Wang and Yang *Journal of Inequalities and Applications* (2018) 2018:38
https://doi.org/10.1186/s13660-018-1634-x

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

On a reverse Mulholland's inequality in the whole plane

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Abstract

By introducing multi-parameters, applying the weight coefficients and Hermite–Hadamard's inequality, we give a reverse of the extended Mulholland inequality in the whole plane with the best possible constant factor. The equivalent forms and a few particular cases are also considered.

MSC: 26D15; 47A07

Keywords: Mulholland's inequality; Parameter; Weight coefficient; Equivalent form; Reverse

1 Introduction

Assuming that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geq 0$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, we have the following Hardy–Hilbert inequality (cf. [1]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{n=1}^{\infty} a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. We still have the following Mulholland inequality with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. Theorem 343 of [1], replacing $\frac{a_m}{m}$, $\frac{b_n}{n}$ by a_m, b_n):

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{n=2}^{\infty} \frac{a_m^p}{m^{1-p}}\right)^{1/p}\left(\sum_{n=2}^{\infty} \frac{b_n^q}{n^{1-q}}\right)^{1/q}. \tag{2}$$

Inequalities (1)–(2) are important in the analysis and its applications (cf. [1, 2]).

In 2007, Yang [3] first gave a Hilbert-type integral inequality in the whole plane as follows:

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{(1+e^{x+y})^{\lambda}}\,dx\,dy\\&\quad< B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\int_{-\infty}^{\infty} e^{-\lambda x}f^2(x)\,dx\int_{-\infty}^{\infty} e^{-\lambda y}f^2(y)\,dy\right)^{\frac{1}{2}},\end{aligned} \tag{3}$$

where the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ $(\lambda > 0)$ is the best possible. Some new results on inequalities (1)–(3) were provided by [4–24]. In 2016, Yang and Chen [25] gave a more accurate extension of (1) in the whole plane as follows:

$$\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{a_m b_n}{(|m-\xi|+|n-\eta|)^{\lambda}}$$
$$< 2B(\lambda_1,\lambda_2)\left[\sum_{|m|=1}^{\infty}|m-\xi|^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}|n-\eta|^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \tag{4}$$

where the constant factor $2B(\lambda_1,\lambda_2)$ ($0<\lambda_1,\lambda_2\leq 1$, $\lambda_1+\lambda_2=\lambda$, $\xi,\eta\in[0,\frac{1}{2}]$) is the best possible. Another result was provided by Xin et al. [26].

In this paper, by introducing multi-parameters, applying the weight coefficients and Hermite–Hadamard's inequality, we give a reverse of the extension of Mulholland's inequality (2) in the whole plane with the best possible constant factor similar to the reverse of (4). The equivalent forms and a few particular cases are also considered.

2 Some lemmas

In the following, we agree that $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda_1,\lambda_2>0$, $\lambda_1+\lambda_2=\lambda\leq 1$, $\alpha,\beta\in[\arccos\frac{1}{3},\frac{\pi}{2}]$, $\xi,\eta\in[0,\frac{1}{2}]$, and

$$H_{\gamma}(\lambda_1):=\frac{2\pi\csc^2\gamma}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}\quad(\gamma=\alpha,\beta). \tag{5}$$

Remark 1 In view of the conditions that $\alpha,\beta\in[\arccos\frac{1}{3},\frac{\pi}{2}]$, $\xi,\eta\in[0,\frac{1}{2}]$, it follows that $(\frac{3}{2}\pm\eta)(1\mp\cos\beta)\geq 1$ and $(\frac{3}{2}\pm\xi)(1\mp\cos\alpha)\geq 1$.

We set the following functions:

$$A_{\alpha}(\xi,x)=|x-\xi|+(x-\xi)\cos\alpha,$$

$B_{\beta}(\eta,y)=|y-\eta|+(y-\eta)\cos\beta$, and

$$H(x,y):=\frac{1}{\ln^{\lambda}A_{\alpha}(\xi,x)+\ln^{\lambda}B_{\beta}(\eta,y)}\quad\left(|x|,|y|>\frac{3}{2}\right). \tag{6}$$

Definition 1 Define two weight coefficients as follows:

$$\omega(\lambda_2,m):=\sum_{|n|=2}^{\infty}\frac{H(m,n)}{B_{\beta}(\eta,n)}\frac{\ln^{\lambda_1}A_{\alpha}(\xi,m)}{\ln^{1-\lambda_2}B_{\beta}(\eta,n)},\quad |m|\in\mathbf{N}\backslash\{1\}, \tag{7}$$

$$\varpi(\lambda_1,n):=\sum_{|m|=2}^{\infty}\frac{H(m,n)}{A_{\alpha}(\xi,m)}\frac{\ln^{\lambda_2}B_{\beta}(\eta,n)}{\ln^{1-\lambda_1}A_{\alpha}(\xi,m)},\quad |n|\in\mathbf{N}\backslash\{1\}, \tag{8}$$

where $\sum_{|j|=2}^{\infty}\cdots=\sum_{j=-2}^{-\infty}\cdots+\sum_{j=2}^{\infty}\cdots$ $(j=m,n)$.

