

RESEARCH

Open Access



Extensions of different type parameterized inequalities for generalized (m, h) -preinvex mappings via k -fractional integrals

Yao Zhang¹, Ting-Song Du^{1,2*} , Hao Wang¹, Yan-Jun Shen³ and Artion Kashuri⁴

*Correspondence:

tingsongdu@ctgu.edu.cn

¹Department of Mathematics, College of Science, China Three Gorges University, Yichang, China

²Three Gorges Mathematical Research Center, China Three Gorges University, Yichang, China
Full list of author information is available at the end of the article

Abstract

The authors discover a general k -fractional integral identity with multi-parameters for twice differentiable functions. By using this integral equation, the authors derive some new bounds on Hermite–Hadamard's and Simpson's inequalities for generalized (m, h) -preinvex functions through k -fractional integrals. By taking the special parameter values for various suitable choices of function h , some interesting results are also obtained.

MSC: 26A33; 26A51; 26D07; 26D20; 41A55

Keywords: Hermite–Hadamard's inequality; Simpson's inequality; Generalized (m, h) -preinvex functions; k -fractional integrals

1 Introduction

The subsequent inequalities are notable in the literature as Hermite–Hadamard's inequality and Simpson's inequality, respectively.

Theorem 1.1 *Suppose that $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex function defined on the interval I of real numbers and $a, b \in I$ along with $a < b$. The following double inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}.$$

Theorem 1.2 *Assume that $f : [a, b] \rightarrow \mathbb{R}$ is a four times continuously differentiable mapping on (a, b) and $\|f^{(4)}\|_\infty = \sup_{x \in (a,b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:*

$$\left| \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b-a)^4.$$

Hermite–Hadamard's inequalities and Simpson's inequalities have remained an area of great interest owing to their extensive applications in mathematics and other sciences. Many researchers generalized these inequalities. For recent results, for example, see [1–8] and the references mentioned in these papers.

In 2013, Sarikaya et al. established the subsequent interesting Hermite–Hadamard’s inequalities by utilizing Riemann–Liouville fractional integrals.

Theorem 1.3 ([9]) *Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function along with $0 \leq a < b$, and let $f \in L^1[a, b]$. Suppose that f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a + b}{2}\right) \leq \frac{\Gamma(\mu + 1)}{2(b - a)^\mu} [J_{a^+}^\mu f(b) + J_{b^-}^\mu f(a)] \leq \frac{f(a) + f(b)}{2}, \tag{1.1}$$

where the symbols $J_{a^+}^\mu f$ and $J_{b^-}^\mu f$ denote respectively the left-sided and right-sided Riemann–Liouville fractional integrals of order $\mu > 0$ defined by

$$J_{a^+}^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x - t)^{\mu-1} f(t) dt, \quad a < x,$$

and

$$J_{b^-}^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t - x)^{\mu-1} f(t) dt, \quad x < b.$$

Here, $\Gamma(\mu)$ is the gamma function and its definition is $\Gamma(\mu) = \int_0^\infty e^{-t} t^{\mu-1} dt$. It is to be noted that $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

In the case of $\mu = 1$, the fractional integral recaptures the classical integral.

Because of the extensive application of Riemann–Liouville fractional integrals, some authors extended their studies to fractional Hermite–Hadamard’s inequalities via mappings of different classes. For example, refer to [10–12] for convex mappings, to [13] for s -convex mappings, to [14] for (s, m) -convex mappings, to [15] for s -Godunova–Levin mappings, to [16] for harmonically convex mappings, to [17] for preinvex mappings, to [18] for MT_m -preinvex mappings, to [19] for h -convex mappings, to [20] for r -convex mappings, and see the references cited therein.

In 2012, Mubeen and Habibullah introduced the following class of fractional integrals.

Definition 1.1 ([21]) *Let $f \in L^1[a, b]$, then the Riemann–Liouville k -fractional integrals ${}_k J_{a^+}^\mu f(x)$ and ${}_k J_{b^-}^\mu f(x)$ of order $\mu > 0$ are given as*

$${}_k J_{a^+}^\mu f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (x - t)^{\frac{\mu}{k}-1} f(t) dt \quad (0 \leq a < x < b)$$

and

$${}_k J_{b^-}^\mu f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (t - x)^{\frac{\mu}{k}-1} f(t) dt \quad (0 \leq a < x < b),$$

respectively, where $k > 0$ and $\Gamma_k(\mu)$ is the k -gamma function defined by $\Gamma_k(\mu) = \int_0^\infty t^{\mu-1} \times e^{-\frac{t^k}{k}} dt$. Furthermore, $\Gamma_k(\mu + k) = \mu\Gamma_k(\mu)$ and ${}_k J_{a^+}^0 f(x) = {}_k J_{b^-}^0 f(x) = f(x)$.

The concept of Riemann–Liouville k -fractional integral is an important generalization of Riemann–Liouville fractional integrals. We would like to stress here that for $k \neq 1$

the properties of Riemann–Liouville k -fractional integrals are very dissimilar to those of classical Riemann–Liouville fractional integrals. Due to this, the Riemann–Liouville k -fractional integrals have aroused many researchers’ interest. Properties and estimations for the integral inequality related to this operator can be sought out in [22–27] and the references cited therein.

The main purpose of the current paper is to establish some new bounds on Hermite–Hadamard’s and Simpson’s inequalities for mappings whose absolute values of second derivatives are generalized (m, h) -preinvex. To do this, the authors derive a general k -fractional integral identity along with multi parameters for twice differentiable mappings. By using this integral identity, the authors derive some new inequalities of Simpson and Hermite–Hadamard type for these mappings.

To end this section, we restate some special functions and definitions as follows.

Let us consider the following special functions:

(1) The beta function:

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} = \int_0^1 t^{x-1}(1 - t)^{y-1} dt, \quad x, y > 0;$$

(2) The incomplete beta function:

$$\beta(a; x, y) = \int_0^a t^{x-1}(1 - t)^{y-1} dt, \quad 0 < a < 1, x, y > 0;$$

(3) The hypergeometric function:

$${}_2F_1(a, b; c; z) = \frac{1}{\beta(b, c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - zt)^{-a} dt, \quad c > b > 0, |z| < 1.$$

Definition 1.2 ([28]) A function $f : [0, \infty) \rightarrow \mathbb{R}$ is named s -convex in the second sense along with $s \in (0, 1]$ if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

holds for all $x, y \in [0, \infty)$ and $\alpha, \beta \geq 0$ along with $\alpha + \beta = 1$.

Definition 1.3 ([29]) A function $f : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called s -Godunova–Levin function of the second kind along with $s \in [0, 1]$ if

$$f(tx + (1 - t)y) \leq \frac{f(x)}{t^s} + \frac{f(y)}{(1 - t)^s}$$

holds for all $x, y \in A$ and $t \in (0, 1)$.

Definition 1.4 ([30]) A function $f : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is named tgs -convex on A if f is non-negative and

$$f(tx + (1 - t)y) \leq t(1 - t)[f(x) + f(y)]$$

holds for all $x, y \in A$ and $t \in (0, 1)$.

Definition 1.5 ([31]) A function $f : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called *MT*-convex if f is non-negative and

$$f(tx + (1 - t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(y)$$

holds for all $x, y \in A$ and $t \in (0, 1)$.

Definition 1.6 ([32]) A set $A \subseteq \mathbb{R}^n$ is called *m*-invex with respect to the mapping $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$ if $mx + \lambda\eta(y, x, m) \in A$ holds for all $x, y \in A$ and $\lambda \in [0, 1]$.

Definition 1.7 ([33]) Let $A \subseteq \mathbb{R}$ be an open *m*-invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}$, and let $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}_0$. A function $f : A \rightarrow \mathbb{R}$ is said to be generalized (m, h_1, h_2) -preinvex if

$$f(mx + t\eta(y, x, m)) \leq mh_1(t)f(x) + h_2(t)f(y) \tag{1.2}$$

is valid for all $x, y \in A$ and $t \in [0, 1]$. If inequality (1.2) reverses, then f is said to be generalized (m, h_1, h_2) -preincave on A .

Clearly, if we take $h_1(t) = h(1 - t)$, $h_2(t) = h(t)$ in Definition 1.7, then f becomes generalized (m, h) -preinvex functions as follows.

Definition 1.8 Let $A \subseteq \mathbb{R}$ be an open *m*-invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}$, and let $h : [0, 1] \rightarrow \mathbb{R}_0$. A function $f : A \rightarrow \mathbb{R}$ is called generalized (m, h) -preinvex if

$$f(mx + t\eta(y, x, m)) \leq h(t)f(y) + mh(1 - t)f(x) \tag{1.3}$$

is valid for all $x, y \in A$ and $t \in [0, 1]$.

Remark 1.1 Let us discuss some special cases of Definition 1.8 as follows:

- (i) choosing $h(t) = 1$, we obtain the definition of generalized (m, P) -preinvex functions;
- (ii) choosing $h(t) = t^s$ for $s \in (0, 1]$, we obtain the definition of generalized (m, s) -Breckner-preinvex functions;
- (iii) choosing $h(t) = t^{-s}$ for $s \in (0, 1)$, we obtain the definition of generalized (m, s) -Godunova–Levin–Dragomir-preinvex functions;
- (iv) choosing $h(t) = t(1 - t)$, we obtain the definition of generalized (m, tgs) -preinvex functions;
- (v) choosing $h(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, we obtain the definition of generalized *m*-*MT*-preinvex functions.

It is worth mentioning here that, as far as we know, all the special cases considered above are new in the literature.

2 Main results

In order to derive our main results, we need the subsequent identity.

Lemma 2.1 *Let $A \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R} \setminus \{0\}$ for some fixed $m \in (0, 1]$, and let $a, b \in A$, $a < b$ with $\eta(b, a, m) > 0$. Assume that $f : A \rightarrow \mathbb{R}$ is a twice differentiable function on A such that f'' is integrable on $[ma, ma + \eta(b, a, m)]$. Then the following identity for Riemann–Liouville k -fractional integrals along with $x \in [a, b]$, $\lambda \in [0, 1]$, $\mu > 0$, and $k > 0$ exists:*

$$\begin{aligned}
 &I_{f,\eta}(\mu, k; x, \lambda, m, a, b) \\
 &= \frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{(\frac{\mu}{k} + 1)\eta(b, a, m)} \int_0^1 t \left[\left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right] f''(ma + t\eta(x, a, m)) dt \\
 &\quad + \frac{(-1)^{\frac{\mu}{k}+2} \eta^{\frac{\mu}{k}+2}(x, b, m)}{(\frac{\mu}{k} + 1)\eta(b, a, m)} \int_0^1 t \left[\left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right] f''(mb + t\eta(x, b, m)) dt, \tag{2.1}
 \end{aligned}$$

where

$$\begin{aligned}
 &I_{f,\eta}(\mu, k; x, \lambda, m, a, b) \\
 &:= \frac{1 - \lambda}{\eta(b, a, m)} \left[\eta^{\frac{\mu}{k}}(x, a, m) f(ma + \eta(x, a, m)) \right. \\
 &\quad \left. + (-1)^{\frac{\mu}{k}} \eta^{\frac{\mu}{k}}(x, b, m) f(mb + \eta(x, b, m)) \right] \\
 &\quad + \frac{\lambda}{\eta(b, a, m)} \left[\eta^{\frac{\mu}{k}}(x, a, m) f(ma) + (-1)^{\frac{\mu}{k}} \eta^{\frac{\mu}{k}}(x, b, m) f(mb) \right] \\
 &\quad + \frac{\frac{1}{k} - \lambda}{\eta(b, a, m)} \left[(-1)^{\frac{\mu}{k}+1} \eta^{\frac{\mu}{k}+1}(x, b, m) f'(mb + \eta(x, b, m)) \right. \\
 &\quad \left. - \eta^{\frac{\mu}{k}+1}(x, a, m) f'(ma + \eta(x, a, m)) \right] \\
 &\quad - \frac{\Gamma_k(\mu + k)}{\eta(b, a, m)} \left[{}_k J_{(ma+\eta(x,a,m))^-}^\mu f(ma) + {}_k J_{(mb+\eta(x,b,m))^+}^\mu f(mb) \right]
 \end{aligned}$$

and Γ_k is the k -gamma function.

