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The Bézier variant of Kantorovich type λ-Bernstein operators

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Abstract

In this paper, we introduce the Bézier variant of Kantorovich type λ-Bernstein operators with parameter $\lambda \in [-1, 1]$. We establish a global approximation theorem in terms of second order modulus of continuity and a direct approximation theorem by means of the Ditzian–Totik modulus of smoothness. Finally, we combine the Bojanic–Cheng decomposition method with some analysis techniques to derive an asymptotic estimate on the rate of convergence for some absolutely continuous functions.

MSC: 41A10; 41A25; 41A36

Keywords: λ-Bernstein operators; Basis functions; Modulus of continuity; Rate of convergence; Absolutely continuous functions

1 Introduction

In 1912, Bernstein [1] proposed the famous polynomials, nowadays called Bernstein polynomials, to prove the Weierstrass approximation theorem as follows:

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x), \tag{1}$$

where $x \in [0,1]$, $n = 1, 2, \ldots$, and Bernstein basis functions $b_{n,k}(x)$ are defined as follows:

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}. \tag{2}$$

Based on this, there are many papers that mention Bernstein type operators, we illustrate some of them [2–13]. In 2010, Ye et al. [14] defined the following new Bernstein bases with shape parameter λ:

$$\begin{cases} \widetilde{b}_{n,0}(\lambda;x) = b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x), \\ \widetilde{b}_{n,i}(\lambda;x) = b_{n,i}(x) + \lambda\left(\frac{n-2i+1}{n^2-1} b_{n+1,i}(x) - \frac{n-2i-1}{n^2-1} b_{n+1,i+1}(x)\right) \quad (1 \le i \le n-1), \\ \widetilde{b}_{n,n}(\lambda;x) = b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x), \end{cases} \tag{3}$$

where $b_{n,i}(x)$ $(i = 0, 1, \ldots, n)$ are defined in (2), $x \in [0,1]$, $\lambda \in [-1,1]$. They discussed some important properties of the basis functions and the corresponding curves and tensor prod-

uct surfaces. It must be pointed out that we have more modeling flexibility when adding the shape parameter λ.

Recently, Cai et al. [15] introduced the λ-Bernstein operators as follows:

$$B_{n,\lambda}(f;x) = \sum_{k=0}^{n} \widetilde{b}_{n,k}(\lambda;x) f\left(\frac{k}{n}\right), \tag{4}$$

where $\widetilde{b}_{n,k}(\lambda;x)$ $(k = 0, 1, \ldots, n)$ are defined in (3) and $\lambda \in [-1, 1]$.

In this paper, we propose the Kantorovich type λ-Bernstein operators

$$K_{n,\lambda}(f;x) = (n+1) \sum_{k=0}^{n} \widetilde{b}_{n,k}(\lambda;x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt, \tag{5}$$

and the Bézier variant of Kantorovich type λ-Bernstein operators

$$L_{n,\lambda,\alpha}(f;x) = (n+1) \sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(\lambda;x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt, \tag{6}$$

where

$$Q_{n,k}^{(\alpha)}(\lambda;x) = \left[J_{n,k}(\lambda;x)\right]^{\alpha} - \left[J_{n,k+1}(\lambda;x)\right]^{\alpha}, \qquad J_{n,k}(\lambda;x) = \sum_{j=k}^{n} \widetilde{b}_{n,k}(\lambda;x),$$

$\widetilde{b}_{n,k}(\lambda;x)$ $(k = 0, 1, \ldots, n)$ are defined in (3), $\alpha \geq 1$, $x \in [0, 1]$, and $\lambda \in [-1, 1]$.

Obviously, when $\alpha = 1$, $L_{n,\lambda,1}(f;x)$ reduce to Kantorovich type λ-Bernstein operators (5); when $\lambda = 0$, $L_{n,0,\alpha}(f;x)$ reduce to Bernstein–Kantorovich–Bézier operators defined in [13]; when $\lambda = 0$, $\alpha = 1$, $L_{n,0,1}(f;x)$ reduce to Bernstein–Kantorovich operators defined in [13].