Lemma 1 (cf. [26]) *Suppose that $g(t)$ (>0) is strictly decreasing in $(1,\infty)$, satisfying $\int_1^{\infty}g(t)\,dt\in\mathbf{R}_+$. We have*

$$\int_2^{\infty}g(t)\,dt<\sum_{n=2}^{\infty}g(n)<\int_1^{\infty}g(t)\,dt. \tag{9}$$

If $(-1)^i g(t) > 0$ $(i = 0, 1, 2; t \in (\frac{3}{2}, \infty))$, $\int_{\frac{3}{2}}^{\infty} g(t)\,dt \in \mathbf{R}_+$, *then we have the following Hermite–Hadamard inequality* (*cf.* [27]):

$$\sum_{n=2}^{\infty} g(n) < \int_{\frac{3}{2}}^{\infty} g(t)\,dt. \tag{10}$$

Lemma 2 *For* $0 < \lambda \leq 1$, $0 < \lambda_2 < 1$, *the following inequalities are valid*:

$$H_\beta(\lambda_1)\bigl(1 - \theta(\lambda_2, m)\bigr) < \omega(\lambda_2, m) < H_\beta(\lambda_1), \quad |m| \in \mathbf{N}\backslash\{1\}, \tag{11}$$

where

$$\begin{aligned} \theta(\lambda_2, m) &:= \frac{\lambda}{\pi} \sin\left(\frac{\pi \lambda_1}{\lambda}\right) \int_0^{\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_\alpha(\xi,m)}} \frac{u^{\lambda_2 - 1}}{1 + u^\lambda}\,du \\ &= O\left(\frac{1}{\ln^{\lambda_2} A_\alpha(\xi, m)}\right) \in (0, 1). \end{aligned} \tag{12}$$

Proof For $|m| \in \mathbf{N}\backslash\{1\}$, we set the following functions:

$$\begin{aligned} H^{(1)}(m, y) &:= \frac{1}{\ln^\lambda A_\alpha(\xi, m) + \ln^\lambda[(y - \eta)(\cos\beta - 1)]}, \quad y < -\frac{3}{2}, \\ H^{(2)}(m, y) &:= \frac{1}{\ln^\lambda A_\alpha(\xi, m) + \ln^\lambda[(y - \eta)(\cos\beta + 1)]}, \quad y > \frac{3}{2}, \end{aligned}$$

wherefrom

$$H^{(1)}(m, -y) = \frac{1}{\ln^\lambda A_\alpha(\xi, m) + \ln^\lambda[(y + \eta)(1 - \cos\beta)]}, \quad y > \frac{3}{2}.$$

We find

$$\begin{aligned} \omega(\lambda_2, m) &= \sum_{n=-2}^{-\infty} \frac{H^{(1)}(m, n) \ln^{\lambda_1} A_\alpha(\xi, m)}{(n - \eta)(\cos\beta - 1) \ln^{1-\lambda_2}[(n - \eta)(\cos\beta - 1)]} \\ &\quad + \sum_{n=2}^{\infty} \frac{H^{(2)}(m, n) \ln^{\lambda_1} A_\alpha(\xi, m)}{(n - \eta)(1 + \cos\beta) \ln^{1-\lambda_2}[(n - \eta)(1 + \cos\beta)]} \\ &= \frac{\ln^{\lambda_1} A_\alpha(\xi, m)}{1 - \cos\beta} \sum_{n=2}^{\infty} \frac{H^{(1)}(m, -n)}{(n + \eta) \ln^{1-\lambda_2}[(n + \eta)(1 - \cos\beta)]} \\ &\quad + \frac{\ln^{\lambda_1} A_\alpha(\xi, m)}{1 + \cos\beta} \sum_{n=2}^{\infty} \frac{H^{(2)}(m, n)}{(n - \eta) \ln^{1-\lambda_2}[(n - \eta)(1 + \cos\beta)]}. \end{aligned} \tag{13}$$

In virtue of $0 < \lambda \leq 1$, $0 < \lambda_2 < 1$, we find that for $y > \frac{3}{2}$,

$$\begin{aligned} &\frac{d}{dy} \frac{H^{(i)}(m, (-1)^i y)}{(y - (-1)^i \eta) \ln^{1-\lambda_2}[(y - (-1)^i \eta)(1 + (-1)^i \cos\beta)]} < 0, \\ &\frac{d^2}{dy^2} \frac{H^{(i)}(m, (-1)^i y)}{(y - (-1)^i \eta) \ln^{1-\lambda_2}[(y - (-1)^i \eta)(1 + (-1)^i \cos\beta)]} > 0 \quad (i = 1.2), \end{aligned}$$

it follows that $\frac{H^{(i)}(m,(-1)^i y)}{(y-(-1)^i\eta)\ln^{1-\lambda_2}[(y-(-1)^i\eta)(1+(-1)^i\cos\beta)]}$ ($i = 1.2$) are strictly decreasing and strictly convex in $(\frac{3}{2},\infty)$. By (10) and (13) we find

$$\omega(\lambda_2,m) < \frac{\ln^{\lambda_1} A_\alpha(\xi,m)}{1-\cos\beta}\int_{3/2}^{\infty}\frac{H^{(1)}(m,-y)\,dy}{(y+\eta)\ln^{1-\lambda_2}[(y+\eta)(1-\cos\beta)]} + \frac{\ln^{\lambda_1} A_\alpha(\xi,m)}{1+\cos\beta}\int_{3/2}^{\infty}\frac{H^{(2)}(m,y)\,dy}{(y-\eta)\ln^{1-\lambda_2}[(y-\eta)(1+\cos\beta)]}.$$

Setting $u = \frac{\ln[(y+\eta)(1-\cos\beta)]}{\ln A_\alpha(\xi,m)}$ ($u = \frac{\ln[(y-\eta)(1+\cos\beta)]}{\ln A_\alpha(\xi,m)}$) in the above first (second) integral, in view of Remark 1, by simplifications, we obtain

$$\begin{aligned}\omega(\lambda_2,m) &< \left(\frac{1}{1-\cos\beta}+\frac{1}{1+\cos\beta}\right)\int_0^{\infty}\frac{u^{\lambda_2-1}}{1+u^{\lambda}}\,du \\ &= \frac{2\csc^2\beta}{\lambda}\int_0^{\infty}\frac{v^{(\lambda_2/\lambda)-1}\,dv}{1+v} = \frac{2\pi\csc^2\beta}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})} = H_\beta(\lambda_1).\end{aligned}$$