Proof By integration by parts and replacing the variable, we can state

$$\begin{aligned}
 &\int_0^1 t \left[\left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right] f''(ma + t\eta(x, a, m)) dt \\
 &= t \left[\left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right] \frac{f'(ma + t\eta(x, a, m))}{\eta(x, a, m)} \Big|_0^1 \\
 &\quad - \frac{\frac{\mu}{k} + 1}{\eta(x, a, m)} \int_0^1 (\lambda - t^{\frac{\mu}{k}}) f'(ma + t\eta(x, a, m)) dt \\
 &= \left[\left(\frac{\mu}{k} + 1 \right) \lambda - 1 \right] \frac{f'(ma + \eta(x, a, m))}{\eta(x, a, m)} \\
 &\quad - \frac{\frac{\mu}{k} + 1}{\eta(x, a, m)} \left[(\lambda - t^{\frac{\mu}{k}}) \frac{f(ma + t\eta(x, a, m))}{\eta(x, a, m)} \Big|_0^1 \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \int_0^1 \frac{\mu}{k} t^{\frac{\mu}{k}-1} \frac{f(ma + t\eta(x, a, m))}{\eta(x, a, m)} dt \Big] \\
 = & \left[\left(\frac{\mu}{k} + 1 \right) \lambda - 1 \right] \frac{f'(ma + \eta(x, a, m))}{\eta(x, a, m)} \\
 & + \frac{\frac{\mu}{k} + 1}{\eta^2(x, a, m)} \left[(1 - \lambda)f(ma + \eta(x, a, m)) + \lambda f(ma) \right. \\
 & \left. - \frac{\frac{\mu}{k}}{\eta^{\frac{\mu}{k}}(x, a, m)} \int_{ma}^{ma+\eta(x, a, m)} (u - ma)^{\frac{\mu}{k}-1} f(u) du \right] \\
 = & \left[\left(\frac{\mu}{k} + 1 \right) \lambda - 1 \right] \frac{f'(ma + \eta(x, a, m))}{\eta(x, a, m)} \\
 & + \frac{\frac{\mu}{k} + 1}{\eta^2(x, a, m)} \left[(1 - \lambda)f(ma + \eta(x, a, m)) + \lambda f(ma) \right. \\
 & \left. - \frac{\Gamma_k(\mu + k)}{\eta^{\frac{\mu}{k}}(x, a, m)} {}_k J_{(ma+\eta(x, a, m))}^\mu f(ma) \right]. \tag{2.2}
 \end{aligned}$$

Similarly, we get

$$\begin{aligned}
 & \int_0^1 t \left[\left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right] f''(mb + t\eta(x, b, m)) dt \\
 = & \left[\left(\frac{\mu}{k} + 1 \right) \lambda - 1 \right] \frac{f'(mb + \eta(x, b, m))}{\eta(x, b, m)} \\
 & + \frac{\frac{\mu}{k} + 1}{\eta^2(x, b, m)} \left[(1 - \lambda)f(mb + \eta(x, b, m)) + \lambda f(mb) \right. \\
 & \left. - \frac{\Gamma_k(\mu + k)}{\eta^{\frac{\mu}{k}}(x, b, m)} {}_k J_{(mb+\eta(x, b, m))}^\mu f(mb) \right]. \tag{2.3}
 \end{aligned}$$

Multiplying both sides of (2.2) and (2.3) by $\frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{(\frac{\mu}{k}+1)\eta(b, a, m)}$ and $\frac{(-1)^{\frac{\mu}{k}+2} \eta^{\frac{\mu}{k}+2}(x, b, m)}{(\frac{\mu}{k}+1)\eta(b, a, m)}$, respectively, and adding the resulting identities together, we obtain the desired result. \square

Remark 2.1 In Lemma 2.1, if we put $k = 1$ and $\eta(b, a, m) = b - ma$ along with $m = 1$, then we get the following identity:

$$\begin{aligned}
 & (1 - \lambda) \left[\frac{(x - a)^\mu + (b - x)^\mu}{b - a} \right] f(x) + \lambda \left[\frac{(x - a)^\mu f(a) + (b - x)^\mu f(b)}{b - a} \right] \\
 & + \left(\frac{1}{\mu + 1} - \lambda \right) \left[\frac{(b - x)^{\mu+1} - (x - a)^{\mu+1}}{b - a} \right] f'(x) - \frac{\Gamma(\mu + 1)}{b - a} [J_x^\mu f(a) + J_{x^+}^\mu f(b)] \\
 = & \frac{(x - a)^{\mu+2}}{(\mu + 1)(b - a)} \int_0^1 t [(\mu + 1)\lambda - t^\mu] f''(tx + (1 - t)a) dt \\
 & + \frac{(b - x)^{\mu+2}}{(\mu + 1)(b - a)} \int_0^1 t [(\mu + 1)\lambda - t^\mu] f''(tx + (1 - t)b) dt,
 \end{aligned}$$

which is proved by İşcan in [34]. Further, if we put $\mu = 1, \lambda = \frac{1}{2}$, and $x = a$ or $x = b$, then the above identity recaptures Lemma 1 in [35].

Using Lemma 2.1, we now state the following theorem.

Theorem 2.1 *Let $A \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R} \setminus \{0\}$ for some fixed $m \in (0, 1]$, and let $a, b \in A, a < b$ with $\eta(b, a, m) > 0$. Assume that $f : A \rightarrow \mathbb{R}$ is a twice differentiable function on A such that f'' is integrable on $[ma, ma + \eta(b, a, m)]$. If $|f''|^q$ for $q \geq 1$ is a generalized (m, h) -preinvex function with respect to η and $h : [0, 1] \rightarrow \mathbb{R}_0$, then the following inequality for k -fractional integrals with $x \in [a, b], \lambda \in [0, 1], \mu > 0, k > 0$ exists:*

$$\begin{aligned}
 & |I_{f,\eta}(\mu, k; x, \lambda, m, a, b)| \\
 & \leq C_0^{1-\frac{1}{q}}(k, \mu, \lambda) \left\{ \left| \frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{(\frac{\mu}{k} + 1)\eta(b, a, m)} \right| [|f''(x)|^q C_1(k, \mu, \lambda; h) \right. \\
 & \quad + m |f''(a)|^q C_2(k, \mu, \lambda; h)]^{\frac{1}{q}} \\
 & \quad + \left| \frac{(-1)^{\frac{\mu}{k}+2} \eta^{\frac{\mu}{k}+2}(x, b, m)}{(\frac{\mu}{k} + 1)\eta(b, a, m)} \right| [|f''(x)|^q C_1(k, \mu, \lambda; h) \\
 & \quad + m |f''(b)|^q C_2(k, \mu, \lambda; h)]^{\frac{1}{q}} \Big\}, \tag{2.4}
 \end{aligned}$$

where

$$\begin{aligned}
 C_0(k, \mu, \lambda) &= \int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right| dt \\
 &= \begin{cases} \frac{\frac{\mu}{k} [(\frac{\mu}{k}+1)\lambda]^{\frac{1+2k}{\mu}}}{(\frac{\mu}{k}+2)} - \frac{(\frac{\mu}{k}+1)\lambda}{2} + \frac{1}{\frac{\mu}{k}+2}, & 0 \leq \lambda \leq \frac{1}{\frac{\mu}{k}+1}, \\ \frac{(\frac{\mu}{k}+1)\lambda}{2} - \frac{1}{\frac{\mu}{k}+2}, & \frac{1}{\frac{\mu}{k}+1} < \lambda \leq 1, \end{cases} \tag{2.5}
 \end{aligned}$$

$$C_1(k, \mu, \lambda; h) = \int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right| h(t) dt, \quad 0 \leq \lambda \leq 1, \tag{2.6}$$

and

$$C_2(k, \mu, \lambda; h) = \int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right| h(1-t) dt, \quad 0 \leq \lambda \leq 1. \tag{2.7}$$

Proof Applying Lemma 2.1 and the power mean inequality, we have

$$\begin{aligned}
 & |I_{f,\eta}(\mu, k; x, \lambda, m, a, b)| \\
 & \leq \left| \frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{(\frac{\mu}{k} + 1)\eta(b, a, m)} \right| \int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right| |f''(ma + t\eta(x, a, m))| dt \\
 & \quad + \left| \frac{(-1)^{\frac{\mu}{k}+2} \eta^{\frac{\mu}{k}+2}(x, b, m)}{(\frac{\mu}{k} + 1)\eta(b, a, m)} \right| \int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right| |f''(mb + t\eta(x, b, m))| dt \\
 & \leq \left| \frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{(\frac{\mu}{k} + 1)\eta(b, a, m)} \right| \left(\int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right| dt \right)^{1-\frac{1}{q}} \\
 & \quad \times \left(\int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right| |f''(ma + t\eta(x, a, m))|^q dt \right)^{\frac{1}{q}} \\
 & \quad + \left| \frac{(-1)^{\frac{\mu}{k}+2} \eta^{\frac{\mu}{k}+2}(x, b, m)}{(\frac{\mu}{k} + 1)\eta(b, a, m)} \right| \left(\int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right| dt \right)^{1-\frac{1}{q}}
 \end{aligned}$$

$$\begin{aligned}
 & \times \left(\int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right| |f''(mb + t\eta(x, b, m))|^q dt \right)^{\frac{1}{q}} \\
 & = C_0^{1-\frac{1}{q}}(k, \mu, \lambda) \\
 & \times \left\{ \left| \frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{\left(\frac{\mu}{k} + 1 \right) \eta(b, a, m)} \right| \left(\int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right| |f''(ma + t\eta(x, a, m))|^q dt \right)^{\frac{1}{q}} \right. \\
 & + \left| \frac{(-1)^{\frac{\mu}{k}+2} \eta^{\frac{\mu}{k}+2}(x, b, m)}{\left(\frac{\mu}{k} + 1 \right) \eta(b, a, m)} \right| \left(\int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda \right. \right. \\
 & \left. \left. - t^{\frac{\mu}{k}} \right| |f''(mb + t\eta(x, b, m))|^q dt \right)^{\frac{1}{q}} \left. \right\}. \tag{2.8}
 \end{aligned}$$

Since $|f''|^q$ is generalized (m, h) -preinvex on $[ma, ma + \eta(b, a, m)]$, we get

$$\begin{aligned}
 & \int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right| |f''(ma + t\eta(x, a, m))|^q dt \\
 & \leq \int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right| [h(t)|f''(x)|^q + mh(1-t)|f''(a)|^q] dt \\
 & = |f''(x)|^q C_1(k, \mu, \lambda; h) + m|f''(a)|^q C_2(k, \mu, \lambda; h) \tag{2.9}
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right| |f''(mb + t\eta(x, b, m))|^q dt \\
 & \leq \int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right| [h(t)|f''(x)|^q + mh(1-t)|f''(b)|^q] dt \\
 & = |f''(x)|^q C_1(k, \mu, \lambda; h) + m|f''(b)|^q C_2(k, \mu, \lambda; h), \tag{2.10}
 \end{aligned}$$

where $C_0(k, \mu, \lambda)$, $C_1(k, \mu, \lambda; h)$, and $C_2(k, \mu, \lambda; h)$ are defined by (2.5)–(2.7), respectively. Hence, if we use (2.9) and (2.10) in (2.8), we can get the desired result. This completes the proof. \square

Let us point out some special cases of Theorem 2.1.