Let

$$P_{n,\lambda,\alpha}(x,t) = (n+1) \sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(\lambda;x) \chi_k(t)$$

and

$$R_{n,\lambda,\alpha}(x,t) = \int_0^t P_{n,\lambda,\alpha}(x,s)\,ds,$$

where $\chi_k(t)$ is the characteristic function on the interval $[\frac{k}{n+1}, \frac{k+1}{n+1}]$ with respect to $[0, 1]$. By the Lebesgue–Stieltjes integral representations, we have

$$L_{n,\lambda,\alpha}(f;x) = \int_0^1 f(t) P_{n,\lambda,\alpha}(x,t)\,dt = \int_0^1 f(t)\,d_t R_{n,\lambda,\alpha}(x,t). \tag{7}$$

The aims of this paper are to study the rate of convergence of operators $L_{n,\lambda,\alpha}$ for $f \in C_{[0,1]}$ and the asymptotic behavior of $L_{n,\lambda,\alpha}$ for some absolutely continuous functions $f \in \Phi_{\mathrm{DB}}$, where the class of functions of Φ_{DB} is defined by

$$\Phi_{\mathrm{DB}} = \left\{ f \,\middle|\, f(x) - f(0) = \int_0^x \phi(u)\,du; x \geq 0; \phi \text{ is bounded on } [0, 1] \right\}. \tag{8}$$

For a bounded function f on $[0,1]$, the following metric forms were first introduced in [12]:

$$\Omega_{x-}(f;\delta_1)=\sup_{t\in[x-\delta_1,x]}\left|f(t)-f(x)\right|;\qquad \Omega_{x+}(f;\delta_2)=\sup_{t\in[x,x+\delta_2]}\left|f(t)-f(x)\right|;$$
$$\Omega_{x}(f;\mu)=\sup_{t\in[x-x/\mu,x+(1-x)/\mu]}\left|f(t)-f(x)\right|,$$

where $x\in[0,1]$ is fixed, $0\le\delta_1\le x$, $0\le\delta_2\le 1-x$, and $\mu\ge 1$. For the basic properties of $\Omega_{x-}(f;\delta_1)$, $\Omega_{x+}(f;\delta_2)$, and $\Omega_{x}(f;\mu)$, refer to [12].

2 Some lemmas

For proving the main results, we need the following lemmas.

Lemma 2.1 ([15]) *Let $e_i=t^i$, $i=0,1,2$, and $n>1$. For the λ-Bernstein operators $B_{n,\lambda}(f;x)$, we have*

$$B_{n,\lambda}(e_0;x)=1;$$
$$B_{n,\lambda}(e_1;x)=x+\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\lambda;$$
$$B_{n,\lambda}(e_2;x)=x^2+\frac{x(1-x)}{n}+\lambda\left[\frac{2x-4x^2+2x^{n+1}}{n(n-1)}+\frac{x^{n+1}+(1-x)^{n+1}-1}{n^2(n-1)}\right].$$

Lemma 2.2 *Let $e_i=t^i$, $i=0,1,2$, and $n>1$, for the Kantorovich type λ-Bernstein operators $K_{n,\lambda}(f;x)$, we have the following equalities*:

$$K_{n,\lambda}(e_0;x)=1; \tag{9}$$
$$K_{n,\lambda}(e_1;x)=x+\frac{1-2x}{2(n+1)}+\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n^2-1}\lambda; \tag{10}$$
$$K_{n,\lambda}(e_2;x)=x^2+\frac{3nx(2-3x)-3x^2+1}{3(n+1)^2}+2\lambda\left[\frac{(x-2x^2+x^{n+1})n+x^{n+1}-x}{(n-1)(n+1)^2}\right]. \tag{11}$$

Proof We can obtain (9) easily by the fact that $\sum_{k=0}^{n}\widetilde{b}_{n,k}(\lambda;x)=1$. Next, by (5) and using Lemma 2.1, we have

$$\begin{aligned}K_{n,\lambda}(e_1;x)&=(n+1)\sum_{k=0}^{n}\widetilde{b}_{n,k}(\lambda;x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}t\,dt\\&=\sum_{k=0}^{n}\widetilde{b}_{n,k}(\lambda;x)\frac{2k+1}{2(n+1)}\\&=\frac{n}{n+1}B_{n,\lambda}(e_1;x)+\frac{1}{2(n+1)}\\&=x+\frac{1-2x}{2(n+1)}+\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n^2-1}\lambda.\end{aligned}$$