By (9) and (13), in the same way, we still have

$$\begin{aligned}\omega(\lambda_2,m) &> \frac{\ln^{\lambda_1} A_\alpha(\xi,m)}{1-\cos\beta}\int_{2}^{\infty}\frac{H^{(1)}(m,-y)\,dy}{(y+\eta)\ln^{1-\lambda_2}[(y+\eta)(1-\cos\beta)]} \\ &\quad + \frac{\ln^{\lambda_1} A_\alpha(\xi,m)}{1+\cos\beta}\int_{2}^{\infty}\frac{H^{(2)}(m,y)\,dy}{(y-\eta)\ln^{1-\lambda_2}[(y-\eta)(1+\cos\beta)]} \\ &\geq \left(\frac{1}{1-\cos\beta}+\frac{1}{1+\cos\beta}\right)\int_{\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_\alpha(\xi,m)}}^{\infty}\frac{u^{\lambda_2-1}}{1+u^{\lambda}}\,du \\ &= H_\beta(\lambda_1) - 2\csc^2\beta\int_0^{\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_\alpha(\xi,m)}}\frac{u^{\lambda_2-1}}{1+u^{\lambda}}\,du \\ &= H_\beta(\lambda_1)\big(1-\theta(\lambda_2,m)\big) > 0,\end{aligned}$$

where $\theta(\lambda_2,m)$ is indicated by (12). It follows that $\theta(\lambda_2,m) < 1$ and

$$\begin{aligned}0 &< \theta(\lambda_2,m) \\ &< \frac{\lambda}{\pi}\sin\left(\frac{\pi\lambda_1}{\lambda}\right)\int_0^{\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_\alpha(\xi,m)}} u^{\lambda_2-1}\,du \\ &= \frac{\lambda}{\pi\lambda_2}\sin\left(\frac{\pi\lambda_1}{\lambda}\right)\left(\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_\alpha(\xi,m)}\right)^{\lambda_2}.\end{aligned}$$

Hence, (11) and (12) are valid. □

In the same way, for $0 < \lambda \leq 1$, $0 < \lambda_1 < 1$, we still have the following.

Lemma 3 *The following inequalities are valid*:

$$H_\alpha(\lambda_1)\big(1-\widetilde{\theta}(\lambda_1,n)\big) < \varpi(\lambda_1,n) < H_\alpha(\lambda_1), \quad |n| \in \mathbf{N}\backslash\{1\}, \tag{14}$$

where

$$\begin{aligned}\widetilde{\theta}(\lambda_1,n) &:= \frac{\lambda}{\pi}\sin\left(\frac{\pi\lambda_1}{\lambda}\right)\int_0^{\frac{\ln[(2+\xi)(1+\cos\alpha)]}{\ln B_\beta(\eta,n)}}\frac{u^{\lambda_1-1}}{1+u^\lambda}\,du \\ &= O\left(\frac{1}{\ln^{\lambda_1}B_\beta(\eta,n)}\right)\in(0,1). \end{aligned} \tag{15}$$

Lemma 4 *If* $(\zeta,\gamma)=(\xi,\alpha)$ (*or* (η,β)), $\rho>0$, *then we have*

$$\begin{aligned}h_\rho(\zeta,\gamma) &:= \sum_{|n|=2}^{\infty}\frac{\ln^{-1-\rho}[|n-\zeta|+(n-\zeta)\cos\gamma]}{[|n-\zeta|+(n-\zeta)\cos\gamma]} \\ &= \frac{1}{\rho}\big(2\csc^2\gamma+o(1)\big)\quad\big(\rho\to0^+\big).\end{aligned} \tag{16}$$

Proof Still by (10) we find

$$\begin{aligned}h_\rho(\zeta,\gamma) &= \sum_{n=-2}^{-\infty}\frac{\ln^{-1-\rho}[(n-\zeta)(\cos\gamma-1)]}{(n-\zeta)(\cos\gamma-1)} \\ &\quad+\sum_{n=2}^{\infty}\frac{\ln^{-1-\rho}[(n-\zeta)(\cos\gamma+1)]}{(n-\zeta)(\cos\gamma+1)} \\ &= \sum_{n=2}^{\infty}\left\{\frac{\ln^{-1-\rho}[(n+\zeta)(1-\cos\gamma)]}{(n+\zeta)(1-\cos\gamma)}+\frac{\ln^{-1-\rho}[(n-\zeta)(\cos\gamma+1)]}{(n-\zeta)(\cos\gamma+1)}\right\} \\ &\le \int_{\frac{3}{2}}^{\infty}\left\{\frac{\ln^{-1-\rho}[(y+\zeta)(1-\cos\gamma)]}{(y+\zeta)(1-\cos\gamma)}+\frac{\ln^{-1-\rho}[(y-\zeta)(\cos\gamma+1)]}{(y-\zeta)(\cos\gamma+1)}\right\}dy \\ &= \frac{1}{\rho}\left\{\frac{\ln^{-\rho}[(\frac{3}{2}+\zeta)(1-\cos\gamma)]}{1-\cos\gamma}+\frac{\ln^{-\rho}[(\frac{3}{2}-\zeta)(1+\cos\gamma)]}{1+\cos\gamma}\right\} \\ &= \frac{1}{\rho}\left\{\frac{1}{1-\cos\gamma}+\frac{1}{1+\cos\gamma}+o_1(1)\right\}\quad\big(\rho\to0^+\big).\end{aligned}$$