I. If $h(t) = t^s$ in Theorem 2.1, then we have the following results.

Corollary 2.1 *In Theorem 2.1, if $|f''|^q$ for $q \geq 1$ is generalized (m, s) -Breckner-preinvex functions, then, for $s \in (0, 1]$ and $m \in (0, 1]$, we have*

$$\begin{aligned}
 & |I_{f,\eta}(\mu, k; x, \lambda, m, a, b)| \\
 & \leq C_0^{1-\frac{1}{q}}(k, \mu, \lambda) \\
 & \times \left\{ \left| \frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{\left(\frac{\mu}{k} + 1 \right) \eta(b, a, m)} \right| \left[|f''(x)|^q T_1(k, \mu, \lambda; s) + m|f''(a)|^q T_2(k, \mu, \lambda; s) \right]^{\frac{1}{q}} \right. \\
 & + \left| \frac{(-1)^{\frac{\mu}{k}+2} \eta^{\frac{\mu}{k}+2}(x, b, m)}{\left(\frac{\mu}{k} + 1 \right) \eta(b, a, m)} \right| \left[|f''(x)|^q T_1(k, \mu, \lambda; s) + m|f''(b)|^q T_2(k, \mu, \lambda; s) \right]^{\frac{1}{q}} \left. \right\},
 \end{aligned}$$

where we use the fact that

$$T_1(k, \mu, \lambda; s) = \begin{cases} \frac{2^{\frac{\mu}{k}} [(\frac{\mu}{k} + 1)\lambda]^{1 + \frac{k(s+2)}{\mu}}}{(s+2)(\frac{\mu}{k} + s+2)} - \frac{(\frac{\mu}{k} + 1)\lambda}{s+2} + \frac{1}{\frac{\mu}{k} + s+2}, & 0 \leq \lambda \leq \frac{1}{\frac{\mu}{k} + 1}, \\ \frac{(\frac{\mu}{k} + 1)\lambda}{s+2} - \frac{1}{\frac{\mu}{k} + s+2}, & \frac{1}{\frac{\mu}{k} + 1} < \lambda \leq 1, \end{cases}$$

$$T_2(k, \mu, \lambda; s) = \begin{cases} \left[\begin{aligned} &\beta(\frac{\mu}{k} + 2, s + 1) - (\frac{\mu}{k} + 1)\lambda\beta(2, s + 1) \\ &+ 2(\frac{\mu}{k} + 1)\lambda\beta([\frac{\mu}{k} + 1]\lambda^{\frac{k}{\mu}}; 2, s + 1) \\ &- 2\beta([\frac{\mu}{k} + 1]\lambda^{\frac{k}{\mu}}; \frac{\mu}{k} + 2, s + 1) \end{aligned} \right], & 0 \leq \lambda \leq \frac{1}{\frac{\mu}{k} + 1}, \\ \beta(\frac{\mu}{k} + 1)\lambda\beta(2, s + 1) - \beta(\frac{\mu}{k} + 2, s + 1), & \frac{1}{\frac{\mu}{k} + 1} < \lambda \leq 1, \end{cases}$$

and $C_0(k, \mu, \lambda)$ is defined by (2.5).

Corollary 2.2 *In Theorem 2.1, if the mapping $\eta(b, a, m) = b - ma$ along with $m = 1$, taking $x = \frac{a+b}{2}$, then for $s \in (0, 1]$, we have the following inequality for s -convex functions:*

$$\begin{aligned} &\left| \frac{2^{\frac{\mu}{k}-1}}{(b-a)^{\frac{\mu}{k}-1}} I_f \left(\mu, k; \frac{a+b}{2}, \lambda, 1, a, b \right) \right| \\ &= \left| (1-\lambda)f\left(\frac{a+b}{2}\right) + \lambda \left[\frac{f(a)+f(b)}{2} \right] \right. \\ &\quad \left. - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{(\frac{a+b}{2})^-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})^+}^{\mu} f(b) \right] \right| \\ &\leq \frac{(b-a)^2}{8(\frac{\mu}{k} + 1)} C_0^{1-\frac{1}{q}}(k, \mu, \lambda) \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q T_1(k, \mu, \lambda; s) + |f''(a)|^q T_2(k, \mu, \lambda; s) \right]^{\frac{1}{q}} \right. \\ &\quad \left. + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q T_1(k, \mu, \lambda; s) + |f''(b)|^q T_2(k, \mu, \lambda; s) \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Remark 2.2 In Corollary 2.2,

(i) taking $\lambda = \frac{1}{2}$, we obtain

$$\begin{aligned} &\left| \frac{2^{\frac{\mu}{k}-1}}{(b-a)^{\frac{\mu}{k}-1}} I_f \left(\mu, k; \frac{a+b}{2}, \frac{1}{2}, 1, a, b \right) \right| \\ &= \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{(\frac{a+b}{2})^-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})^+}^{\mu} f(b) \right] \right| \\ &\leq \frac{(b-a)^2}{8(\frac{\mu}{k} + 1)} C_0^{1-\frac{1}{q}} \left(k, \mu, \frac{1}{2} \right) \\ &\quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q T_1\left(k, \mu, \frac{1}{2}; s\right) + |f''(a)|^q T_2\left(k, \mu, \frac{1}{2}; s\right) \right]^{\frac{1}{q}} \right. \\ &\quad \left. + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q T_1\left(k, \mu, \frac{1}{2}; s\right) + |f''(b)|^q T_2\left(k, \mu, \frac{1}{2}; s\right) \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(ii) taking $\lambda = \frac{1}{3}$, we obtain

$$\begin{aligned} & \left| \frac{2^{\frac{\mu}{k}-1}}{(b-a)^{\frac{\mu}{k}-1}} I_f \left(\mu, k; \frac{a+b}{2}, \frac{1}{3}, 1, a, b \right) \right| \\ &= \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right. \\ &\quad \left. - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{\left(\frac{a+b}{2}\right)^-}^{\mu} f(a) + {}_k J_{\left(\frac{a+b}{2}\right)^+}^{\mu} f(b) \right] \right| \\ &\leq \frac{(b-a)^2}{8\left(\frac{\mu}{k}+1\right)} C_0^{1-\frac{1}{q}} \left(k, \mu, \frac{1}{3} \right) \\ &\quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q T_1\left(k, \mu, \frac{1}{3}; s\right) + |f''(a)|^q T_2\left(k, \mu, \frac{1}{3}; s\right) \right]^{\frac{1}{q}} \right. \\ &\quad \left. + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q T_1\left(k, \mu, \frac{1}{3}; s\right) + |f''(b)|^q T_2\left(k, \mu, \frac{1}{3}; s\right) \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = \mu = s = 1$, then we obtain

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ &\leq \frac{(b-a)^2}{162} \left\{ \left[\frac{59}{96} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{37}{96} |f''(a)|^q \right]^{\frac{1}{q}} \right. \\ &\quad \left. + \left[\frac{59}{96} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{37}{96} |f''(b)|^q \right]^{\frac{1}{q}} \right\} \\ &\leq \frac{(b-a)^2}{162} \left\{ \left(\frac{59|f''(a)|^q + 133|f''(b)|^q}{192} \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\frac{59|f''(b)|^q + 133|f''(a)|^q}{192} \right)^{\frac{1}{q}} \right\}. \tag{2.11} \end{aligned}$$

It is noted that the result of the first inequality in (2.11) is proved by İşcan in [34], which is better than the result presented by Sarikaya et al. in [36, Theorem 6].

Remark 2.3 In Corollary 2.2, if we take $k = 1$ and $\lambda = 0, 1$, we have the results (f) and (h) in [34, Corollary 2.3], respectively. Further, if we take $\mu = 1$, we have the results (g) and (i) in [34, Corollary 2.3], respectively.

II. If $h(t) = t^{-s}$ in Theorem 2.1, then we have the following results.

Corollary 2.3 *In Theorem 2.1, if $|f''|^q$ for $q \geq 1$ is generalized (m, s) -Godunova–Levin–Dragomir-preinvex functions, then, for $s \in (0, 1)$ and $m \in (0, 1]$, we have*

$$\begin{aligned} & |I_{f,\eta}(\mu, k; x, \lambda, m, a, b)| \\ &\leq C_0^{1-\frac{1}{q}}(k, \mu, \lambda) \end{aligned}$$

$$\begin{aligned} & \times \left\{ \left| \frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{(\frac{\mu}{k} + 1)\eta(b, a, m)} \right| \left[|f''(x)|^q U_1(k, \mu, \lambda; s) + m |f''(a)|^q U_2(k, \mu, \lambda; s) \right]^{\frac{1}{q}} \right. \\ & \left. + \left| \frac{(-1)^{\frac{\mu}{k}+2} \eta^{\frac{\mu}{k}+2}(x, b, m)}{(\frac{\mu}{k} + 1)\eta(b, a, m)} \right| \left[|f''(x)|^q U_1(k, \mu, \lambda; s) + m |f''(b)|^q U_2(k, \mu, \lambda; s) \right]^{\frac{1}{q}} \right\}, \end{aligned}$$

where we use the fact that

$$\begin{aligned} U_1(k, \mu, \lambda; s) &= \begin{cases} \frac{2\frac{\mu}{k}[(\frac{\mu}{k}+1)\lambda]^{1+\frac{k(2-s)}{\mu}}}{(2-s)(\frac{\mu}{k}-s+2)} - \frac{(\frac{\mu}{k}+1)\lambda}{2-s} + \frac{1}{\frac{\mu}{k}-s+2}, & 0 \leq \lambda \leq \frac{1}{\frac{\mu}{k}+1}, \\ \frac{(\frac{\mu}{k}+1)\lambda}{2-s} - \frac{1}{\frac{\mu}{k}-s+2}, & \frac{1}{\frac{\mu}{k}+1} < \lambda \leq 1, \end{cases} \\ U_2(k, \mu, \lambda; s) &= \begin{cases} \left[\begin{aligned} & \beta(\frac{\mu}{k} + 2, 1 - s) - (\frac{\mu}{k} + 1)\lambda\beta(2, 1 - s) \\ & + 2(\frac{\mu}{k} + 1)\lambda\beta([\frac{\mu}{k} + 1]\lambda^{\frac{k}{\mu}}; 2, 1 - s) \\ & - 2\beta([\frac{\mu}{k} + 1]\lambda^{\frac{k}{\mu}}; \frac{\mu}{k} + 2, 1 - s) \end{aligned} \right], & 0 \leq \lambda \leq \frac{1}{\frac{\mu}{k}+1}, \\ \left[\begin{aligned} & (\frac{\mu}{k} + 1)\lambda\beta(2, 1 - s) - \beta(\frac{\mu}{k} + 2, 1 - s), \end{aligned} \right], & \frac{1}{\frac{\mu}{k}+1} < \lambda \leq 1, \end{cases} \end{aligned}$$

and $C_0(k, \mu, \lambda)$ is defined by (2.5).