Finally,

$$\begin{aligned} K_{n,\lambda}(e_2;x) &= (n+1)\sum_{k=0}^{n}\widetilde{b}_{n,k}(\lambda;x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} t^2\,dt \\ &= \sum_{k=0}^{n}\widetilde{b}_{n,k}(\lambda;x)\frac{3k^2+3k+1}{3(n+1)^2} \\ &= \frac{n^2}{(n+1)^2}B_{n,\lambda}(e_2;x)+\frac{n}{(n+1)^2}B_{n,\lambda}(e_1;x)+\frac{1}{3(n+1)^2} \\ &= x^2+\frac{3nx(2-3x)-3x^2+1}{3(n+1)^2}+2\lambda\left[\frac{(x-2x^2+x^{n+1})n+x^{n+1}-x}{(n-1)(n+1)^2}\right]. \end{aligned}$$

Lemma 2.2 is proved. □

Lemma 2.3 *For the Kantorovich type λ-Bernstein operators $K_{n,\lambda}(f;x)$ and $n>1$, using Lemma* 2.2, *we have*

$$\begin{aligned} K_{n,\lambda}(t-x;x) &= \frac{1-2x}{2(n+1)}+\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n^2-1}\lambda, \\ K_{n,\lambda}\big((t-x)^2;x\big) &= \frac{nx(1-x)}{(n+1)^2}+\frac{1-3x(1-x)}{3(n+1)^2}+\frac{2\lambda[x^{n+1}(1-x)+x(1-x)^{n+1}]}{n^2-1} \\ &\quad -\frac{4x(1-x)\lambda}{(n+1)^2(n-1)} \\ &\le \frac{4}{n+1}. \end{aligned} \tag{12}$$

Lemma 2.4 *For the Bézier variant of Kantorovich type λ-Bernstein operators $L_{n,\lambda,\alpha}(f;x)$ and $f\in C_{[0,1]}$ with the sup-norm $\|f\|:=\sup_{x\in[0,1]}|f(x)|$, we have*

$$\big\|L_{n,\lambda,\alpha}(f)\big\|\le\alpha\|f\|.$$

Proof Since, for $\alpha\ge 1$, we have

$$0<\big[J_{n,k}(\lambda;x)\big]^{\alpha}-\big[J_{n,k+1}(\lambda;x)\big]^{\alpha}\le\alpha\big[J_{n,k}(\lambda;x)-J_{n,k+1}(\lambda;x)\big]=\alpha\widetilde{b}_{n,k}(\lambda;x).$$

Then, from (9) and the definition of $L_{n,\lambda,\alpha}(f;x)$, we have

$$\big\|L_{n,\lambda,\alpha}(f)\big\|\le\alpha\big\|K_{n,\lambda}(f)\big\|\le\alpha\|f\|.$$ □

Lemma 2.5

(i) *For $0\le y\le x<1$, we have*

$$R_{n,\lambda,\alpha}(x,y)=\int_0^y P_{n,\lambda,\alpha}(x,t)\,dt\le\frac{4\alpha}{(n+1)(x-y)^2}. \tag{13}$$

(ii) *For $0<x<z\le 1$, we have*

$$1-R_{n,\lambda,\alpha}(x,z)=\int_z^1 P_{n,\lambda,\alpha}(x,t)\,dt\le\frac{4\alpha}{(n+1)(z-x)^2}. \tag{14}$$

Proof (i) Using (7) and (12), we have

$$\begin{aligned} R_{n,\lambda,\alpha}(x,y) &= \int_0^y P_{n,\lambda,\alpha}(x,t)\,dt \\ &\le \int_0^y \left(\frac{x-t}{x-y}\right)^2 P_{n,\lambda,\alpha}(x,t)\,dt \\ &\le \frac{1}{(x-y)^2}\int_0^1 (t-x)^2 P_{n,\lambda,\alpha}(x,t)\,dt \\ &= \frac{1}{(x-y)^2} L_{n,\lambda,\alpha}\big((t-x)^2;x\big) \\ &\le \frac{\alpha}{(x-y)^2} K_{n,\lambda}\big((t-x)^2;x\big) \\ &\le \frac{4\alpha}{(n+1)(x-y)^2}. \end{aligned}$$