By (9), we still can find that

$$\begin{aligned}&h_\rho(\zeta,\gamma) \\ &\quad= \sum_{n=2}^{\infty}\left\{\frac{\ln^{-1-\rho}[(n+\zeta)(1-\cos\gamma)]}{(n+\zeta)(1-\cos\gamma)}+\frac{\ln^{-1-\rho}[(n-\zeta)(\cos\gamma+1)]}{(n-\zeta)(\cos\gamma+1)}\right\} \\ &\quad\ge \int_{2}^{\infty}\left\{\frac{\ln^{-1-\rho}[(y+\zeta)(1-\cos\gamma)]}{(y+\zeta)(1-\cos\gamma)}+\frac{\ln^{-1-\rho}[(y-\zeta)(\cos\gamma+1)]}{(y-\zeta)(\cos\gamma+1)}\right\}dy \\ &\quad= \frac{1}{\rho}\left\{\frac{\ln^{-\rho}[(2+\zeta)(1-\cos\gamma)]}{1-\cos\gamma}+\frac{\ln^{-\rho}[(2-\zeta)(1+\cos\gamma)]}{1+\cos\gamma}\right\} \\ &\quad= \frac{1}{\rho}\left\{\frac{1}{1-\cos\gamma}+\frac{1}{1+\cos\gamma}+o_2(1)\right\}\quad\big(\rho\to0^+\big).\end{aligned}$$

Hence, we prove that (16) is valid. □

3 Main results and a few particular cases

We also define

$$H(\lambda_1) := H_\beta^{1/p}(\lambda_1) H_\alpha^{1/q}(\lambda_1) = \frac{2\pi \csc^{2/p}\beta \csc^{2/q}\alpha}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})}. \tag{17}$$

Theorem 1 *Suppose that* $a_m, b_n \geq 0$ $(|m|, |n| \in \mathbf{N}\backslash\{1\})$ *and*

$$0 < \sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} A_\alpha(\xi, m)}{(A_\alpha(\xi, m))^{1-p}} a_m^p < \infty,$$

$$0 < \sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_\beta(\eta, n)}{(B_\beta(\eta, n))^{1-q}} b_n^q < \infty.$$

We have the following reverse equivalent inequalities:

$$\begin{aligned} I &:= \sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} \frac{1}{\ln^\lambda A_\alpha(\xi, m) + \ln^\lambda B_\beta(\eta, n)} a_m b_n \\ &> H(\lambda_1) \left[\sum_{|m|=2}^{\infty} \bigl(1 - \theta(\lambda_2, m)\bigr) \frac{\ln^{p(1-\lambda_1)-1} A_\alpha(\xi, m)}{(A_\alpha(\xi, m))^{1-p}} a_m^p \right]^{\frac{1}{p}} \\ &\quad \times \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_\beta(\eta, n)}{(B_\beta(\eta, n))^{1-q}} b_n^q \right]^{\frac{1}{q}}, \end{aligned} \tag{18}$$

$$\begin{aligned} J &:= \left[\sum_{|n|=2}^{\infty} \frac{\ln^{p\lambda_2-1} B_\beta(\eta, n)}{B_\beta(\eta, n)} \left(\sum_{|m|=2}^{\infty} \frac{a_m}{\ln^\lambda A_\alpha(\xi, m) + \ln^\lambda B_\beta(\eta, n)} \right)^p \right]^{\frac{1}{p}} \\ &> H(\lambda_1) \left[\sum_{|m|=2}^{\infty} \bigl(1 - \theta(\lambda_2, m)\bigr) \frac{\ln^{p(1-\lambda_1)-1} A_\alpha(\xi, m)}{(A_\alpha(\xi, m))^{1-p}} a_m^p \right]^{1/p}, \end{aligned} \tag{19}$$

$$\begin{aligned} L &:= \left[\sum_{|m|=2}^{\infty} \frac{\ln^{q\lambda_1-1} A_\alpha(\xi, m)}{(1 - \theta(\lambda_2, m))^{q-1} A_\alpha(\xi, m)} \right. \\ &\quad \left. \times \left(\sum_{|n|=2}^{\infty} \frac{1}{\ln^\lambda A_\alpha(\xi, m) + \ln^\lambda B_\beta(\eta, n)} b_n \right)^q \right]^{\frac{1}{q}} \\ &> H(\lambda_1) \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_\beta(\eta, n)}{(B_\beta(\eta, n))^{1-q}} b_n^q \right]^{1/q}. \end{aligned} \tag{20}$$

In particular, (i) *for* $\alpha = \beta = \frac{\pi}{2}$, $\xi, \eta \in [0, \frac{1}{2}]$, *setting*

$$\begin{aligned} \theta_1(\lambda_2, m) &:= \frac{\lambda}{\pi} \sin\left(\frac{\pi\lambda_1}{\lambda}\right) \int_0^{\frac{\ln(2+\eta)}{\ln|m-\xi|}} \frac{u^{\lambda_2-1}}{1 + u^\lambda} \, du \\ &= O\left(\frac{1}{\ln^{\lambda_2}|m - \xi|}\right) \in (0, 1), \end{aligned} \tag{21}$$

we have the following reverse equivalent inequalities:

$$\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}\frac{1}{\ln^{\lambda}|m-\xi|+\ln^{\lambda}|n-\eta|}a_m b_n$$
$$>\frac{2\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}\left[\sum_{|m|=2}^{\infty}\big(1-\theta_1(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-1}|m-\xi|}{|m-\xi|^{1-p}}a_m^p\right]^{\frac{1}{p}}$$
$$\times\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}|n-\eta|}{|n-\eta|^{1-q}}b_n^q\right]^{\frac{1}{q}}, \tag{22}$$

