb{R}^n}\chi_{\{\epsilon_{i+1}<|x_1-y|<\epsilon_i\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right|^t\frac{|f_2(y)|^t}{|x_1-y|^{n+t-1}}\,dy\Bigg)^{1/t}\Bigg\}_{\beta=\{\epsilon_i\}\in\Theta}\Bigg\|_{F_\rho}\\
&= N_{21}+N_{22}.
\end{aligned}$$

A straight computation deduces that

$$\begin{aligned}
N_{21} &\lesssim l^{1/t'}\sup_{\epsilon_i\searrow 0}\Bigg(\sum_i\Bigg(\int_{\mathbb{R}^n}\chi_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right|^t\frac{|f_2(y)|^t}{|x_1-y|^{n+t-1}}\,dy\Bigg)^{\rho/t}\Bigg)^{1/\rho}\\
&\lesssim l^{1/t'}\Bigg(\int_{(10B)^c}\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right|^t\frac{|f(y)|^t}{|x_1-y|^{n+t-1}}\,dy\Bigg)^{1/t}\\
&\lesssim l^{1/t'}\Bigg(\sum_{k=1}^{\infty}\int_{\widetilde{E}_k}\left|\frac{R_{m+1}(\widetilde{b}_k;x_1,y)}{|x_1-y|^m}\right|^t\frac{|f(y)|^t}{|x_1-y|^{n+t-1}}\,dy\Bigg)^{1/t}.
\end{aligned}$$

Similar to the estimates for $N_{13}$, for $x\in B, y\in\widetilde{E}_k$, we have

$$\big|R_{m+1}(\widetilde{b}_k;x_1,y)\big|\lesssim|x_1-y|^m\sum_{|\alpha|=m}\big(\|D^\alpha b\|_*+\big|D^\alpha b(y)-(D^\alpha b)_{\widetilde{E}_k}\big|\big).$$

Then

$$\begin{aligned}
N_{21} &\lesssim l^{1/t'}\sup_{\epsilon_i\searrow 0}\Bigg(\sum_i\Bigg(\int_{\mathbb{R}^n}\chi_{\{\epsilon_{i+1}<|x-y|<\epsilon_i\}}(y)\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right|^t\frac{|f_2(y)|^t}{|x_1-y|^{n+t-1}}\,dy\Bigg)^{\rho/t}\Bigg)^{1/\rho}\\
&\lesssim l^{1/t'}\Bigg(\int_{(10B)^c}\left|\frac{R_{m+1}(b;x_1,y)}{|x_1-y|^m}\right|^t\frac{|f(y)|^t}{|x_1-y|^{n+t-1}}\,dy\Bigg)^{1/t}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim l^{1/t'}\left(\sum_{k=1}^{\infty}\int_{\widetilde{E}_k}\sum_{|\alpha|=m}\left(\|D^\alpha b\|_*^t+\left|D^\alpha b(y)-(D^\alpha b)_{\widetilde{F}_k}\right|^t\right)\frac{|f(y)|^t}{|x_1-y|^{n+t-1}}\,dy\right)^{1/t}\\
&\lesssim l^{1/t'}\sum_{|\alpha|=m}\|D^\alpha b\|_*\sum_{k=1}^{\infty}\left(\int_{\widetilde{E}_k}\frac{|f(y)|^t}{|x_1-y|^{n+t-1}}\,dy\right)^{1/t}\\
&\quad+l^{1/t'}\sum_{k=1}^{\infty}\left(\int_{\widetilde{E}_k}\sum_{|\alpha|=m}\left|D^\alpha b(y)-(D^\alpha b)_{\widetilde{F}_k}\right|^t\frac{|f(y)|^t}{|x_1-y|^{n+t-1}}\,dy\right)^{1/t}.
\end{aligned}
$$

However,

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\left(\int_{\widetilde{E}_k}\frac{|f(y)|^t}{|x_1-y|^{n+t-1}}\,dy\right)^{1/t}\\
&\quad\lesssim\sum_{k=1}^{\infty}\frac{1}{(2^k\cdot 3l)^{n/t+1/t'}}\left(\int_{|x_0-y|<2^{k+1}\cdot 3l}\left|f(y)\right|^t dt\right)^{1/t}\\
&\quad\lesssim M_tf(x_0)\sum_{k=1}^{\infty}\frac{(2^k\cdot 3l)^{n/t}}{(2^k\cdot 3l)^{n/t+1/t'}}\\
&\quad\lesssim l^{-1/t'}M_{r'}f(x_0)\lesssim l^{-1/t'}M_sf(x_0)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\left(\int_{\widetilde{E}_k}\sum_{|\alpha|=m}\left|D^\alpha b(y)-(D^\alpha b)_{\widetilde{F}_k}\right|^t\frac{|f(y)|^t}{|x_1-y|^{n+t-1}}\,dy\right)^{1/t}\\
&\quad\lesssim\sum_{k=1}^{\infty}\frac{1}{(2^k\cdot 3l)^{n/t+1/t'}}\left(\int_{\widetilde{E}_k}\sum_{|\alpha|=m}\left|D^\alpha b(y)-(D^\alpha b)_{\widetilde{F}_k}\right|^t\left|f(y)\right|^t dy\right)^{1/t}\\
&\quad\lesssim\sum_{k=1}^{\infty}\frac{1}{(2^k\cdot 3l)^{n/t+1/t'}}\left(\int_{\widetilde{F}_k}\left|f(y)\right|^{t^2}dy\right)^{1/t^2}\left(\sum_{|\alpha|=m}\int_{\widetilde{F}_k}\left|D^\alpha b(y)-(D^\alpha b)_{\widetilde{F}_k}\right|^{t't}dy\right)^{1/(tt')}\\
&\quad\lesssim\sum_{|\alpha|=m}\|D^\alpha b\|_*M_{r'}f(x_0)\sum_{k=1}^{\infty}\frac{(2^k\cdot 3l)^{n/t^2+1/(tt')}}{(2^k\cdot 3l)^{n/t+1/t'}}\\
&\quad=\sum_{|\alpha|=m}\|D^\alpha b\|_*M_{r'}f(x_0)\sum_{k=1}^{\infty}\frac{(2^k\cdot 3l)^{n/t}}{(2^k\cdot 3l)^{n/t+1/t'}}\\
&\quad\lesssim l^{-1/t'}\sum_{|\alpha|=m}\|D^\alpha b\|_*M_sf(x_0).
\end{aligned}
$$

Then

$$
N_{21}\lesssim\sum_{|\alpha|=m}\|D^\alpha b\|_*M_sf(x_0).
$$

Similarly,

$$
N_{22}\lesssim\sum_{|\alpha|=m}\|D^\alpha b\|_*M_sf(x_0).
$$

This completes the proof of Theorem 1.

## 4 Conclusion

In this paper, we have established the weighted $L^p$-boundedness of variation and oscillation operators for a family of multilinear singular integrals with kernels satisfying certain Hörmander type conditions. These results extend the corresponding work in [9] and [15].

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**Competing interests**
The author declares that she has no competing interests.

**Authors' contributions**
YHX proved the main result of this article. She read and approved the final manuscript.

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