Similarly, (ii) is proved. □

3 Main results

As we know, the space $C_{[0,1]}$ of all continuous functions on $[0,1]$ is a Banach space with sup-norm $\|f\| := \sup_{x\in[0,1]} |f(x)|$. Let $f \in C[0,1]$, the Peetre's K-functional is defined by $K_2(f;t) := \inf_{g\in C^2_{[0,1]}} \{\|f-g\| + t\|g'\| + t^2\|g''\|\}$, where $t > 0$ and $C^2_{[0,1]} := \{g \in C_{[0,1]} : g', g'' \in C_{[0,1]}\}$. By [16], there exists an absolute constant $C > 0$ such that

$$K_2(f;t) \le C\omega_2(f;\sqrt{t}), \tag{15}$$

where $\omega_2(f;t) := \sup_{0<h\le t} \sup_{x,x+h,x+2h\in[0,1]} |f(x+2h) - 2f(x+h) + f(x)|$ is the second order modulus of smoothness of $f \in C_{[0,1]}$. We also denote the usual modulus of continuity of $f \in C_{[0,1]}$ by $\omega(f;t) := \sup_{0<h\le t} \sup_{x,x+h\in[0,1]} |f(x+h) - f(x)|$.

Theorem 3.1 *For $f \in C_{[0,1]}$, $\lambda \in [-1,1]$, we have*

$$\big|L_{n,\lambda,\alpha}(f;x) - f(x)\big| \le C\omega_2\left(f;\sqrt{\frac{\alpha}{n+1}}\right), \tag{16}$$

where C is a positive constant.

Proof Let $g \in C^2_{[0,1]}$, by Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\,du.$$

As we know, $L_{n,\lambda,\alpha}(1;x) = 1$. Applying $L_{n,\lambda,\alpha}(\cdot;x)$ to both sides of the above equation, we get

$$L_{n,\lambda,\alpha}(g;x) = g(x) + g'(x)L_{n,\lambda,\alpha}(t-x;x) + L_{n,\lambda,\alpha}\left(\int_x^t (t-u)g''(u)\,du;x\right).$$

By the Cauchy–Schwarz inequality, (12) and Lemma 2.4, we have

$$\begin{aligned}\left|L_{n,\lambda,\alpha}(g;x)-g(x)\right| &\le \left|g'(x)\right|\left|L_{n,\lambda,\alpha}\left(|t-x|;x\right)\right| + \left|L_{n,\lambda,\alpha}\left(\int_x^t (t-u)g''(u)\,du;x\right)\right| \\ &\le \left\|g'\right\| L_{n,\lambda,\alpha}\left(|t-x|;x\right) + \frac{\|g''\|}{2}L_{n,\lambda,\alpha}\left((t-x)^2;x\right) \\ &\le \left\|g'\right\|\sqrt{L_{n,\lambda,\alpha}\left((t-x)^2;x\right)} + \frac{\|g''\|}{2}L_{n,\lambda,\alpha}\left((t-x)^2;x\right) \\ &\le \sqrt{\alpha}\left\|g'\right\|\sqrt{K_{n,\lambda}\left((t-x)^2;x\right)} + \frac{\alpha\|g''\|}{2}K_{n,\lambda}\left((t-x)^2;x\right) \\ &\le \frac{2\sqrt{\alpha}\|g'\|}{\sqrt{n+1}} + \frac{2\alpha\|g''\|}{n+1}.\end{aligned}$$

Then, using the above inequality, we have

$$\begin{aligned}\left|L_{n,\lambda,\alpha}(f;x)-f(x)\right| &\le \left|L_{n,\lambda,\alpha}(f-g;x)\right| + \left|(f-g)(x)\right| + \left|L_{n,\lambda,\alpha}(g;x)-g(x)\right| \\ &\le 2\left(\|f-g\| + \sqrt{\frac{\alpha}{n+1}}\left\|g'\right\| + \frac{\alpha}{n+1}\left\|g'\right\|\right).\end{aligned}$$

Hence, taking infimum on the right-hand side over all $g \in C^2_{[0,1]}$, we get

$$\left|L_{n,\lambda,\alpha}(f;x)-f(x)\right| \le 2K_2\left(f;\frac{\alpha}{n+1}\right).$$

By (15), we obtain

$$\left|L_{n,\lambda,\alpha}(f;x)-f(x)\right| \le C\omega_2\left(f;\sqrt{\frac{\alpha}{n+1}}\right).$$