$$\left\{\sum_{|n|=2}^{\infty}\frac{\ln^{p\lambda_2-1}|n-\eta|}{|n-\eta|}\left[\sum_{|m|=2}^{\infty}\frac{a_m}{\ln^{\lambda}|m-\xi|+\ln^{\lambda}|n-\eta|}\right]^p\right\}^{\frac{1}{p}}$$
$$>\frac{2\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}\left[\sum_{|m|=2}^{\infty}\big(1-\theta_1(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-1}|m-\xi|}{|m-\xi|^{1-p}}a_m^p\right]^{\frac{1}{p}}, \tag{23}$$

$$\left[\sum_{|m|=2}^{\infty}\frac{\ln^{q\lambda_1-1}|m-\xi|}{(1-\theta_1(\lambda_2,m))^{q-1}|m-\xi|}\left(\sum_{|n|=2}^{\infty}\frac{1}{\ln^{\lambda}|m-\xi|+\ln^{\lambda}|n-\eta|}b_n\right)^q\right]^{\frac{1}{q}}$$
$$>\frac{2\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}|n-\eta|}{|n-\eta|^{1-q}}b_n^q\right]^{\frac{1}{q}}. \tag{24}$$

(ii) *For* $\xi,\eta=0$, $\alpha,\beta\in[\arccos\frac{1}{3},\frac{\pi}{2}]$, *setting*

$$\theta_2(\lambda_2,m):=\frac{\lambda}{\pi}\sin\left(\frac{\pi\lambda_1}{\lambda}\right)\int_0^{\frac{\ln 2(1+\cos\beta)}{\ln(|m|+m\cos\alpha)}}\frac{u^{\lambda_2-1}}{1+u^{\lambda}}\,du$$
$$=O\left(\frac{1}{\ln^{\lambda_2}(|m|+m\cos\alpha)}\right)\in(0,1), \tag{25}$$

we have the following equivalent inequalities:

$$\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}\frac{a_m b_n}{\ln^{\lambda}(|m|+m\cos\alpha)+\ln^{\lambda}(|n|+n\cos\beta)}$$
$$>H(\lambda_1)\left[\sum_{|m|=2}^{\infty}\big(1-\theta_2(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-1}(|m|+m\cos\alpha)}{(|m|+m\cos\alpha)^{1-p}}a_m^p\right]^{1/p}$$
$$\times\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}(|n|+n\cos\beta)}{(|n|+n\cos\beta)^{1-q}}b_n^q\right]^{1/q}, \tag{26}$$

$$\left\{\sum_{|n|=2}^{\infty}\frac{\ln^{p\lambda_2-1}(|n|+n\cos\beta)}{|n|+n\cos\beta}\left[\sum_{|m|=2}^{\infty}\frac{a_m}{\ln^{\lambda}(|m|+m\cos\alpha)+\ln^{\lambda}(|n|+n\cos\beta)}\right]^p\right\}^{\frac{1}{p}}$$
$$>H(\lambda_1)\left[\sum_{|m|=2}^{\infty}\big(1-\theta_2(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-1}(|m|+m\cos\alpha)}{(|m|+m\cos\alpha)^{1-p}}a_m^p\right]^{\frac{1}{p}}, \tag{27}$$

$$\Bigg[\sum_{|m|=2}^{\infty}\frac{\ln^{q\lambda_1-1}(|m|+m\cos\alpha)}{(1-\theta_2(\lambda_2,m))^{q-1}(|m|+m\cos\alpha)}$$
$$\times\left(\sum_{|n|=2}^{\infty}\frac{1}{\ln^{\lambda}(|m|+m\cos\alpha)+\ln^{\lambda}(|n|+n\cos\beta)}b_n\right)^q\Bigg]^{\frac{1}{q}}$$
$$> H(\lambda_1)\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}(|n|+n\cos\beta)}{(|n|+n\cos\beta)^{1-q}}b_n^q\right]^{\frac{1}{q}}. \tag{28}$$

Proof By the reverse Hölder inequality with weight (cf. [28]) and (8), we find

$$\left(\sum_{|m|=2}^{\infty}H(m,n)a_m\right)^p$$
$$=\Bigg\{\sum_{|m|=2}^{\infty}H(m,n)\left[\frac{(A_\alpha(\xi,m))^{1/q}\ln^{(1-\lambda_1)/q}A_\alpha(\xi,m)}{\ln^{(1-\lambda_2)/p}B_\beta(\eta,n)}a_m\right]$$
$$\times\left[\frac{\ln^{(1-\lambda_2)/p}B_\beta(\eta,n)}{(A_\alpha(\xi,m))^{1/q}\ln^{(1-\lambda_1)/q}A_\alpha(\xi,m)}\right]\Bigg\}^p$$
$$\geq\sum_{|m|=2}^{\infty}H(m,n)\frac{(A_\alpha(\xi,m))^{p/q}\ln^{(1-\lambda_1)p/q}A_\alpha(\xi,m)}{\ln^{1-\lambda_2}B_\beta(\eta,n)}a_m^p$$
$$\times\left[\sum_{|m|=2}^{\infty}H(m,n)\frac{\ln^{(1-\lambda_2)q/p}B_\beta(\eta,n)}{A_\alpha(\xi,m)\ln^{1-\lambda_1}A_\alpha(\xi,m)}\right]^{p-1}$$
$$=\frac{(\varpi(\lambda_1,n))^{p-1}B_\beta(\eta,n)}{\ln^{p\lambda_2-1}B_\beta(\eta,n)}\sum_{|m|=2}^{\infty}\frac{H(m,n)(A_\alpha(\xi,m))^{\frac{p}{q}}\ln^{(1-\lambda_1)\frac{p}{q}}A_\alpha(\xi,m)}{B_\beta(\eta,n)\ln^{1-\lambda_2}B_\beta(\eta,n)}a_m^p.$$

