Huang et al. *Journal of Inequalities and Applications* (2018) 2018:52
https://doi.org/10.1186/s13660-018-1644-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Some Wilker and Cusa type inequalities for generalized trigonometric and hyperbolic functions

Li-Guo Huang[1*], Li Yin[1], Yong-Li Wang[2] and Xiu-Li Lin[3]

*Correspondence:
liguoh123@sina.com
[1]College of Science, Binzhou University, Binzhou, China
Full list of author information is available at the end of the article

Abstract
The authors obtain some Wilker and Cusa type inequalities for generalized trigonometric and hyperbolic functions and generalize some known inequalities.

MSC: 33B10

Keywords: Wilker type inequality; Cusa type inequality; Arithmetic–geometric means inequality; Power means inequality; Generalized trigonometric function; Generalized hyperbolic function

1 Introduction

It is well known from basic calculus that

$$\arcsin x = \int_0^x \frac{1}{(1-t^2)^{1/2}}\,dt \tag{1.1}$$

for $0 \le x \le 1$ and

$$\frac{\pi}{2} = \arcsin 1 = \int_0^1 \frac{1}{(1-t^2)^{1/2}}\,dt. \tag{1.2}$$

For $1 < p < \infty$ and $0 \le x \le 1$, the arc sine may be generalized as

$$\arcsin_p x = \int_0^x \frac{1}{(1-t^p)^{1/p}}\,dt \tag{1.3}$$

and

$$\frac{\pi_p}{2} = \arcsin_p 1 = \int_0^1 \frac{1}{(1-t^p)^{1/p}}\,dt. \tag{1.4}$$

The inverse of $\arcsin_p$ on $[0, \frac{\pi_p}{2}]$ is called the generalized sine function, denoted by $\sin_p$, and may be extended to $(-\infty, \infty)$. In the same way, we can define the generalized cosine function, the generalized tangent function, and their inverses, and also the corresponding hyperbolic functions. For their definitions and formulas, one may see recent references [1–3].

In [2], some classical inequalities for generalized trigonometric and hyperbolic functions, such as Mitrinović–Adamović inequality, Huygens' inequality, and Wilker's inequality, were generalized. In [3], some new second Wilker type inequalities for generalized trigonometric and hyperbolic functions were established. In [4], some Turán type inequalities for generalized trigonometric and hyperbolic functions were presented. Very recently, a conjecture posed in [5] was verified in [1]. For more about the Wilker type inequality and Huygens type inequalities, the reader may see [6–13].

In this paper, we establish some new Wilker and Cusa type inequalities for the generalized trigonometric and hyperbolic functions. Some known inequalities in [3] are the special cases of our results.

2 Lemmas

Lemma 2.1 ([3, Lemma 2.7]) *For $p \in (1,\infty)$, we have*

$$\cos_p^\alpha x < \frac{\sin_p x}{x} < 1, \quad x \in \left(0, \frac{\pi_p}{2}\right) \tag{2.1}$$

and

$$\cosh_p^\alpha x < \frac{\sinh_p x}{x} < \cosh_p^\beta x, \quad x > 0, \tag{2.2}$$

where the constants $\alpha = \frac{1}{p+1}$ and $\beta = 1$ are the best possible.

Lemma 2.2 ([3, Theorem 3.5]) *For $p \in (1,2]$, then*

$$\left(\frac{x}{\sin_p x}\right)^p + \frac{x}{\tan_p x} > 2, \quad x \in \left(0, \frac{\pi_p}{2}\right). \tag{2.3}$$

Lemma 2.3 ([14]) *Let $a > 0, b > 0$ and $r \geq 1$, then*

$$(a+b)^r \leq 2^{r-1}\left(a^r + b^r\right). \tag{2.4}$$

Lemma 2.4 ([15]) *Let $a_k > 0, k = 1,2,\ldots,n$, then*

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{(1+a_1)(1+a_2)\cdots(1+a_n)} - 1 \geq \sqrt[n]{a_1 a_2 \cdots a_n}. \tag{2.5}$$