This completes the proof of Theorem 3.1. □

Next, we recall some definitions of the Ditzian–Totik first order modulus of smoothness and K-functional, which can be found in [17]. Let $f \in C_{[0,1]}$, and $\varphi(x) := \sqrt{x(1-x)}$, the first order modulus of smoothness is given by

$$\omega_\varphi(f;t) := \sup_{0<h\le t, x\pm\frac{h\varphi(x)}{2}\in[0,1]}\left|f\left(x+\frac{h\varphi(x)}{2}\right)-f\left(x-\frac{h\varphi(x)}{2}\right)\right|.$$

The K-functional $K_\varphi(f;t)$ is defined by $K_\varphi(f;t) := \inf_{g\in C^\varphi_{[0,1]}}\{\|f-g\| + t\|\varphi g'\|\}$, where $t>0$, $C^\varphi_{[0,1]} := \{g : g \in AC_{[0,1]}, \|\varphi g'\| < \infty\}$, $AC_{[0,1]}$ is the class of all absolutely continuous functions on $[0,1]$. Besides, from [17], there exists a constant $C>0$ such that

$$K_\varphi(f;t) \le C\omega_\varphi(f;t). \tag{17}$$

Theorem 3.2 *For $f \in C_{[0,1]}$, $\lambda \in [-1,1]$, and $\varphi(x) = \sqrt{x(1-x)}$, we have*

$$\left|L_{n,\lambda,\alpha}(f;x)-f(x)\right| \le C\omega_\varphi\left(f;\frac{2\sqrt{2\alpha}}{\sqrt{n+1}\varphi(x)}\right),$$

where C is a positive constant.

Proof Since

$$g(t) = g(x) + \int_x^t g'(u)\,du,$$

applying $L_{n,\lambda,\alpha}(f;x)$ to the above equality, we have

$$L_{n,\lambda,\alpha}(g;x) = g(x) + L_{n,\lambda,\alpha}\left(\int_x^t g'(u)\,du;x\right). \tag{18}$$

We will estimate $\int_x^t g'(u)\,du$: For any $x,t \in (0,1)$, we have

$$\begin{aligned}
\left|\int_x^t g'(u)\,du\right| &\le \|\varphi g'\|\left|\int_x^t \frac{1}{\varphi(u)}\,du\right| \\
&= \|\varphi g'\|\left|\int_x^t \frac{1}{\sqrt{u(1-u)}}\,du\right| \\
&\le \|\varphi g'\|\left|\int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}}\right)du\right| \\
&\le 2\|\varphi g'\|\left(|\sqrt{t}-\sqrt{x}| + |\sqrt{1-t}-\sqrt{1-x}|\right) \\
&= 2\|\varphi g'\||t-x|\left(\frac{1}{\sqrt{t}+\sqrt{x}} + \frac{1}{\sqrt{1-t}+\sqrt{1-x}}\right) \\
&\le 2\|\varphi g'\||t-x|\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\right) \\
&\le 2\sqrt{2}\|\varphi g'\|\frac{|t-x|}{\varphi(x)}.
\end{aligned}$$

From (18), using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\left|L_{n,\lambda,\alpha}(g;x) - g(x)\right| &\le 2\sqrt{2}\frac{\|\varphi g'\|}{\varphi(x)}L_{n,\lambda,\alpha}\left(|t-x|;x\right) \\
&\le 2\sqrt{2}\frac{\|\varphi g'\|}{\varphi(x)}\sqrt{L_{n,\lambda,\alpha}\left((t-x)^2;x\right)} \\
&\le 2\sqrt{2\alpha}\frac{\|\varphi g'\|}{\varphi(x)}\sqrt{K_{n,\lambda}\left((t-x)^2;x\right)} \\
&\le \frac{4\sqrt{2\alpha}\|\varphi g'\|}{\sqrt{n+1}\varphi(x)}.
\end{aligned}$$

Hence, using the above inequality, we have

$$\begin{aligned}
\left|L_{n,\lambda,\alpha}(f;x) - f(x)\right| &\le \left|L_{n,\lambda,\alpha}(f-g;x)\right| + \left|(f-g)(x)\right| + \left|L_{n,\lambda,\alpha}(f;x) - g(x)\right| \\
&\le 2\left(\|f-g\| + \frac{2\sqrt{2\alpha}}{\sqrt{n+1}\varphi(x)}\|\varphi g'\|\right).
\end{aligned}$$