By (14), in view of $p-1<0$, it follows that

$$J > H_\alpha^{1/q}(\lambda_1)\left[\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}\frac{H(m,n)(A_\alpha(\xi,m))^{p/q}\ln^{(1-\lambda_1)p/q}A_\alpha(\xi,m)}{B_\beta(\eta,n)\ln^{1-\lambda_2}B_\beta(\eta,n)}a_m^p\right]^{\frac{1}{p}}$$
$$= H_\alpha^{1/q}(\lambda_1)\left[\sum_{|m|=2}^{\infty}\sum_{|n|=2}^{\infty}\frac{H(m,n)(A_\alpha(\xi,m))^{p/q}\ln^{(1-\lambda_1)p/q}A_\alpha(\xi,m)}{B_\beta(\eta,n)\ln^{1-\lambda_2}B_\beta(\eta,n)}a_m^p\right]^{\frac{1}{p}}$$
$$= H_\alpha^{1/q}(\lambda_1)\left[\sum_{|m|=2}^{\infty}\omega(\lambda_2,m)\frac{\ln^{p(1-\lambda_1)-1}A_\alpha(\xi,m)}{(A_\alpha(\xi,m))^{1-p}}a_m^p\right]^{\frac{1}{p}}. \tag{29}$$

By (11) and (17), we have (19).

Using the reverse Hölder inequality again, we have

$$I=\sum_{|n|=2}^{\infty}\left[\frac{\ln^{\lambda_2-(1/p)}B_\beta(\eta,n)}{(B_\beta(\eta,n))^{1/p}}\sum_{|m|=2}^{\infty}H(m,n)a_m\right]\left[\frac{\ln^{(1/p)-\lambda_2}B_\beta(\eta,n)}{(B_\beta(\eta,n))^{-1/p}}b_n\right]$$
$$\geq J\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}B_\beta(\eta,n)}{(B_\beta(\eta,n))^{1-q}}b_n^q\right]^{\frac{1}{q}}, \tag{30}$$

and then by (19) we have (18).

On the other hand, assuming that (18) is valid, we set

$$b_n := \frac{\ln^{p\lambda_2-1} B_\beta(\eta,n)}{B_\beta(\eta,n)} \left(\sum_{|m|=2}^{\infty} H(m,n) a_m \right)^{p-1}, \quad |n| \in \mathbf{N}\backslash\{1\},$$

and find

$$J = \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_\beta(\eta,n)}{(B_\beta(\eta,n))^{1-q}} b_n^q \right]^{\frac{1}{p}}.$$

By (29) it follows that $J > 0$. If $J = \infty$, then (19) is trivially valid; if $J < \infty$, then we have

$$\begin{aligned}
&\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_\beta(\eta,n)}{(B_\beta(\eta,n))^{1-q}} b_n^q \\
&\quad = J^p = I \\
&\quad > H(\lambda_1) \left[\sum_{|m|=2}^{\infty} \left(1-\theta(\lambda_2,m)\right) \frac{\ln^{p(1-\lambda_1)-1} A_\alpha(\xi,m)}{(A_\alpha(\xi,m))^{1-p}} a_m^p \right]^{\frac{1}{p}} \\
&\qquad \times \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_\beta(\eta,n)}{(B_\beta(\eta,n))^{1-q}} b_n^q \right]^{\frac{1}{q}}, \\
&J = \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_\beta(\eta,n)}{(B_\beta(\eta,n))^{1-q}} b_n^q \right]^{\frac{1}{p}} \\
&\quad > H(\lambda_1) \left[\sum_{|m|=2}^{\infty} \left(1-\theta(\lambda_2,m)\right) \frac{\ln^{p(1-\lambda_1)-1} A_\alpha(\xi,m)}{(A_\alpha(\xi,m))^{1-p}} a_m^p \right]^{\frac{1}{p}}.
\end{aligned}$$

Hence (19) is valid, which is equivalent to (18).

We have proved that (18) is valid. Then we set

$$a_m := \frac{\ln^{q\lambda_1-1} A_\alpha(\xi,m)}{(1-\theta(\lambda_2,m))^{q-1} A_\alpha(\xi,m)} \left(\sum_{|n|=2}^{\infty} H(m,n) b_n \right)^{q-1}, \quad |m| \in \mathbf{N}\backslash\{1\},$$

and find

$$L = \left[\sum_{|m|=2}^{\infty} \left(1-\theta(\lambda_2,m)\right) \frac{\ln^{p(1-\lambda_1)-1} A_\alpha(\xi,m)}{(A_\alpha(\xi,m))^{1-p}} a_m^p \right]^{\frac{1}{q}}.$$