Lemma 2.5 ([2, Theorem 3.4]) *For $p \in [2,\infty)$ and $x \in (0, \frac{\pi_p}{2})$, then*

$$\frac{\sin_p x}{x} < \frac{x}{\sinh_p x}. \tag{2.6}$$

Lemma 2.6 *For $p \in [2,\infty)$ and $x \in (0, \frac{\pi_p}{2})$, we have*

$$\left(\frac{\sin_p x}{x}\right)^p < \frac{x}{\sinh_p x}. \tag{2.7}$$

Proof Using Lemma 2.5 and $\frac{\sin_p x}{x} < 1$, we have

$$\frac{x}{\sinh_p x} > \frac{\sin_p x}{x} > \left(\frac{\sin_p x}{x}\right)^p. \tag{2.8}$$

This implies inequality (2.7). □

Lemma 2.7 ([2, Corollary 3.10]) *For $p \in [2,\infty)$ and $x \in (0, \frac{\pi_p}{2})$, then*

$$\left(\frac{x}{\sinh_p x}\right)^{p+1} < \frac{\sin_p x}{x}. \tag{2.9}$$

Lemma 2.8 ([2, Theorem 3.22]) *For $p \in (1,2]$, the double inequality*

$$\frac{\sin_p x}{x} < \frac{\cos_p x + p}{1+p} \leq \frac{\cos_p x + 2}{3} \tag{2.10}$$

holds for all $x \in (0, \frac{\pi_p}{2}]$.

3 Main results

Theorem 3.1 *For $x \in (0, \frac{\pi_p}{2})$, $p \in (1,\infty)$, and $\alpha - p\beta \leq 0$, $\beta > 0$, we have*

$$\left(\frac{\sin_p x}{x}\right)^{\alpha} + \left(\frac{\tan_p x}{x}\right)^{\beta} > 2. \tag{3.1}$$

Proof From the arithmetic geometric means inequality and Lemma 2.1, it follows that

$$\begin{aligned}
\left(\frac{\sin_p x}{x}\right)^{\alpha} + \left(\frac{\tan_p x}{x}\right)^{\beta} &\geq 2\left(\frac{\sin_p x}{x}\right)^{\frac{\alpha}{2}}\left(\frac{\tan_p x}{x}\right)^{\frac{\beta}{2}} \\
&= 2\left(\frac{\sin_p x}{x}\right)^{\frac{\alpha+\beta}{2}}\left(\frac{1}{\cos_p x}\right)^{\frac{\beta}{2}} \\
&> 2\left(\frac{\sin_p x}{x}\right)^{\frac{\alpha+\beta}{2}}\left(\frac{\sin_p x}{x}\right)^{-\frac{(p+1)\beta}{2}} \\
&= 2\left(\frac{\sin_p x}{x}\right)^{\frac{\alpha-p\beta}{2}} \\
&\geq 2.
\end{aligned}$$

□

Remark 3.1 If $p = \alpha = 2, \beta = 1$, inequality (3.1) turns into

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{3.2}$$

Inequality (3.2) is called the first Wilker inequality in [16].

Remark 3.2 If $\alpha = 2p, \beta = p$, and $p \geq 2$, then $\alpha - p\beta = 2p - p^2 \leq 0$. So, inequality (3.1) reduces to

$$\left(\frac{\sin_p x}{x}\right)^{2p} + \left(\frac{\tan_p x}{x}\right)^{p} > 2. \tag{3.3}$$

Theorem 3.2 *For* $p \in (1,2], x \in (0, \frac{\pi_p}{2})$, *and* $\alpha - p\beta \le 0, \beta \le -1$, *we have*

$$\left(\frac{\sin_p x}{x}\right)^{\alpha} + \left(\frac{\tan_p x}{x}\right)^{\beta} > 2. \tag{3.4}$$