Corollary 2.4 *In Theorem 2.1, if the mapping $\eta(b, a, m) = b - ma$ together with $m = 1$, taking $x = \frac{a+b}{2}$, for $s \in (0, 1)$, we have the following inequality for s -Godunova–Levin functions:*

$$\begin{aligned} & \left| (1 - \lambda)f\left(\frac{a + b}{2}\right) + \lambda \left[\frac{f(a) + f(b)}{2} \right] \right. \\ & \quad \left. - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu + k)}{(b - a)^{\frac{\mu}{k}}} \left[{}_k J_{(\frac{a+b}{2})-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b - a)^2}{8(\frac{\mu}{k} + 1)} C_0^{1-\frac{1}{q}}(k, \mu, \lambda) \left\{ \left[\left| f''\left(\frac{a + b}{2}\right) \right|^q U_1(k, \mu, \lambda; s) + |f''(a)|^q U_2(k, \mu, \lambda; s) \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\left| f''\left(\frac{a + b}{2}\right) \right|^q U_1(k, \mu, \lambda; s) + |f''(b)|^q U_2(k, \mu, \lambda; s) \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Remark 2.4 In Corollary 2.4,

(a) if $\lambda = \frac{1}{3}$, then we obtain

$$\begin{aligned} & \left| \frac{2^{\frac{\mu}{k}-1}}{(b - a)^{\frac{\mu}{k}-1}} I_f\left(\mu, k; \frac{a + b}{2}, \frac{1}{3}, 1, a, b\right) \right| \\ & = \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a + b}{2}\right) + f(b) \right] - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu + k)}{(b - a)^{\frac{\mu}{k}}} \left[{}_k J_{(\frac{a+b}{2})-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b - a)^2}{8(\frac{\mu}{k} + 1)} C_0^{1-\frac{1}{q}}\left(k, \mu, \frac{1}{3}\right) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a + b}{2}\right) \right|^q U_1\left(k, \mu, \frac{1}{3}; s\right) + |f''(a)|^q U_2\left(k, \mu, \frac{1}{3}; s\right) \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\left| f''\left(\frac{a + b}{2}\right) \right|^q U_1\left(k, \mu, \frac{1}{3}; s\right) + |f''(b)|^q U_2\left(k, \mu, \frac{1}{3}; s\right) \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we obtain a Simpson-type inequality:

$$\begin{aligned} & \left| I_f \left(1, 1; \frac{a+b}{2}, \frac{1}{3}, 1, a, b \right) \right| \\ &= \left| \frac{1}{6} \left[f(a) + 4f \left(\frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ &\leq \frac{(b-a)^2}{16} \left(\frac{8}{81} \right)^{1-\frac{1}{q}} \left\{ \left[\left| f'' \left(\frac{a+b}{2} \right) \right|^q U_1 \left(1, 1, \frac{1}{3}; s \right) \right. \right. \\ &\quad \left. \left. + |f''(a)|^q U_2 \left(1, 1, \frac{1}{3}; s \right) \right]^{\frac{1}{q}} \right. \\ &\quad \left. + \left[\left| f'' \left(\frac{a+b}{2} \right) \right|^q U_1 \left(1, 1, \frac{1}{3}; s \right) + |f''(b)|^q U_2 \left(1, 1, \frac{1}{3}; s \right) \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(b) if $\lambda = \frac{1}{2}$, then we obtain

$$\begin{aligned} & \left| \frac{2^{\frac{\mu}{k}-1}}{(b-a)^{\frac{\mu}{k}-1}} I_f \left(\mu, k; \frac{a+b}{2}, \frac{1}{2}, 1, a, b \right) \right| \\ &= \left| \frac{1}{4} \left[f(a) + 2f \left(\frac{a+b}{2} \right) + f(b) \right] - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{\left(\frac{a+b}{2}\right)-}^{\mu} f(a) + {}_k J_{\left(\frac{a+b}{2}\right)+}^{\mu} f(b) \right] \right| \\ &\leq \frac{(b-a)^2}{8 \left(\frac{\mu}{k} + 1 \right)} C_0^{1-\frac{1}{q}} \left(k, \mu, \frac{1}{2} \right) \\ &\quad \times \left\{ \left[\left| f'' \left(\frac{a+b}{2} \right) \right|^q U_1 \left(k, \mu, \frac{1}{2}; s \right) + |f''(a)|^q U_2 \left(k, \mu, \frac{1}{2}; s \right) \right]^{\frac{1}{q}} \right. \\ &\quad \left. + \left[\left| f'' \left(\frac{a+b}{2} \right) \right|^q U_1 \left(k, \mu, \frac{1}{2}; s \right) + |f''(b)|^q U_2 \left(k, \mu, \frac{1}{2}; s \right) \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we obtain an averaged midpoint-trapezoid-type inequality:

$$\begin{aligned} & \left| I_f \left(1, 1; \frac{a+b}{2}, \frac{1}{2}, 1, a, b \right) \right| \\ &= \left| \frac{1}{4} \left[f(a) + 2f \left(\frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ &\leq \frac{(b-a)^2}{16} \left(\frac{1}{6} \right)^{1-\frac{1}{q}} \left\{ \left[\left| f'' \left(\frac{a+b}{2} \right) \right|^q U_1 \left(1, 1, \frac{1}{2}; s \right) + |f''(a)|^q U_2 \left(1, 1, \frac{1}{2}; s \right) \right]^{\frac{1}{q}} \right. \\ &\quad \left. + \left[\left| f'' \left(\frac{a+b}{2} \right) \right|^q U_1 \left(1, 1, \frac{1}{2}; s \right) + |f''(b)|^q U_2 \left(1, 1, \frac{1}{2}; s \right) \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(c) if $\lambda = 0$, then we obtain

$$\begin{aligned} & \left| \frac{2^{\frac{\mu}{k}-1}}{(b-a)^{\frac{\mu}{k}-1}} I_f \left(\mu, k; \frac{a+b}{2}, 0, 1, a, b \right) \right| \\ &= \left| f \left(\frac{a+b}{2} \right) - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{\left(\frac{a+b}{2}\right)-}^{\mu} f(a) + {}_k J_{\left(\frac{a+b}{2}\right)+}^{\mu} f(b) \right] \right| \end{aligned}$$

$$\begin{aligned} &\leq \frac{(b-a)^2}{8\left(\frac{\mu}{k}+1\right)} \left(\frac{1}{\frac{\mu}{k}+2}\right)^{1-\frac{1}{q}} \left\{ \left[\frac{|f''(\frac{a+b}{2})|^q}{\frac{\mu}{k}-s+2} + |f''(a)|^q \beta\left(\frac{\mu}{k}+2, 1-s\right) \right]^{\frac{1}{q}} \right. \\ &\quad \left. + \left[\frac{|f''(\frac{a+b}{2})|^q}{\frac{\mu}{k}-s+2} + |f''(b)|^q \beta\left(\frac{\mu}{k}+2, 1-s\right) \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we obtain a midpoint-type inequality:

$$\begin{aligned} &\left| I_f\left(1, 1; \frac{a+b}{2}, 0, 1, a, b\right) \right| \\ &= \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ &\leq \frac{(b-a)^2}{16} \left(\frac{1}{3}\right)^{1-\frac{1}{q}} \left\{ \left[\frac{|f''(\frac{a+b}{2})|^q}{3-s} + |f''(a)|^q \beta(3, 1-s) \right]^{\frac{1}{q}} \right. \\ &\quad \left. + \left[\frac{|f''(\frac{a+b}{2})|^q}{3-s} + |f''(b)|^q \beta(3, 1-s) \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(d) if $\lambda = 1$, then we obtain

$$\begin{aligned} &\left| \frac{2^{\frac{\mu}{k}-1}}{(b-a)^{\frac{\mu}{k}-1}} I_f\left(\mu, k; \frac{a+b}{2}, 1, 1, a, b\right) \right| \\ &= \left| \frac{f(a)+f(b)}{2} - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{(\frac{a+b}{2})-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})+}^{\mu} f(b) \right] \right| \\ &\leq \frac{(b-a)^2}{8\left(\frac{\mu}{k}+1\right)} \left[\frac{\mu}{k} \left(\frac{\mu}{k} + 3 \right) \right]^{1-\frac{1}{q}} \\ &\quad \times \left\{ \left[\frac{\frac{\mu}{k}(\frac{\mu}{k}-s+3)|f''(\frac{a+b}{2})|^q}{(2-s)(\frac{\mu}{k}-s+2)} \right. \right. \\ &\quad \left. \left. + |f''(a)|^q \left(\left(\frac{\mu}{k} + 1 \right) \beta(2, 1-s) - \beta\left(\frac{\mu}{k}+2, 1-s\right) \right) \right]^{\frac{1}{q}} \right. \\ &\quad \left. + \left[\frac{\frac{\mu}{k}(\frac{\mu}{k}-s+3)|f''(\frac{a+b}{2})|^q}{(2-s)(\frac{\mu}{k}-s+2)} \right. \right. \\ &\quad \left. \left. + |f''(b)|^q \left(\left(\frac{\mu}{k} + 1 \right) \beta(2, 1-s) - \beta\left(\frac{\mu}{k}+2, 1-s\right) \right) \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we obtain a trapezoid-type inequality:

$$\begin{aligned} &\left| I_f\left(1, 1; \frac{a+b}{2}, 1, 1, a, b\right) \right| \\ &= \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ &\leq \frac{(b-a)^2}{16} \left(\frac{2}{3}\right)^{1-\frac{1}{q}} \left\{ \left[\frac{(4-s)|f''(\frac{a+b}{2})|^q}{(2-s)(3-s)} + |f''(a)|^q (2\beta(2, 1-s) - \beta(3, 1-s)) \right]^{\frac{1}{q}} \right. \\ &\quad \left. + \left[\frac{(4-s)|f''(\frac{a+b}{2})|^q}{(2-s)(3-s)} + |f''(b)|^q (2\beta(2, 1-s) - \beta(3, 1-s)) \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

III. If $h(t) = t(1 - t)$ in Theorem 2.1, then we have the following results.

Corollary 2.5 *In Theorem 2.1, if $|f''|^q$ for $q \geq 1$ is generalized (m, tgs) -preinvex functions, then, for $m \in (0, 1]$, we have*

$$\begin{aligned} & |I_{f,\eta}(\mu, k; x, \lambda, m, a, b)| \\ & \leq C_0^{1-\frac{1}{q}}(k, \mu, \lambda) \theta^{\frac{1}{q}}(k, \mu, \lambda) \left\{ \left| \frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{(\frac{\mu}{k} + 1)\eta(b, a, m)} \right| \left[|f''(x)|^q + m|f''(a)|^q \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left| \frac{(-1)^{\frac{\mu}{k}+2} \eta^{\frac{\mu}{k}+2}(x, b, m)}{(\frac{\mu}{k} + 1)\eta(b, a, m)} \right| \left[|f''(x)|^q + m|f''(b)|^q \right]^{\frac{1}{q}} \right\}, \end{aligned}$$

where we use the fact that

$$\theta(k, \mu, \lambda) = \begin{cases} \frac{2\mu}{k} \frac{[(\frac{\mu}{k}+1)\lambda]^{\frac{3k}{\mu}+1}}{3(\frac{\mu}{k}+3)} - \frac{\mu}{k} \frac{[(\frac{\mu}{k}+1)\lambda]^{\frac{4k}{\mu}+1}}{2(\frac{\mu}{k}+4)} - \frac{(\frac{\mu}{k}+1)\lambda}{12} + \frac{1}{(\frac{\mu}{k}+3)(\frac{\mu}{k}+4)}, & 0 \leq \lambda \leq \frac{1}{\frac{\mu}{k}+1}, \\ \frac{(\frac{\mu}{k}+1)\lambda}{12} - \frac{1}{(\frac{\mu}{k}+3)(\frac{\mu}{k}+4)}, & \frac{1}{\frac{\mu}{k}+1} < \lambda \leq 1 \end{cases}$$

and $C_0(k, \mu, \lambda)$ is defined by (2.5).