If $L = 0$, then (20) is impossible, namely $L > 0$. If $L = \infty$, then (20) is trivially valid; if $L < \infty$, then we have

$$\begin{aligned}
&\sum_{|m|=2}^{\infty}\bigl(1-\theta(\lambda_2,m)\bigr)\frac{\ln^{p(1-\lambda_1)-1}A_\alpha(\xi,m)}{(A_\alpha(\xi,m))^{1-p}}a_m^p \\
&\quad = L^q = I \\
&\quad > H(\lambda_1)\Biggl[\sum_{|m|=2}^{\infty}\bigl(1-\theta(\lambda_2,m)\bigr)\frac{\ln^{p(1-\lambda_1)-1}A_\alpha(\xi,m)}{(A_\alpha(\xi,m))^{1-p}}a_m^p\Biggr]^{\frac{1}{p}} \\
&\quad\quad \times \Biggl[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}B_\beta(\eta,n)}{(B_\beta(\eta,n))^{1-q}}b_n^q\Biggr]^{\frac{1}{q}}, \\
&L = \Biggl[\sum_{|m|=2}^{\infty}\bigl(1-\theta(\lambda_2,m)\bigr)\frac{\ln^{p(1-\lambda_1)-1}A_\alpha(\xi,m)}{(A_\alpha(\xi,m))^{1-p}}a_m^p\Biggr]^{\frac{1}{q}} \\
&\quad > H(\lambda_1)\Biggl[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}B_\beta(\eta,n)}{(B_\beta(\eta,n))^{1-q}}b_n^q\Biggr]^{\frac{1}{q}}.
\end{aligned}$$

Hence, (20) is valid.

On the other hand, assuming that (20) is valid, using the reverse Hölder inequality, we have

$$\begin{aligned}
I &= \sum_{|m|=2}^{\infty}\biggl[\frac{\ln^{(1/q)-\lambda_1}A_\alpha(\xi,m)}{(1-\theta(\lambda_2,m))^{-1/p}(A_\alpha(\xi,m))^{-1/q}}a_m\biggr] \\
&\quad \times \biggl[\frac{\ln^{\lambda_1-(1/q)}A_\alpha(\xi,m)}{(1-\theta(\lambda_2,m))^{1/p}(A_\alpha(\xi,m))^{1/q}}\sum_{|n|=2}^{\infty}H(m,n)b_n\biggr] \\
&\geq \Biggl[\sum_{|m|=2}^{\infty}\bigl(1-\theta(\lambda_2,m)\bigr)\frac{\ln^{p(1-\lambda_1)-1}A_\alpha(\xi,m)}{(A_\alpha(\xi,m))^{1-p}}a_m^p\Biggr]^{\frac{1}{p}}L, \qquad (31)
\end{aligned}$$

and then by (20) we have (18), which is equivalent to (20).

Therefore, inequalities (18), (19), and (20) are equivalent.

The theorem is proved. □

Theorem 2 *With regards to the assumptions of Theorem* 1, *the constant factor* $H(\lambda_1)$ *in* (18), (19), *and* (20) *is the best possible.*

Proof For $0 < \varepsilon < p\lambda_1$, we set $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$ ($\in (0,1)$), $\widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$ (> 0), and

$$\begin{aligned}
\widetilde{a}_m &:= \frac{\ln^{\lambda_1-(\varepsilon/p)-1}A_\alpha(\xi,m)}{A_\alpha(\xi,m)} = \frac{\ln^{\widetilde{\lambda}_1-1}A_\alpha(\xi,m)}{A_\alpha(\xi,m)} \quad \bigl(|m| \in \mathbf{N}\backslash\{1\}\bigr), \\
\widetilde{b}_n &:= \frac{\ln^{\lambda_2-(\varepsilon/q)-1}B_\beta(\eta,n)}{B_\beta(\eta,n)} = \frac{\ln^{\widetilde{\lambda}_2-\varepsilon-1}B_\beta(\eta,n)}{B_\beta(\eta,n)} \quad \bigl(|n| \in \mathbf{N}\backslash\{1\}\bigr).
\end{aligned}$$

By (16) and (14) we find

$$\begin{aligned}
\widetilde{I}_1 &:= \left[\sum_{|m|=2}^{\infty}\bigl(1-\theta(\lambda_2,m)\bigr)\frac{\ln^{p(1-\lambda_1)-1}A_\alpha(\xi,m)}{(A_\alpha(\xi,m))^{1-p}}\widetilde{a}_m^p\right]^{\frac{1}{p}} \\
&\quad\times\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}B_\beta(\eta,n)}{(B_\beta(\eta,n))^{1-q}}\widetilde{b}_n^q\right]^{\frac{1}{q}} \\
&= \left[\sum_{|m|=2}^{\infty}\frac{\ln^{-1-\varepsilon}A_\alpha(\xi,m)}{A_\alpha(\xi,m)}-\sum_{|m|=2}^{\infty}\frac{O(\ln^{-1-(\lambda_2+\varepsilon)}A_\alpha(\xi,m))}{A_\alpha(\xi,m)}\right]^{\frac{1}{p}} \\
&\quad\times\left[\sum_{|n|=2}^{\infty}\frac{\ln^{-1-\varepsilon}B_\beta(\eta,n)}{B_\beta(\eta,n)}\right]^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\bigl(2\csc^2\alpha+o(1)-\varepsilon O(1)\bigr)^{1/p}\bigl(2\csc^2\beta+\widetilde{o}(1)\bigr)^{1/q}\quad\bigl(\varepsilon\to 0^+\bigr), \\
\widetilde{I} &:= \sum_{|m|=2}^{\infty}\sum_{|n|=2}^{\infty}H(m,n)\widetilde{a}_m\widetilde{b}_n \\
&= \sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}H(m,n)\frac{\ln^{\widetilde{\lambda}_1-1}A_\alpha(\xi,m)\ln^{\widetilde{\lambda}_2-\varepsilon-1}B_\beta(\eta,n)}{A_\alpha(\xi,m)B_\beta(\eta,n)} \\
&= \sum_{|n|=2}^{\infty}\varpi(\widetilde{\lambda}_1,n)\frac{\ln^{-1-\varepsilon}B_\beta(\eta,n)}{B_\beta(\eta,n)} < H_\alpha(\widetilde{\lambda}_1)\sum_{|n|=2}^{\infty}\frac{\ln^{-1-\varepsilon}B_\beta(\eta,n)}{B_\beta(\eta,n)} \\
&= \frac{1}{\varepsilon}H_\alpha\left(\lambda_1-\frac{\varepsilon}{p}\right)\bigl(2\csc^2\beta+\widetilde{o}(1)\bigr).
\end{aligned}$$