Proof Using $\frac{x}{\sin_p x} \ge 1$ and $\alpha - p\beta \le 0$, we have

$$\begin{aligned}\left(\frac{\sin_p x}{x}\right)^{\alpha} + \left(\frac{\tan_p x}{x}\right)^{\beta} &= \left(\frac{x}{\sin_p x}\right)^{-\alpha} + \left(\frac{x}{\tan_p x}\right)^{-\beta} \\ &= \left(\frac{x}{\sin_p x}\right)^{-p\beta}\left(\frac{x}{\sin_p x}\right)^{p\beta-\alpha} + \left(\frac{x}{\tan_p x}\right)^{-\beta} \\ &\ge \left[\left(\frac{x}{\sin_p x}\right)^{p}\right]^{-\beta} + \left(\frac{x}{\tan_p x}\right)^{-\beta}.\end{aligned}$$

Applying Lemmas 2.2 and 2.3, we obtain

$$\left(\frac{\sin_p x}{x}\right)^{\alpha} + \left(\frac{\tan_p x}{x}\right)^{\beta} \ge 2^{1+\beta}\left[\left(\frac{x}{\sin_p x}\right)^{p} + \frac{x}{\tan_p x}\right]^{-\beta} > 2.$$

This completes the proof. □

Using the same method as that in Theorem 3.1, we can easily obtain the following Theorem 3.3 by Lemma 2.1 and the arithmetic and geometric means inequality. We omit the proof for the sake of simplicity.

Theorem 3.3 *For* $p \in (1,\infty), x \in (0,\infty)$, *and* $\alpha - p\beta \le 0, \beta > 0$, *then*

$$\left(\frac{\sinh_p x}{x}\right)^{\alpha} + \left(\frac{\tanh_p x}{x}\right)^{\beta} > 2. \tag{3.5}$$

Remark 3.3 Taking $\alpha = 2, \beta = 1$ and $p = 2$ in inequality (3.5), we have

$$\left(\frac{\sinh x}{x}\right)^{2} + \frac{\tanh x}{x} > 2, \tag{3.6}$$

which is the (4) in Theorem 1 of [7]. Inequality (3.6) is called the first hyperbolic Wilker inequality.

Remark 3.4 Taking $\alpha = 2p, \beta = p$, and $p \in [2,\infty)$, we have

$$\left(\frac{\sinh_p x}{x}\right)^{2p} + \left(\frac{\tanh_p x}{x}\right)^{p} > 2. \tag{3.7}$$

Theorem 3.4 *For all* $x \in (0, \frac{\pi_p}{2})$ *and* $\alpha - p\beta \le 0, \beta > 0$, *we have*

$$\left[1 + \left(\frac{\sin_p x}{x}\right)^{\alpha}\right]\left[1 + \left(\frac{\tan_p x}{x}\right)^{\beta}\right] > 4 \tag{3.8}$$

and

$$\left(\frac{\sin_p x}{x}\right)^{\alpha}+\left(\frac{\tan_p x}{x}\right)^{\beta}>2\sqrt{\left[1+\left(\frac{\sin_p x}{x}\right)^{\alpha}\right]\left[1+\left(\frac{\tan_p x}{x}\right)^{\beta}\right]}-2>2. \tag{3.9}$$

Proof Setting $n=2, a_1=(\frac{\sin_p x}{x})^{\alpha}$ and $a_2=(\frac{\tan_p x}{x})^{\beta}$ in Lemma 2.4, we have

$$\begin{aligned}&\left[1+\left(\frac{\sin_p x}{x}\right)^{\alpha}\right]\left[1+\left(\frac{\tan_p x}{x}\right)^{\beta}\right]\\ &\quad\geq\left[\left(\frac{\sin_p x}{x}\right)^{\frac{\alpha}{2}}\left(\frac{\tan_p x}{x}\right)^{\frac{\beta}{2}}+1\right]^2\\ &\quad>\left[\left(\frac{\sin_p x}{x}\right)^{\frac{\alpha-p\beta}{2}}+1\right]^2\\ &\quad>4.\end{aligned}$$

Then it follows from Lemma 2.1 that

$$\left(\frac{\sin_p x}{x}\right)^{\alpha}+\left(\frac{\tan_p x}{x}\right)^{\beta}>2\sqrt{\left[1+\left(\frac{\sin_p x}{x}\right)^{\alpha}\right]\left[1+\left(\frac{\tan_p x}{x}\right)^{\beta}\right]}-2>2.$$