Corollary 2.6 *In Theorem 2.1, if the mapping $\eta(b, a, m) = b - ma$ along with $m = 1$, taking $x = \frac{a+b}{2}$, we get the following inequality for tgs-convex functions:*

$$\begin{aligned} & \left| (1 - \lambda)f\left(\frac{a+b}{2}\right) + \lambda \left[\frac{f(a)+f(b)}{2} \right] \right. \\ & \quad \left. - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{(\frac{a+b}{2})-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)} C_0^{1-\frac{1}{q}}(k, \mu, \lambda) \theta^{\frac{1}{q}}(k, \mu, \lambda) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Remark 2.5 In Corollary 2.6,

(a) if $\lambda = \frac{1}{3}$, then we obtain

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{(\frac{a+b}{2})-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)} C_0^{1-\frac{1}{q}}\left(k, \mu, \frac{1}{3}\right) \theta^{\frac{1}{q}}\left(k, \mu, \frac{1}{3}\right) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we obtain a Simpson-type inequality:

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\begin{aligned} &\leq \frac{(b-a)^2}{162} \left(\frac{23}{160}\right)^{\frac{1}{q}} \\ &\quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(b) if $\lambda = \frac{1}{2}$, then we obtain

$$\begin{aligned} &\left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{\left(\frac{a+b}{2}\right)^-}^{\mu} f(a) + {}_k J_{\left(\frac{a+b}{2}\right)^+}^{\mu} f(b) \right] \right| \\ &\leq \frac{(b-a)^2}{8\left(\frac{\mu}{k}+1\right)} C_0^{1-\frac{1}{q}} \left(k, \mu, \frac{1}{2}\right) \theta^{\frac{1}{q}} \left(k, \mu, \frac{1}{2}\right) \\ &\quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we derive an averaged midpoint-trapezoid-type inequality:

$$\begin{aligned} &\left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ &\leq \frac{(b-a)^2}{96} \left(\frac{1}{5}\right)^{\frac{1}{q}} \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} \right. \\ &\quad \left. + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(c) if $\lambda = 0$, then we obtain

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{\left(\frac{a+b}{2}\right)^-}^{\mu} f(a) + {}_k J_{\left(\frac{a+b}{2}\right)^+}^{\mu} f(b) \right] \right| \\ &\leq \frac{(b-a)^2}{8\left(\frac{\mu}{k}+1\right)} \left(\frac{1}{\frac{\mu}{k}+2}\right)^{1-\frac{1}{q}} \left[\frac{1}{\left(\frac{\mu}{k}+3\right)\left(\frac{\mu}{k}+4\right)} \right]^{\frac{1}{q}} \\ &\quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we derive a midpoint-type inequality:

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ &\leq \frac{(b-a)^2}{48} \left(\frac{3}{20}\right)^{\frac{1}{q}} \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} \right. \\ &\quad \left. + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(d) if $\lambda = 1$, then we obtain

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} [{}_k J_{(\frac{a+b}{2})}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})}^{\mu} f(b)] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)} \left[\frac{\mu}{2(\frac{\mu}{k}+2)} \right]^{1-\frac{1}{q}} \left[\frac{\mu}{12(\frac{\mu}{k}+3)(\frac{\mu}{k}+4)} \right]^{\frac{1}{q}} \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we obtain a trapezoid-type inequality:

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{24} \left(\frac{7}{40} \right)^{\frac{1}{q}} \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

IV. If $h(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$ in Theorem 2.1, then we have the following results.

Corollary 2.7 *In Theorem 2.1, if $|f''|^q$ for $q \geq 1$ is generalized m -MT-preinvex functions, then, for $m \in (0, 1]$, we have*

$$\begin{aligned} & |I_{f,\eta}(\mu, k; x, \lambda, m, a, b)| \\ & \leq C_0^{1-\frac{1}{q}}(k, \mu, \lambda) \left\{ \left| \frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{(\frac{\mu}{k}+1)\eta(b, a, m)} \right| \left[|f''(x)|^q \Phi_1(k, \mu, \lambda) + m|f''(a)|^q \Phi_2(k, \mu, \lambda) \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left| \frac{(-1)^{\frac{\mu}{k}+2}\eta^{\frac{\mu}{k}+2}(x, b, m)}{(\frac{\mu}{k}+1)\eta(b, a, m)} \right| \left[|f''(x)|^q \Phi_1(k, \mu, \lambda) + m|f''(b)|^q \Phi_2(k, \mu, \lambda) \right]^{\frac{1}{q}} \right\}, \end{aligned}$$

where we use the fact that

$$\begin{aligned} & \Phi_1(k, \mu, \lambda) \\ & = \begin{cases} \left[\frac{1}{2}\beta\left(\frac{\mu}{k} + \frac{5}{2}, \frac{1}{2}\right) - \frac{3\lambda\pi(\frac{\mu}{k}+1)}{16} + \left(\frac{\mu}{k} + 1\right)\lambda\beta\left(\left[\left(\frac{\mu}{k} + 1\right)\lambda\right]^{\frac{k}{\mu}}; \frac{5}{2}, \frac{1}{2}\right) \right. \\ \quad \left. - \beta\left(\left[\left(\frac{\mu}{k} + 1\right)\lambda\right]^{\frac{k}{\mu}}; \frac{\mu}{k} + \frac{5}{2}, \frac{1}{2}\right) \right], & 0 \leq \lambda \leq \frac{1}{\frac{\mu}{k}+1}, \\ \left[\frac{3\lambda\pi(\frac{\mu}{k}+1)}{16} - \frac{1}{2}\beta\left(\frac{\mu}{k} + \frac{5}{2}, \frac{1}{2}\right), \right. & \frac{1}{\frac{\mu}{k}+1} < \lambda \leq 1, \end{cases} \\ & \Phi_2(k, \mu, \lambda) \\ & = \begin{cases} \left[\frac{1}{2}\beta\left(\frac{\mu}{k} + \frac{3}{2}, \frac{3}{2}\right) - \frac{\lambda\pi(\frac{\mu}{k}+1)}{16} + \left(\frac{\mu}{k} + 1\right)\lambda\beta\left(\left[\left(\frac{\mu}{k} + 1\right)\lambda\right]^{\frac{k}{\mu}}; \frac{3}{2}, \frac{3}{2}\right) \right. \\ \quad \left. - \beta\left(\left[\left(\frac{\mu}{k} + 1\right)\lambda\right]^{\frac{k}{\mu}}; \frac{\mu}{k} + \frac{3}{2}, \frac{3}{2}\right) \right], & 0 \leq \lambda \leq \frac{1}{\frac{\mu}{k}+1}, \\ \left[\frac{\lambda\pi(\frac{\mu}{k}+1)}{16} - \frac{1}{2}\beta\left(\frac{\mu}{k} + \frac{3}{2}, \frac{3}{2}\right), \right. & \frac{1}{\frac{\mu}{k}+1} < \lambda \leq 1, \end{cases} \end{aligned}$$

and $C_0(k, \mu, \lambda)$ is defined by (2.5).

Corollary 2.8 *In Theorem 2.1, if the mapping $\eta(b, a, m) = b - ma$ together with $m = 1$, taking $x = \frac{a+b}{2}$, we get the following inequality for MT-convex functions:*

$$\begin{aligned} & \left| (1 - \lambda)f\left(\frac{a+b}{2}\right) + \lambda \left[\frac{f(a)+f(b)}{2} \right] \right. \\ & \quad \left. - \frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{(\frac{a+b}{2})-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)} C_0^{1-\frac{1}{q}}(k, \mu, \lambda) \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q \Phi_1(k, \mu, \lambda) + |f''(a)|^q \Phi_2(k, \mu, \lambda) \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[|f''\left(\frac{a+b}{2}\right)|^q \Phi_1(k, \mu, \lambda) + |f''(b)|^q \Phi_2(k, \mu, \lambda) \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Remark 2.6 In Corollary 2.8,

(a) if $\lambda = \frac{1}{3}$, then we obtain

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{(\frac{a+b}{2})-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)} C_0^{1-\frac{1}{q}}\left(k, \mu, \frac{1}{3}\right) \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q \Phi_1\left(k, \mu, \frac{1}{3}\right) \right. \right. \\ & \quad \left. \left. + |f''(a)|^q \Phi_2\left(k, \mu, \frac{1}{3}\right) \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q \Phi_1\left(k, \mu, \frac{1}{3}\right) + |f''(b)|^q \Phi_2\left(k, \mu, \frac{1}{3}\right) \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we derive a Simpson-type inequality:

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16} \left(\frac{8}{81}\right)^{1-\frac{1}{q}} \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q \Phi_1\left(1, 1, \frac{1}{3}\right) + |f''(a)|^q \Phi_2\left(1, 1, \frac{1}{3}\right) \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q \Phi_1\left(1, 1, \frac{1}{3}\right) + |f''(b)|^q \Phi_2\left(1, 1, \frac{1}{3}\right) \right]^{\frac{1}{q}} \right\}, \end{aligned}$$

where

$$\Phi_1\left(1, 1, \frac{1}{3}\right) = \frac{71\sqrt{2}}{648} + \frac{1}{8} \arcsin \frac{\sqrt{3}}{3} - \frac{\pi}{32}$$

and

$$\Phi_2\left(1, 1, \frac{1}{3}\right) = \frac{19\sqrt{2}}{648} + \frac{1}{12} \arcsin \frac{1}{3} + \frac{1}{16} \arctan 2\sqrt{2} - \frac{\pi}{32};$$

(b) if $\lambda = \frac{1}{2}$, then we obtain

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{\left(\frac{a+b}{2}\right)^-}^{\mu} f(a) + {}_k J_{\left(\frac{a+b}{2}\right)^+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8\left(\frac{\mu}{k}+1\right)} C_0^{1-\frac{1}{q}} \left(k, \mu, \frac{1}{2}\right) \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q \Phi_1\left(k, \mu, \frac{1}{2}\right) \right. \right. \\ & \quad \left. \left. + |f''(a)|^q \Phi_2\left(k, \mu, \frac{1}{2}\right) \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q \Phi_1\left(k, \mu, \frac{1}{2}\right) + |f''(b)|^q \Phi_2\left(k, \mu, \frac{1}{2}\right) \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we derive an averaged midpoint-trapezoid-type inequality:

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ & \leq \frac{(b-a)^2}{96} \left(\frac{3\pi}{16}\right)^{\frac{1}{q}} \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(c) if $\lambda = 0$, then we obtain

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{\left(\frac{a+b}{2}\right)^-}^{\mu} f(a) + {}_k J_{\left(\frac{a+b}{2}\right)^+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8\left(\frac{\mu}{k}+1\right)} \left(\frac{1}{\frac{\mu}{k}+2}\right)^{1-\frac{1}{q}} \left(\frac{1}{2}\right)^{\frac{1}{q}} \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q \beta\left(\frac{\mu}{k} + \frac{5}{2}, \frac{1}{2}\right) + |f''(a)|^q \beta\left(\frac{\mu}{k} + \frac{3}{2}, \frac{3}{2}\right) \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q \beta\left(\frac{\mu}{k} + \frac{5}{2}, \frac{1}{2}\right) + |f''(b)|^q \beta\left(\frac{\mu}{k} + \frac{3}{2}, \frac{3}{2}\right) \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we obtain a midpoint-type inequality:

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ & \leq \frac{(b-a)^2}{48} \left(\frac{3}{2}\right)^{\frac{1}{q}} \\ & \quad \times \left\{ \left[\frac{5\pi}{16} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{\pi}{16} |f''(a)|^q \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\frac{5\pi}{16} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{\pi}{16} |f''(b)|^q \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(d) if $\lambda = 1$, then we obtain

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} [{}_k J_{(\frac{a+b}{2})-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})+}^{\mu} f(b)] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)} \left(\frac{\frac{\mu}{k}(\frac{\mu}{k}+3)}{2(\frac{\mu}{k}+2)} \right)^{1-\frac{1}{q}} \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q \left(\frac{3(\frac{\mu}{k}+1)\pi}{16} - \frac{1}{2}\beta\left(\frac{\mu}{k} + \frac{5}{2}, \frac{1}{2}\right) \right) \right. \right. \\ & \quad \left. \left. + |f''(a)|^q \left(\frac{(\frac{\mu}{k}+1)\pi}{16} - \frac{1}{2}\beta\left(\frac{\mu}{k} + \frac{3}{2}, \frac{3}{2}\right) \right) \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q \left(\frac{3(\frac{\mu}{k}+1)\pi}{16} - \frac{1}{2}\beta\left(\frac{\mu}{k} + \frac{5}{2}, \frac{1}{2}\right) \right) \right. \right. \\ & \quad \left. \left. + |f''(b)|^q \left(\frac{(\frac{\mu}{k}+1)\pi}{16} - \frac{1}{2}\beta\left(\frac{\mu}{k} + \frac{3}{2}, \frac{3}{2}\right) \right) \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we obtain a trapezoid-type inequality:

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16} \left(\frac{2}{3} \right)^{1-\frac{1}{q}} \\ & \quad \times \left\{ \left[\frac{7\pi}{32} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{3\pi}{32} |f''(a)|^q \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\frac{7\pi}{32} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{3\pi}{32} |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Now, we get ready to state the second theorem as follows.

Theorem 2.2 *Let $A \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R} \setminus \{0\}$ for some fixed $m \in (0, 1]$, and let $a, b \in A, a < b$ with $\eta(b, a, m) > 0$. Assume that $f : A \rightarrow \mathbb{R}$ is a twice differentiable function on A such that f'' is integrable on $[ma, ma + \eta(b, a, m)]$. If $|f''|^q$ for $q > 1$ is a generalized (m, h) -preinvex function with respect to η and $h : [0, 1] \rightarrow \mathbb{R}_0$, then the following inequality for k -fractional integrals with $x \in [a, b], \lambda \in [0, 1], \mu > 0, k > 0$ exists:*

$$\begin{aligned} & |I_{f,\eta}(\mu, k; x, \lambda, m, a, b)| \\ & \leq C_3^{\frac{1}{p}}(k, \mu, \lambda, p) \\ & \quad \times \left\{ \left| \frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{(\frac{\mu}{k}+1)\eta(b, a, m)} \left[|f''(x)|^q \int_0^1 h(t) dt + m |f''(a)|^q \int_0^1 h(1-t) dt \right]^{\frac{1}{q}} \right. \right. \\ & \quad \left. \left. + \left| \frac{(-1)^{\frac{\mu}{k}+2} \eta^{\frac{\mu}{k}+2}(x, b, m)}{(\frac{\mu}{k}+1)\eta(b, a, m)} \right| \right. \right. \\ & \quad \left. \left. \times \left[|f''(x)|^q \int_0^1 h(t) dt + m |f''(b)|^q \int_0^1 h(1-t) dt \right]^{\frac{1}{q}} \right\}, \tag{2.12} \end{aligned}$$

where $p = \frac{q}{q-1}$ and

$$C_3(k, \mu, \lambda, p) = \begin{cases} \frac{1}{p(\frac{\mu}{k}+1)+1}, & \lambda = 0, \\ \left[\frac{k[(\frac{\mu}{k}+1)\lambda]^{\frac{k+kp(\frac{\mu}{k}+1)}{\mu}}}{\mu} \beta(\frac{k(1+p)}{\mu}, 1+p) + \frac{k[1-(\frac{\mu}{k}+1)\lambda]^{p+1}}{\mu(p+1)} {}_2F_1(1 - \frac{k(1+p)}{\mu}, 1; p+2; 1 - (\frac{\mu}{k}+1)\lambda) \right], & 0 < \lambda \leq \frac{1}{\frac{\mu}{k}+1}, \\ \frac{k[(\frac{\mu}{k}+1)\lambda]^{\frac{k+kp(\frac{\mu}{k}+1)}{\mu}}}{\mu} \beta(\frac{1}{(\frac{\mu}{k}+1)\lambda}; \frac{k(1+p)}{\mu}, 1+p), & \frac{1}{\frac{\mu}{k}+1} < \lambda \leq 1. \end{cases}$$

Proof Using Lemma 2.1 and Hölder’s inequality, we have

$$\begin{aligned} & |I_{f,\eta}(\mu, k; x, \lambda, m, a, b)| \\ & \leq \left| \frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{(\frac{\mu}{k}+1)\eta(b, a, m)} \right| \int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right| |f''(ma + t\eta(x, a, m))| dt \\ & \quad + \left| \frac{(-1)^{\frac{\mu}{k}+2}\eta^{\frac{\mu}{k}+2}(x, b, m)}{(\frac{\mu}{k}+1)\eta(b, a, m)} \right| \int_0^1 t \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right| |f''(mb + t\eta(x, b, m))| dt \\ & \leq \left| \frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{(\frac{\mu}{k}+1)\eta(b, a, m)} \right| \left[\int_0^1 t^p \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right|^p dt \right]^{\frac{1}{p}} \\ & \quad \times \left[\int_0^1 |f''(ma + t\eta(x, a, m))|^q dt \right]^{\frac{1}{q}} \\ & \quad + \left| \frac{(-1)^{\frac{\mu}{k}+2}\eta^{\frac{\mu}{k}+2}(x, b, m)}{(\frac{\mu}{k}+1)\eta(b, a, m)} \right| \left[\int_0^1 t^p \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right|^p dt \right]^{\frac{1}{p}} \\ & \quad \times \left[\int_0^1 |f''(mb + t\eta(x, b, m))|^q dt \right]^{\frac{1}{q}}. \end{aligned} \tag{2.13}$$

Since $|f''|^q$ is generalized (m, h) -preinvex on $[ma, ma + \eta(b, a, m)]$, we get

$$\begin{aligned} \int_0^1 |f''(ma + t\eta(x, a, m))|^q dt & \leq \int_0^1 [h(t)|f''(x)|^q + mh(1-t)|f''(a)|^q] dt \\ & = |f''(x)|^q \int_0^1 h(t) dt + m|f''(a)|^q \int_0^1 h(1-t) dt, \end{aligned} \tag{2.14}$$

$$\begin{aligned} \int_0^1 |f''(mb + t\eta(x, b, m))|^q dt & \leq \int_0^1 [h(t)|f''(x)|^q + mh(1-t)|f''(b)|^q] dt \\ & = |f''(x)|^q \int_0^1 h(t) dt + m|f''(b)|^q \int_0^1 h(1-t) dt, \end{aligned} \tag{2.15}$$

and

$$C_3(k, \mu, \lambda, p) = \int_0^1 t^p \left| \left(\frac{\mu}{k} + 1 \right) \lambda - t^{\frac{\mu}{k}} \right|^p dt$$

$$\begin{aligned}
 &= \begin{cases} \int_0^1 t^{(\frac{\mu}{k}+1)^p} dt, & \lambda = 0, \\ \left[\begin{aligned} &\int_0^{[(\frac{\mu}{k}+1)\lambda]^{\frac{k}{\mu}}} t^p [(\frac{\mu}{k}+1)\lambda - t^{\frac{\mu}{k}}]^p dt \\ &+ \int_{[(\frac{\mu}{k}+1)\lambda]^{\frac{k}{\mu}}}^1 t^p [t^{\frac{\mu}{k}} - (\frac{\mu}{k}+1)\lambda]^p dt \end{aligned} \right], & 0 < \lambda \leq \frac{1}{\frac{\mu}{k}+1}, \\ \int_0^1 t^p [(\frac{\mu}{k}+1)\lambda - t^{\frac{\mu}{k}}]^p dt, & \frac{1}{\frac{\mu}{k}+1} < \lambda \leq 1 \end{cases} \\
 &= \begin{cases} \frac{1}{p(\frac{\mu}{k}+1)+1}, & \lambda = 0, \\ \left[\begin{aligned} &\frac{k[(\frac{\mu}{k}+1)\lambda]^{\frac{k+kp(\frac{\mu}{k}+1)}{\mu}}}{\mu} \beta\left(\frac{k(1+p)}{\mu}, 1+p\right) \\ &+ \frac{k[1-(\frac{\mu}{k}+1)\lambda]^{p+1}}{\mu(p+1)} {}_2F_1\left(1 - \frac{k(1+p)}{\mu}, 1; p+2; 1 - (\frac{\mu}{k}+1)\lambda\right) \end{aligned} \right], & 0 < \lambda \leq \frac{1}{\frac{\mu}{k}+1}, \\ \frac{k[(\frac{\mu}{k}+1)\lambda]^{\frac{k+kp(\frac{\mu}{k}+1)}{\mu}}}{\mu} \beta\left(\frac{1}{(\frac{\mu}{k}+1)\lambda}, \frac{k(1+p)}{\mu}, 1+p\right), & \frac{1}{\frac{\mu}{k}+1} < \lambda \leq 1. \end{cases} \quad (2.16)
 \end{aligned}$$

Hence, if we use (2.14)–(2.16) in (2.13), we can get the desired result. This completes the proof. □

Let us point out some special cases of Theorem 2.2.

I. If $h(t) = t^s$ in Theorem 2.2, then we have the following results.

Corollary 2.9 *In Theorem 2.2, if we use the generalized (m, s) -Breckner-preinvexity of $|f''|^q$ along with $q > 1$ and $p = \frac{q}{q-1}$, then, for $s \in (0, 1]$ and $m \in (0, 1]$, we have the following inequality:*

$$\begin{aligned}
 &|I_{f,\eta}(\mu, k; x, \lambda, m, a, b)| \\
 &\leq C_3^{\frac{1}{p}}(k, \mu, \lambda, p) \left\{ \left| \frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{(\frac{\mu}{k}+1)\eta(b, a, m)} \left[\frac{|f''(x)|^q + m|f''(a)|^q}{s+1} \right]^{\frac{1}{q}} \right. \right. \\
 &\quad \left. \left. + \left| \frac{(-1)^{\frac{\mu}{k}+2}\eta^{\frac{\mu}{k}+2}(x, b, m)}{(\frac{\mu}{k}+1)\eta(b, a, m)} \left[\frac{|f''(x)|^q + m|f''(b)|^q}{s+1} \right]^{\frac{1}{q}} \right| \right\}.
 \end{aligned}$$

Corollary 2.10 *In Theorem 2.2, if the mapping $\eta(b, a, m) = b - ma$ together with $m = 1$, choosing $x = \frac{a+b}{2}$, for $s \in (0, 1]$, we have the following inequality for s -convex functions:*

$$\begin{aligned}
 &\left| \frac{2^{\frac{\mu}{k}-1}}{(b-a)^{\frac{\mu}{k}-1}} I_f\left(\mu, k; \frac{a+b}{2}, \lambda, 1, a, b\right) \right| \\
 &\leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)} C_3^{\frac{1}{p}}(k, \mu, \lambda, p) \left\{ \left[\frac{|f''(x)|^q + |f''(a)|^q}{s+1} \right]^{\frac{1}{q}} + \left[\frac{|f''(x)|^q + |f''(b)|^q}{s+1} \right]^{\frac{1}{q}} \right\}.
 \end{aligned}$$

Remark 2.7 In Corollary 2.10,

(i) if $\lambda = \frac{1}{2}$, then we obtain

$$\begin{aligned}
 &\left| \frac{2^{\frac{\mu}{k}-1}}{(b-a)^{\frac{\mu}{k}-1}} I_f\left(\mu, k; \frac{a+b}{2}, \frac{1}{2}, 1, a, b\right) \right| \\
 &\leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)} C_3^{\frac{1}{p}}\left(k, \mu, \frac{1}{2}, p\right) \left\{ \left[\frac{|f''(x)|^q + |f''(a)|^q}{s+1} \right]^{\frac{1}{q}} + \left[\frac{|f''(x)|^q + |f''(b)|^q}{s+1} \right]^{\frac{1}{q}} \right\};
 \end{aligned}$$

(ii) if $k = 1$ and $\lambda = \frac{1}{3}, 1$, then we have the results (c) and (f) of Corollary 2.3 in [34], respectively. Further, if we choose $\mu = 1$, then we have the results (d) and (g) of Corollary 2.3 in [34], respectively.