Taking infimum on the right-hand side over all $g \in C^{\varphi}_{[0,1]}$, we get

$$\left|L_{n,\lambda,\alpha}(f;x) - f(x)\right| \le 2K_{\varphi}\left(f;\frac{2\sqrt{2\alpha}}{\sqrt{n+1}\varphi(x)}\right).$$

By (17), we obtain

$$\left|L_{n,\lambda,\alpha}(f;x)-f(x)\right| \le C\omega_{\varphi}\left(f;\frac{2\sqrt{2\alpha}}{\sqrt{n+1}\varphi(x)}\right).$$

Theorem 3.2 is proved. □

Finally, we study the approximation properties of $L_{n,\lambda,\alpha}(f;x)$ for some absolutely continuous functions $f \in \Phi_{\mathrm{DB}}$.

Theorem 3.3 *Let f be a function in Φ_{DB}. If $\phi(x+)$ and $\phi(x-)$ exist at a fixed point $x \in (0,1)$, then we have*

$$\left|L_{n,\lambda,\alpha}(f;x)-f(x)\right| \le \frac{2\alpha(|\phi(x+)|+|\phi(x-)|)}{\sqrt{n+1}} + \frac{8\alpha+2x(1-x)}{nx(1-x)}\sum_{k=1}^{[\sqrt{n}]}\Omega_x\left(\phi_x;\frac{1}{k}\right),$$

where $[n]$ denotes the greatest integer not exceeding n, and

$$\phi_x(u)=\begin{cases}\phi(u)-\phi(x+), & x<u\le 1;\\ 0, & u=x;\\ \phi(u)-\phi(x-), & 0\le u<x.\end{cases} \tag{19}$$

Proof By the fact that $L_{n,\lambda,\alpha}(1;x)=1$, using (7) and (8), we have

$$\begin{aligned}L_{n,\lambda,\alpha}(f;x)-f(x) &= \int_0^1 \left[f(t)-f(x)\right] d_tR_{n,\lambda,\alpha}(x,t)\\ &= \int_0^1\left(\int_x^t \phi(u)\,du\right) d_tR_{n,\lambda,\alpha}(x,t).\end{aligned}$$

By the Bojanic–Cheng decomposition [18], we have

$$\begin{aligned}\phi(u) &= \frac{\phi(x+)+\phi(x-)}{2}+\phi_x(u)+\frac{\phi(x+)-\phi(x-)}{2}\operatorname{sgn}(u-x)\\ &\quad+\delta_x(u)\left(\phi(x)-\frac{\phi(x+)+\phi(x-)}{2}\right),\end{aligned} \tag{20}$$

where $\phi_x(u)$ is defined in (19), $\operatorname{sgn}(u)$ is a sign function and $\delta_x(u)=\begin{cases}1, & u=x;\\ 0, & u\ne x.\end{cases}$ By direct integrations, we find that

$$\begin{aligned}L_{n,\lambda,\alpha}(f;x)-f(x) &= \frac{\phi(x+)-\phi(x-)}{2}L_{n,\lambda,\alpha}\left(|t-x|;x\right)-U_{n,\lambda,\alpha}(\phi_x;x)+T_{n,\lambda,\alpha}(\phi_x;x)\\ &\quad+\frac{\phi(x+)+\phi(x-)}{2}L_{n,\lambda,\alpha}(t-x;x),\end{aligned} \tag{21}$$

where

$$U_{n,\lambda,\alpha}(\phi_x;x)=\int_0^x\left(\int_t^x \phi_x(u)\,du\right) d_tR_{n,\lambda,\alpha}(x,t),$$

$$T_{n,\lambda,\alpha}(\phi_x;x)=\int_x^1\left(\int_x^t \phi_x(u)\,du\right) d_tR_{n,\lambda,\alpha}(x,t).$$

Integration by parts derives

$$\begin{aligned} U_{n,\lambda,\alpha}(\phi_x;x) &= \int_0^x \left(\int_t^x \phi_x(u)\,du \right) d_t R_{n,\lambda,\alpha}(x,t) \\ &= \int_t^x \phi_x(u)\,du R_{n,\lambda,\alpha}(x,t)\Big|_0^x + \int_0^x R_{n,\lambda,\alpha}(x,t)\phi_x(t)\,dt \\ &= \int_0^x R_{n,\lambda,\alpha}(x,t)\phi_x(t)\,dt \\ &= \left(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^x \right) R_{n,\lambda,\alpha}(x,t)\phi_x(t)\,dt. \end{aligned}$$