□

Remark 3.5 If $n=3$ and $a_1=a_2=(\frac{\sin_p x}{x})^{\alpha}, a_3=(\frac{\tan_p x}{x})^{\beta}$ in Lemma 2.4, it can be easily obtained that

$$\left[1+\left(\frac{\sin_p x}{x}\right)^{\alpha}\right]^2\left[1+\left(\frac{\tan_p x}{x}\right)^{\beta}\right]>8 \tag{3.10}$$

and

$$2\left(\frac{\sin_p x}{x}\right)^{\alpha}+\left(\frac{\tan_p x}{x}\right)^{\beta}>3\sqrt[3]{\left[1+\left(\frac{\sin_p x}{x}\right)^{\alpha}\right]^2\left[1+\left(\frac{\tan_p x}{x}\right)^{\beta}\right]}-3>3, \tag{3.11}$$

by a similar method to that in Theorem 3.4 when changing the condition $\alpha-p\beta\leq 0$ to $2\alpha-p\beta\leq 0$.

Theorem 3.5 *For $p\in[2,\infty), t>0$, and $x\in(0,\frac{\pi_p}{2}]$, then*

$$\left(\frac{x}{\sin_p x}\right)^{pt}+\left(\frac{x}{\sinh_p x}\right)^{t}>2. \tag{3.12}$$

Proof Applying the AGM inequality $a+b\geq 2\sqrt{ab}$ and Lemma 2.6 for $a=(\frac{x}{\sin_p x})^{pt}$ and $b=(\frac{x}{\sinh_p x})^{t}$, we obtain

$$a+b\geq 2\sqrt{\left(\frac{x}{\sin_p x}\right)^{pt}\left(\frac{x}{\sinh_p x}\right)^{t}}>2.$$

The proof is completed. □

Theorem 3.6 *For $p \in [2, \infty)$, $t > 0$ and $x \in (0, \frac{\pi_p}{2}]$, then*

$$(p+1)\left(\frac{x}{\sin_p x}\right)^t + \left(\frac{x}{\sinh_p x}\right)^t > p+1. \tag{3.13}$$

Proof From the AGM inequality $(n+1)a + b \geq (n+1)\sqrt[n+1]{a^n b}$ and Lemma 2.6, for $a = (\frac{x}{\sin_p x})^t$ and $b = (\frac{x}{\sinh_p x})^t$, inequality (3.13) follows readily. □

Applying AGM inequality and Lemma 2.7, Theorems 3.7 and 3.8 can be easily obtained by the similar method as before.

Theorem 3.7 *For $p \in [2, \infty)$, $t > 0$, and $x \in (0, \frac{\pi_p}{2}]$, then*

$$\left(\frac{\sinh_p x}{x}\right)^{(p+1)t} + \left(\frac{\sin_p x}{x}\right)^t > 2. \tag{3.14}$$

Theorem 3.8 *For $p \in [2, \infty)$, $t > 0$, and $x \in (0, \frac{\pi_p}{2}]$, then*

$$(p+2)\left(\frac{\sinh_p x}{x}\right)^t + \left(\frac{\sin_p x}{x}\right)^t > p+2. \tag{3.15}$$

Finally, we give a Cusa type inequality.

Theorem 3.9 *For $p \in (1, 2]$ and $x \in (0, \frac{\pi_p}{2}]$, the function $f(x) = \frac{\ln \frac{\sin_p x}{x}}{\ln \frac{p+\cos_p x}{p+1}}$ is strictly increasing. Consequently, we have the following inequality*:

$$\left(\frac{p+\cos_p x}{p+1}\right)^\alpha < \frac{\sin_p x}{x} < \left(\frac{p+\cos_p x}{p+1}\right)^\beta \tag{3.16}$$

with the best constants $\alpha = \frac{\ln \frac{2\sin_p \frac{\pi_p}{2}}{\pi_p}}{\ln \frac{p+\cos_p \frac{\pi_p}{2}}{p+1}}$ and $\beta = 1$.