II. If $h(t) = t^{-s}$ in Theorem 2.2, then we have the following results.

Corollary 2.11 *In Theorem 2.2, if we use the generalized (m, s) -Godunova–Levin–Dragomir-preinvexity of $|f''|^q$ along with $q > 1$ and $p = \frac{q}{q-1}$, then, for $s \in (0, 1)$ and $m \in (0, 1]$, we have the following inequality:*

$$\begin{aligned}
 & |I_{f,\eta}(\mu, k; x, \lambda, m, a, b)| \\
 & \leq C_3^{\frac{1}{p}}(k, \mu, \lambda, p) \frac{1}{(1-s)^{\frac{1}{q}}} \left\{ \left| \frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{(\frac{\mu}{k} + 1)\eta(b, a, m)} \right| [|f''(x)|^q + m|f''(a)|^q]^{\frac{1}{q}} \right. \\
 & \quad \left. + \left| \frac{(-1)^{\frac{\mu}{k}+2}\eta^{\frac{\mu}{k}+2}(x, b, m)}{(\frac{\mu}{k} + 1)\eta(b, a, m)} \right| [|f''(x)|^q + m|f''(b)|^q]^{\frac{1}{q}} \right\}.
 \end{aligned}$$

Corollary 2.12 *In Theorem 2.2, if the mapping $\eta(b, a, m) = b - ma$ along with $m = 1$, choosing $x = \frac{a+b}{2}$, for $s \in (0, 1)$, we have the following inequality for s -Godunova–Levin functions:*

$$\begin{aligned}
 & \left| (1-\lambda)f\left(\frac{a+b}{2}\right) + \lambda\left[\frac{f(a)+f(b)}{2}\right] - \frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} [kJ_{(\frac{a+b}{2})^-}^\mu f(a) + kJ_{(\frac{a+b}{2})^+}^\mu f(b)] \right| \\
 & \leq \frac{(b-a)^2}{8(\frac{\mu}{k} + 1)(1-s)^{\frac{1}{q}}} C_3^{\frac{1}{p}}(k, \mu, \lambda, p) \\
 & \quad \times \left\{ \left[|f''\left(\frac{a+b}{2}\right)|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[|f''\left(\frac{a+b}{2}\right)|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}.
 \end{aligned}$$

Remark 2.8 In Corollary 2.12,

(a) if $\lambda = \frac{1}{3}$, then we obtain

$$\begin{aligned}
 & \left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} [kJ_{(\frac{a+b}{2})^-}^\mu f(a) + kJ_{(\frac{a+b}{2})^+}^\mu f(b)] \right| \\
 & \leq \frac{(b-a)^2}{8(\frac{\mu}{k} + 1)(1-s)^{\frac{1}{q}}} C_3^{\frac{1}{p}}\left(k, \mu, \frac{1}{3}, p\right) \\
 & \quad \times \left\{ \left[|f''\left(\frac{a+b}{2}\right)|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[|f''\left(\frac{a+b}{2}\right)|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}.
 \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we derive a Simpson-type inequality:

$$\begin{aligned}
 & \left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{(b-a)^2}{16(1-s)^{\frac{1}{q}}} C_3^{\frac{1}{p}}\left(1, 1, \frac{1}{3}, p\right) \\
 & \quad \times \left\{ \left[|f''\left(\frac{a+b}{2}\right)|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[|f''\left(\frac{a+b}{2}\right)|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\};
 \end{aligned}$$

(b) if $\lambda = \frac{1}{2}$, then we obtain

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{(\frac{a+b}{2})-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)(1-s)^{\frac{1}{q}}} C_3^{\frac{1}{p}} \left(k, \mu, \frac{1}{2}, p \right) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we derive an averaged midpoint-trapezoid-type inequality:

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ & \leq \frac{(b-a)^2}{16(1-s)^{\frac{1}{q}}} \beta^{\frac{1}{p}} (1+p, 1+p) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(c) if $\lambda = 0$, then we obtain

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{(\frac{a+b}{2})-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)(1-s)^{\frac{1}{q}}} \left[\frac{1}{p(\frac{\mu}{k}+1)+1} \right]^{\frac{1}{p}} \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we derive a midpoint-type inequality:

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ & \leq \frac{(b-a)^2}{16(1-s)^{\frac{1}{q}}} \left(\frac{1}{2p+1} \right)^{\frac{1}{p}} \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(d) if $\lambda = 1$, then we obtain

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{(\frac{a+b}{2})-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)(1-s)^{\frac{1}{q}}} C_3^{\frac{1}{p}} (k, \mu, 1, p) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we derive a trapezoid-type inequality:

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16(1-s)^{\frac{1}{q}}} \left[2^{1+2p} \beta\left(\frac{1}{2}; 1+p, 1+p\right) \right]^{\frac{1}{p}} \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

III. If $h(t) = t(1-t)$ in Theorem 2.2, then we have the following results.

Corollary 2.13 *In Theorem 2.2, if we use the generalized (m, tgs) -preinvexity of $|f''|^q$ along with $q > 1$ and $p = \frac{q}{q-1}$, then, for $m \in (0, 1]$, we have the following inequality:*

$$\begin{aligned} & |I_{f,\eta}(\mu, k; x, \lambda, m, a, b)| \\ & \leq C_3^{\frac{1}{p}}(k, \mu, \lambda, p) \left(\frac{1}{6}\right)^{\frac{1}{q}} \left\{ \left| \frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{(\frac{\mu}{k}+1)\eta(b, a, m)} \right| \left[|f''(x)|^q + m|f''(a)|^q \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left| \frac{(-1)^{\frac{\mu}{k}+2} \eta^{\frac{\mu}{k}+2}(x, b, m)}{(\frac{\mu}{k}+1)\eta(b, a, m)} \right| \left[|f''(x)|^q + m|f''(b)|^q \right]^{\frac{1}{q}} \right\}, \end{aligned}$$

where $C_3(k, \mu, \lambda, p)$ is defined by (2.16).

Corollary 2.14 *In Theorem 2.2, if the mapping $\eta(b, a, m) = b - ma$ together with $m = 1$, choosing $x = \frac{a+b}{2}$, we get the following inequality for tgs -convex functions:*

$$\begin{aligned} & \left| (1-\lambda)f\left(\frac{a+b}{2}\right) + \lambda \left[\frac{f(a)+f(b)}{2} \right] - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{\left(\frac{a+b}{2}\right)^-}^{\mu} f(a) + {}_k J_{\left(\frac{a+b}{2}\right)^+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)} \left(\frac{1}{6}\right)^{\frac{1}{q}} C_3^{\frac{1}{p}}(k, \mu, \lambda, p) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Remark 2.9 In Corollary 2.14,

(a) if $\lambda = \frac{1}{3}$, then we obtain

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{\left(\frac{a+b}{2}\right)^-}^{\mu} f(a) + {}_k J_{\left(\frac{a+b}{2}\right)^+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)} \left(\frac{1}{6}\right)^{\frac{1}{q}} C_3^{\frac{1}{p}}\left(k, \mu, \frac{1}{3}, p\right) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we derive a Simpson-type inequality:

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ & \leq \frac{(b-a)^2}{16} \left(\frac{1}{6}\right)^{\frac{1}{q}} C_3^{\frac{1}{p}} \left(1, 1, \frac{1}{3}, p\right) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(b) if $\lambda = \frac{1}{2}$, then we obtain

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[kJ_{\left(\frac{a+b}{2}\right)^-}^{\mu} f(a) + kJ_{\left(\frac{a+b}{2}\right)^+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8\left(\frac{\mu}{k}+1\right)} \left(\frac{1}{6}\right)^{\frac{1}{q}} C_3^{\frac{1}{p}} \left(k, \mu, \frac{1}{2}, p\right) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we derive an averaged midpoint-trapezoid-type inequality:

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ & \leq \frac{(b-a)^2}{16} \left(\frac{1}{6}\right)^{\frac{1}{q}} \beta^{\frac{1}{p}} (1+p, 1+p) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(c) if $\lambda = 0$, then we obtain

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[kJ_{\left(\frac{a+b}{2}\right)^-}^{\mu} f(a) + kJ_{\left(\frac{a+b}{2}\right)^+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8\left(\frac{\mu}{k}+1\right)} \left(\frac{1}{6}\right)^{\frac{1}{q}} \left[\frac{1}{p\left(\frac{\mu}{k}+1\right)+1} \right]^{\frac{1}{p}} \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we derive a midpoint-type inequality:

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ & \leq \frac{(b-a)^2}{16} \left(\frac{1}{6}\right)^{\frac{1}{q}} \left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(d) if $\lambda = 1$, then we obtain

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{(\frac{a+b}{2})-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)} \left(\frac{1}{6} \right)^{\frac{1}{q}} C_3^{\frac{1}{p}}(k, \mu, 1, p) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we derive a trapezoid-type inequality:

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16} \left(\frac{1}{6} \right)^{\frac{1}{q}} \left[2^{1+2p} \beta\left(\frac{1}{2}; 1+p, 1+p\right) \right]^{\frac{1}{p}} \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

IV. If $h(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$ in Theorem 2.2, then we have the following results.

Corollary 2.15 *In Theorem 2.2, if we use the generalized m -MT-preinvexity of $|f''|^q$ along with $q > 1$ and $p = \frac{q}{q-1}$, then, for $m \in (0, 1]$, we have the following inequality:*

$$\begin{aligned} & |I_{f,\eta}(\mu, k; x, \lambda, m, a, b)| \\ & \leq C_3^{\frac{1}{p}}(k, \mu, \lambda, p) \left(\frac{\pi}{4} \right)^{\frac{1}{q}} \left\{ \left| \frac{\eta^{\frac{\mu}{k}+2}(x, a, m)}{(\frac{\mu}{k}+1)\eta(b, a, m)} \right| \left[|f''(x)|^q + m|f''(a)|^q \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left| \frac{(-1)^{\frac{\mu}{k}+2} \eta^{\frac{\mu}{k}+2}(x, b, m)}{(\frac{\mu}{k}+1)\eta(b, a, m)} \right| \left[|f''(x)|^q + m|f''(b)|^q \right]^{\frac{1}{q}} \right\}, \end{aligned}$$

where we use the fact that

$$\int_0^1 \frac{\sqrt{t}}{2\sqrt{1-t}} dt = \int_0^1 \frac{\sqrt{1-t}}{2\sqrt{t}} dt = \frac{\pi}{4}.$$