Note that $R_{n,\lambda,\alpha}(x,t) \le 1$ and $\phi_x(x) = 0$, it follows that

$$\left| \int_{x-x/\sqrt{n}}^x R_{n,\lambda,\alpha}(x,t)\phi_x(t)\,dt \right| \le \frac{x}{\sqrt{n}} \Omega_x\left(\phi_x; \frac{x}{\sqrt{n}}\right) \le \frac{2x}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega_x\left(\phi_x; \frac{x}{k}\right).$$

From Lemma 2.5 (i) and change of variable $t = x - x/u$, we have

$$\begin{aligned} \left| \int_0^{x-x/\sqrt{n}} R_{n,\lambda,\alpha}(x,t)\phi_x(t)\,dt \right| &\le \frac{4\alpha}{n+1} \int_0^{x-x/\sqrt{n}} \frac{\Omega_x(\phi_x, x-t)}{(x-t)^2}\,dt \\ &= \frac{4\alpha}{(n+1)x} \int_1^{\sqrt{n}} \Omega_x\left(\phi_x; \frac{x}{u}\right) du \\ &\le \frac{8\alpha}{(n+1)x} \sum_{k=1}^{[\sqrt{n}]} \Omega_x\left(\phi_x; \frac{x}{k}\right). \end{aligned}$$

Thus, it follows that

$$\begin{aligned} \left| U_{n,\lambda,\alpha}(\phi_x;x) \right| &\le \frac{8\alpha}{(n+1)x} \sum_{k=1}^{[\sqrt{n}]} \Omega_x\left(\phi_x; \frac{x}{k}\right) + \frac{2x}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega_x\left(\phi_x; \frac{x}{k}\right) \\ &\le \frac{8\alpha + 2x^2}{nx} \sum_{k=1}^{[\sqrt{n}]} \Omega_x\left(\phi_x; \frac{1}{k}\right). \end{aligned} \tag{22}$$

From Lemma 2.5(ii), using a similar method, we also obtain

$$\left| T_{n,\lambda,\alpha}(\phi_x;x) \right| \le \frac{8\alpha + 2(1-x)^2}{n(1-x)} \sum_{k=1}^{[\sqrt{n}]} \Omega_x\left(\phi_x; \frac{1}{k}\right). \tag{23}$$

By the Cauchy–Schwarz inequality, (12), and Lemma 2.4, we have

$$L_{n,\lambda,\alpha}\left(|t-x|;x\right) \le \alpha K_{n,\lambda}\left(|t-x|;x\right) \le \alpha \sqrt{K_{n,\lambda}\left((t-x)^2;x\right)} \le \frac{2\alpha}{\sqrt{n+1}}. \tag{24}$$

Hence, by (22), (23), (24), and (21), we have

$$\left| L_{n,\lambda,\alpha}(f;x) - f(x) \right| \le \frac{2\alpha(|\phi(x+)| + |\phi(x-)|)}{\sqrt{n+1}} + \frac{8\alpha + 2x(1-x)}{nx(1-x)} \sum_{k=1}^{[\sqrt{n}]} \Omega_x\left(\phi_x; \frac{1}{k}\right).$$

Theorem 3.3 is proved. □

4 Conclusion

In this paper, we have presented a Bézier variant of Kantorovich type λ-Bernstein operators $L_{n,\lambda,\alpha}(f;x)$, and established approximation theorems by using the usual second order modulus of smoothness and the Ditzian–Totik modulus of smoothness. From Theorem 3.3 of Sect. 3, we know that the rate of convergence of operators $L_{n,\lambda,\alpha}(f;x)$ for $f \in \Phi_{\mathrm{DB}}$ is $\frac{1}{\sqrt{n+1}}$. Furthermore, we might consider the approximation of these operators $L_{n,\lambda,\alpha}(f;x)$ for locally bounded functions.

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Competing interests
The author declares that there are no competing interests.

Authors' contributions
The author carried out the work and wrote the whole manuscript. All authors read and approved the final manuscript.

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