Proof A simple computation yields

$$\begin{aligned}
&f'(x)\ln^2 \frac{p+\cos_p x}{p+1} \\
&\quad = \frac{x\cos_p x - \sin_p x}{x\sin_p x}\ln\frac{p+\cos_p x}{p+1} + \frac{\cos_p x \tan_p^{p-1} x}{p+\cos_p x}\ln\frac{\sin_p x}{x} \\
&\quad > \frac{x\cos_p x - \sin_p x}{x\sin_p x} + \frac{\cos_p x \tan_p^{p-1} x}{p+\cos_p x}\ln\frac{\sin_p x}{x} \\
&\quad = \frac{(x\cos_p x - \sin_p x)(p+\cos_p x) + x\sin_p x\cos_p x\tan_p^{p-1} x}{x\sin_p x(p+\cos_p x)}\ln\frac{\sin_p x}{x} \\
&\quad = \frac{\ln\frac{\sin_p x}{x}}{x\sin_p x(p+\cos_p x)} g(x),
\end{aligned}$$

where

$$g(x) = x\cos_p^2 x \sec_p^p x + px\cos_p x - p\sin_p x - \sin_p x\cos_p x.$$

Since

$$g'(x) = \cos_p x \tan_p^{p-1} x h(x),$$

where

$$h(x) = 2\sin_p x - px - (2-p)x\sec_p^{p-1} x,$$

with

$$h'(x) = 2\cos_p x - p - (2-p)\sec_p^{p-1} x - (2-p)(p-1)x\sec_p^{p-1} x\tan_p^{p-1} x$$

and

$$\begin{aligned} h''(x) &= -2\cos_p x\tan_p^{p-1} x - 2(2-p)(p-1)\sec_p^{p-1} x\tan_p^{p-1} x \\ &\quad - (2-p)(p-1)^2 x\sec_p^{p-1} x\tan_p^{p-1} x\big(\tan_p^{p-1} x + \csc_p x\sec_p^{p-1} x\big) < 0. \end{aligned}$$

Hence $h'(x)$ is decreasing on $(0, \frac{\pi_p}{2})$. It then follows that $h'(x) < h'(0) = 0$, which also implies that $h(x) < h(0) = 0$. Hence, $g'(x) < 0$, which shows that the function $g(x)$ is also decreasing on $(0, \frac{\pi_p}{2})$. The inequality $g(x) < g(0) = 0$ indicates that $f'(x) > 0$. Hence, $f(x)$ is strictly increasing for $x \in (0, \frac{\pi_p}{2})$. As a result, we have $f(0) < f(x) \le f(\frac{\pi_p}{2})$.

Using L'Hôspital's rule, we obtain that

$$\begin{aligned} f(0^+) &= \lim_{x\to 0^+} \frac{\ln\frac{\sin_p x}{x}}{\ln\frac{p+\cos_p x}{p+1}} \\ &= \lim_{x\to 0^+} -\frac{x\cos_p x - \sin_p x}{x\sin_p x}\frac{p+\cos_p x}{\cos_p x\tan_p^{p-1} x} \\ &= -(p+1)\lim_{x\to 0^+}\frac{x\cos_p x - \sin_p x}{x^{p+1}} \\ &= 1 \end{aligned}$$

and

$$f\left(\frac{\pi_p}{2}\right) = \frac{\ln\frac{2\sin_p\frac{\pi_p}{2}}{\pi_p}}{\ln\frac{p+\cos_p\frac{\pi_p}{2}}{p+1}}.$$

The proof is completed. □

4 A conjecture

Conjecture 4.1 *For all* $x \in (0, \frac{\pi_p}{2}]$ *and* $p \in (1,2]$, *is the function* $\frac{\ln\frac{x}{\sin_p x}}{\ln\cosh_p x}$ *strictly increasing?*