Corollary 2.16 *In Theorem 2.2, if the mapping $\eta(b, a, m) = b - ma$ together with $m = 1$, choosing $x = \frac{a+b}{2}$, we get the following inequality for MT-convex functions:*

$$\begin{aligned} & \left| (1-\lambda)f\left(\frac{a+b}{2}\right) + \lambda \left[\frac{f(a)+f(b)}{2} \right] \right. \\ & \quad \left. - \frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{(\frac{a+b}{2})-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)} \left(\frac{\pi}{4} \right)^{\frac{1}{q}} C_3^{\frac{1}{p}}(k, \mu, \lambda, p) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Remark 2.10 In Corollary 2.16,

(a) if $\lambda = \frac{1}{3}$, then we obtain

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right. \\ & \quad \left. - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{(\frac{a+b}{2})-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)} \left(\frac{\pi}{4}\right)^{\frac{1}{q}} C_3^{\frac{1}{p}} \left(k, \mu, \frac{1}{3}, p\right) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we derive a Simpson-type inequality:

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ & \leq \frac{(b-a)^2}{16} \left(\frac{\pi}{4}\right)^{\frac{1}{q}} C_3^{\frac{1}{p}} \left(1, 1, \frac{1}{3}, p\right) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(b) if $\lambda = \frac{1}{2}$, then we obtain

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] \right. \\ & \quad \left. - \frac{2^{\frac{\mu}{k}-1} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k J_{(\frac{a+b}{2})-}^{\mu} f(a) + {}_k J_{(\frac{a+b}{2})+}^{\mu} f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)} \left(\frac{\pi}{4}\right)^{\frac{1}{q}} C_3^{\frac{1}{p}} \left(k, \mu, \frac{1}{2}, p\right) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we derive an averaged midpoint-trapezoid-type inequality:

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ & \leq \frac{(b-a)^2}{16} \left(\frac{\pi}{4}\right)^{\frac{1}{q}} \beta^{\frac{1}{p}} (1+p, 1+p) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(c) if $\lambda = 0$, then we obtain

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} [{}_k J_{(\frac{a+b}{2})^-}^\mu f(a) + {}_k J_{(\frac{a+b}{2})^+}^\mu f(b)] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)} \left(\frac{\pi}{4}\right)^{\frac{1}{q}} \left[\frac{1}{p(\frac{\mu}{k}+1)+1} \right]^{\frac{1}{p}} \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we derive a midpoint-type inequality:

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16} \left(\frac{\pi}{4}\right)^{\frac{1}{q}} \left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}; \end{aligned}$$

(d) if $\lambda = 1$, then we obtain

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} [{}_k J_{(\frac{a+b}{2})^-}^\mu f(a) + {}_k J_{(\frac{a+b}{2})^+}^\mu f(b)] \right| \\ & \leq \frac{(b-a)^2}{8(\frac{\mu}{k}+1)} \left(\frac{\pi}{4}\right)^{\frac{1}{q}} C_3^{\frac{1}{p}}(k, \mu, 1, p) \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Specially, if we put $k = 1 = \mu$, then we derive a trapezoid-type inequality:

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16} \left(\frac{\pi}{4}\right)^{\frac{1}{q}} \left[2^{1+2p} \beta\left(\frac{1}{2}; 1+p, 1+p\right) \right]^{\frac{1}{p}} \\ & \quad \times \left\{ \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(a)|^q \right]^{\frac{1}{q}} + \left[\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Acknowledgements

This work was partially supported by the National Natural Science Foundation of China under Grant No. 61374028.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details

¹Department of Mathematics, College of Science, China Three Gorges University, Yichang, China. ²Three Gorges Mathematical Research Center, China Three Gorges University, Yichang, China. ³Hubei Provincial Collaborative Innovation

Center for New Energy Microgrid, China Three Gorges University, Yichang, China. ⁴Department of Mathematics, Faculty of Technical Science, University "Ismael Qemali", Vloera, Albania.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 15 October 2017 Accepted: 9 February 2018 Published online: 21 February 2018

References

- Du, T.S., Li, Y.J., Yang, Z.Q.: A generalization of Simpson's inequality via differentiable mapping using extended (s, m) -convex functions. *Appl. Math. Comput.* **293**, 358–369 (2017)
- Hussain, S., Qaisar, S.: More results on Simpson's type inequality through convexity for twice differentiable continuous mappings. *SpringerPlus* **5**, Article ID 77 (2016)
- Hwang, D.Y., Dragomir, S.S.: Extensions of the Hermite–Hadamard inequality for r -preinvex functions on an invex set. *Bull. Aust. Math. Soc.* **95**(3), 412–423 (2017)
- Khan, M.A., Ali, T., Dragomir, S.S., Sarikaya, M.Z.: Hermite–Hadamard type inequalities for conformable fractional integrals. *Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat.* (2017). <https://doi.org/10.1007/s13398-017-0408-5>
- Latif, M.A.: On some new inequalities of Hermite–Hadamard type for functions whose derivatives are s -convex in the second sense in the absolute value. *Ukr. Math. J.* **67**(10), 1552–1571 (2016)
- Latif, M.A., Dragomir, S.S.: Generalization of Hermite–Hadamard type inequalities for n -times differentiable functions through preinvexity. *Georgian Math. J.* **23**(1), 97–104 (2016)
- Qi, F., Xi, B.Y.: Some integral inequalities of Simpson type for GA- ε -convex functions. *Georgian Math. J.* **20**, 775–788 (2013)
- Wu, S.H., Baloch, I.A., İşcan, İ.: On harmonically (p, h, m) -preinvex functions. *J. Funct. Spaces* **2017**, Article ID 2148529 (2017)
- Sarikaya, M.Z., Set, E., Yaldiz, H., Başağ, N.: Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **57**, 2403–2407 (2013)
- Dragomir, S.S., Bhatti, M.I., Iqbal, M., Muddassar, M.: Some new Hermite–Hadamard's type fractional integral inequalities. *J. Comput. Anal. Appl.* **18**(4), 655–661 (2015)
- Hwang, S.R., Tseng, K.L., Hsu, K.C.: New inequalities for fractional integrals and their applications. *Turk. J. Math.* **40**, 471–486 (2016)
- Iqbal, M., Bhatti, M.I., Nazeer, K.: Generalization of inequalities analogous to Hermite–Hadamard inequality via fractional integrals. *Bull. Korean Math. Soc.* **52**(3), 707–716 (2015)
- Set, E., İşcan, İ., Kara, H.H.: Hermite–Hadamard–Fejér type inequalities for s -convex function in the second sense via fractional integrals. *Filomat* **30**(12), 3131–3138 (2016)
- Anastassiou, G.A.: Generalised fractional Hermite–Hadamard inequalities involving m -convexity and (s, m) -convexity. *Facta Univ., Ser. Math. Inform.* **28**(2), 107–126 (2013)
- Awan, M.U., Noor, M.A., Mihai, M.V., Noor, K.I.: Fractional Hermite–Hadamard inequalities for differentiable s -Godunova–Levin functions. *Filomat* **30**(12), 3235–3241 (2016)
- İşcan, İ., Wu, S.H.: Hermite–Hadamard type inequalities for harmonically convex functions via fractional integrals. *Appl. Math. Comput.* **238**, 237–244 (2014)
- Qaisar, S., Iqbal, M., Muddassar, M.: New Hermite–Hadamard's inequalities for preinvex functions via fractional integrals. *J. Comput. Anal. Appl.* **20**(7), 1318–1328 (2016)
- Kashuri, A., Liko, R.: Generalizations of Hermite–Hadamard and Ostrowski type inequalities for MT_m -preinvex functions. *Proyecciones* **36**(1), 45–80 (2017)
- Matloka, M.: Some inequalities of Hadamard type for mappings whose second derivatives are h -convex via fractional. *J. Fract. Calc. Appl.* **6**(1), 110–119 (2015)
- Wang, J., Deng, J., Fečkan, M.: Hermite–Hadamard-type inequalities for r -convex functions based on the use of Riemann–Liouville fractional integrals. *Ukr. Math. J.* **65**(2), 193–211 (2013)
- Mubeen, S., Habibullah, G.M.: k -fractional integrals and application. *Int. J. Contemp. Math. Sci.* **7**(2), 89–94 (2012)
- Agarwal, P.: Some inequalities involving Hadamard-type k -fractional integral operators. *Math. Methods Appl. Sci.* **40**(11), 3882–3891 (2017)
- Ali, A., Gulshan, G., Hussain, R., Latif, A., Muddassar, M.: Generalized inequalities of the type of Hermite–Hadamard–Fejér with quasi-convex functions by way of k -fractional derivatives. *J. Comput. Anal. Appl.* **22**(7), 1208–1219 (2017)
- Tomar, M., Mubeen, S., Choi, J.: Certain inequalities associated with Hadamard k -fractional integral operators. *J. Inequal. Appl.* **2016**, Article ID 234 (2016)
- Romero, L.G., Luque, L.L., Dorrego, G.A., Cerutti, R.A.: On the k -Riemann–Liouville fractional derivative. *Int. J. Contemp. Math. Sci.* **8**(1), 41–51 (2013)
- Set, E., Tomar, M., Sarikaya, M.Z.: On generalized Grüss type inequalities for k -fractional integrals. *Appl. Math. Comput.* **269**, 29–34 (2015)
- Tariboon, J., Ntouyas, S.K., Tomar, M.: Some new integral inequalities for k -fractional integrals. *Malaya J. Mat.* **4**(1), 100–110 (2016)
- Hudzik, H., Maligranda, L.: Some remarks on s -convex functions. *Aequ. Math.* **48**, 100–111 (1994)
- Noor, M.A., Noor, K.I., Awan, M.U., Khan, S.: Fractional Hermite–Hadamard inequalities for some new classes of Godunova–Levin functions. *Appl. Math. Inf. Sci.* **8**(6), 2865–2872 (2014)
- Tunç, M., Göv, E., Şanal, Ü.: On tgs -convex function and their inequalities. *Facta Univ., Ser. Math. Inform.* **30**(5), 679–691 (2015)
- Tunç, M., Şubaş, Y., Karabayır, İ.: On some Hadamard type inequalities for MT -convex functions. *Int. J. Open Probl. Comput. Sci. Math.* **6**(2), 102–113 (2013)

32. Du, T.S., Liao, J.G., Li, Y.J.: Properties and integral inequalities of Hadamard–Simpson type for the generalized (s, m) -preinvex functions. *J. Nonlinear Sci. Appl.* **9**, 3112–3126 (2016)
33. Peng, C., Zhou, C., Du, T.S.: Riemann–Liouville fractional Simpson's inequalities through generalized (m, h_1, h_2) -preinvexity. *Ital. J. Pure Appl. Math.* **38**, 345–367 (2017)
34. İşcan, İ.: Generalization of different type integral inequalities for s -convex functions via fractional integrals. *Appl. Anal.* **93**(9), 1846–1862 (2014)
35. Özdemir, M.E., Avcı, M., Kavurmacı, H.: Hermite–Hadamard-type inequalities via (α, m) -convexity. *Comput. Math. Appl.* **61**, 2614–2620 (2011)
36. Sarikaya, M.Z., Set, E., Özdemir, M.E.: On new inequalities of Simpson's type for functions whose second derivatives absolute values are convex. *J. Appl. Math. Stat. Inform.* **9**(1), 37–45 (2013)

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com