Acknowledgements
The work was supported by NSFC11401041, 51674038, NSF of Shandong Province under grant numbers ZR2017MA019, Science and Technology Project of Shandong Province under grant J16li52, and by the Science Foundation of Binzhou University under grant BZXYL1704. Also the authors would like to thank the editor and the anonymous referees for their valuable suggestions and comments which helped us to improve this paper greatly.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Author details
[1]College of Science, Binzhou University, Binzhou, China. [2]College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, China. [3]School of Mathematical Sciences, Qufu Normal University, Qufu, China.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 6 December 2017 Accepted: 22 February 2018 Published online: 02 March 2018

References

1. Jiang, W.-D., Wang, M.-K., Chu, Y.-M., Jiang, Y.-P., Qi, F.: Convexity of the generalized sine function and the generalized hyperbolic sine function. J. Approx. Theory **174**, 1–9 (2013). https://doi.org/10.1016/j.jat.2013.06.005
2. Klén, R., Vuorinen, M., Zhang, X.-H.: Inequalities for the generalized trigonometric and hyperbolic functions. J. Math. Anal. Appl. **409**(5), 21–529 (2014). https://doi.org/10.1016/j.jmaa.2013.07.021
3. Yin, L., Huang, L.-G., Qi, F.: Some inequalities for the generalized trigonometric and hyperbolic functions. Turk. J. Anal. Number Theory **2**(3), 96–101 (2014). https://doi.org/10.12691/tjant-2-3-8
4. Baricz, Á., Bhayo, B.A., Vuorinen, M.: Turán type inequalities for generalized inverse trigonometric functions. Filomat **29**(2), 303–313 (2015). https://doi.org/10.2298/FIL1502303B
5. Bhayo, B.A., Vuorinen, M.: Inequalities for eigenfunctions of the *p*-Laplacian. Probl. Anal. Issues Anal. **2**(20), 13–35 (2013). https://doi.org/10.15393/j3.art.2013.2322
6. Jiang, W.-D., Luo, Q.-M., Qi, F.: Refinements and sharpening of some Huygens and Wilker type inequalities. Turk. J. Anal. Number Theory **2**(4), 134–139 (2014). https://doi.org/10.12691/tjant-2-4-6
7. Zhu, L.: On Wilker-type inequalities. Math. Inequal. Appl. **10**(4), 727–731 (2007). https://doi.org/10.5402/2011/681702
8. Zhu, L.: Some new Wilker type inequalities for circular and hyperbolic functions. Abstr. Appl. Anal. **2009**, 485842 (2009). https://doi.org/10.1155/2009/485842
9. Wu, S.-H., Srivastava, H.M.: A further refinement of Wilker's inequality. Integral Transforms Spec. Funct. **19**(10), 757–765 (2008). https://doi.org/10.1080/10652460802340931
10. Wu, S.-H., Debnath, L.: Wilker-type inequalities for hyperbolic functions. Appl. Math. Lett. **25**(5), 837–842 (2012). https://doi.org/10.1016/j.aml.2011.10.028
11. Wu, S.-H.: On extension and refinement of Wilker inequality. Rocky Mt. J. Math. **39**(2), 683–687 (2009). https://doi.org/10.1216/RMJ-2009-39-2-683
12. Neuman, E.: Wilker and Huygens-type inequalities for the generalized trigonometric and for the generalized hyperbolic functions. Appl. Math. Comput. **230**, 211–217 (2014). https://doi.org/10.1016/j.amc.2013.12.136
13. Neuman, E.: On Wilker and Huygens type inequalities. Math. Inequal. Appl. **15**(2), 271–279 (2012). https://doi.org/10.7153/mia-15-22
14. Mitrinovic, D.S.: Analytic Inequalities. Springer, New York (1970)
15. Neumann, E., Sándor, J.: On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities. Math. Inequal. Appl. **13**(4), 715–723 (2010). https://doi.org/10.7153/mia-13-50
16. Qi, F., Niu, D.-W., Guo, B.-N.: Refinements, generalizations, and applications of Jordan's inequality and related problems. J. Inequal. Appl. **2009**, 271923 (2009). https://doi.org/10.1155/2009/271923