If there exists a positive number $K \geq H(\lambda_1)$ such that (18) is still valid when replacing $H(\lambda_1)$ by K, then, in particular, we have

$$\varepsilon\widetilde{I} = \varepsilon\sum_{|m|=2}^{\infty}\sum_{|n|=2}^{\infty}H(m,n)\widetilde{a}_m\widetilde{b}_n > \varepsilon K\widetilde{I}_1.$$

In view of the above results, it follows that

$$H_\alpha\left(\lambda_1-\frac{\varepsilon}{p}\right)\bigl(2\csc^2\beta+\widetilde{o}(1)\bigr) > K\cdot\bigl(2\csc^2\alpha+o(1)-\varepsilon O(1)\bigr)^{1/p}\bigl(2\csc^2\beta+\widetilde{o}(1)\bigr)^{1/q},$$

and then

$$\frac{4\pi\csc^2\alpha}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}\csc^2\beta \geq 2K\csc^{2/p}\alpha\csc^{2/q}\beta\quad\bigl(\varepsilon\to 0^+\bigr),$$

namely

$$H(\lambda_1) = \frac{2\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}\csc^{2/p}\beta\csc^{2/q}\alpha \geq K.$$

Hence, $K = H(\lambda_1)$ is the best possible constant factor in (18).

The constant factor $H(\lambda_1)$ in (19) ((20)) is still the best possible. Otherwise we would reach a contradiction by (30) ((31)) that the constant factor in (18) is not the best possible. □

Remark 2 (i) For $\xi = \eta = 0$ in (22), setting

$$\begin{aligned}\widetilde{\theta}_1(\lambda_2, m) &:= \frac{\lambda}{\pi} \sin\left(\frac{\pi\lambda_1}{\lambda}\right) \int_0^{\frac{\ln 2}{\ln|m|}} \frac{u^{\lambda_2 - 1}}{1 + u^\lambda} \, du \\ &= O\left(\frac{1}{\ln^{\lambda_2}|m|}\right) \in (0,1),\end{aligned}$$

we have the following new inequality with the best possible constant factor $\frac{2\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})}$:

$$\begin{aligned}&\sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} \frac{a_m b_n}{\ln^\lambda |m| + \ln^\lambda |n|} \\ &\quad > \frac{2\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})} \left[\sum_{|m|=2}^{\infty} \left(1 - \widetilde{\theta}_1(\lambda_2, m)\right) \frac{\ln^{p(1-\lambda_1)-1}|m|}{|m|^{1-p}} a_m^p\right]^{\frac{1}{p}} \\ &\quad \times \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1}|n|}{|n|^{1-q}} b_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{32}$$

It follows that (20) ((22)) is an extension of (29).

(ii) If $a_{-m} = a_m$ and $b_{-n} = b_n$ ($m, n \in \mathbf{N} \backslash \{1\}$), setting

$$\theta_2(\lambda_2, m) := \frac{\lambda}{\pi} \sin\left(\frac{\pi\lambda_1}{\lambda}\right) \int_0^{\frac{\ln(2+\eta)}{\ln(m-\xi)}} \frac{u^{\lambda_2 - 1}}{1 + u^\lambda} \, du = O\left(\frac{1}{\ln^{\lambda_2}(m - \xi)}\right) \in (0,1),$$

$$\theta_3(\lambda_2, m) := \frac{\lambda}{\pi} \sin\left(\frac{\pi\lambda_1}{\lambda}\right) \int_0^{\frac{\ln(2+\eta)}{\ln(m+\xi)}} \frac{u^{\lambda_2 - 1}}{1 + u^\lambda} \, du = O\left(\frac{1}{\ln^{\lambda_2}(m + \xi)}\right) \in (0,1),$$

then (22) reduces to

$$\begin{aligned}&\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \left[\frac{1}{\ln^\lambda(m-\xi) + \ln^\lambda(n-\eta)} + \frac{1}{\ln^\lambda(m-\xi) + \ln^\lambda(n+\eta)}\right. \\ &\quad \left. + \frac{1}{\ln^\lambda(m+\xi) + \ln^\lambda(n-\eta)} + \frac{1}{\ln^\lambda(m+\xi) + \ln^\lambda(n+\eta)}\right] a_m b_n \\ &\quad > \frac{2\pi}{\lambda \sin(\pi\lambda_1/\lambda)} \left\{\sum_{m=2}^{\infty} \left[\left(1 - \theta_2(\lambda_2, m)\right) \frac{\ln^{p(1-\lambda_1)-1}(m-\xi)}{(m-\xi)^{1-p}}\right.\right. \\ &\quad \left.\left. + \left(1 - \theta_3(\lambda_2, m)\right) \frac{\ln^{p(1-\lambda_1)-1}(m+\xi)}{(m+\xi)^{1-p}}\right] a_m^p\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\sum_{n=2}^{\infty} \left[\frac{\ln^{q(1-\lambda_2)-1}(n-\eta)}{(n-\eta)^{1-q}} + \frac{\ln^{q(1-\lambda_2)-1}(n+\eta)}{(n+\eta)^{1-q}}\right] b_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{33}$$

In particular, for $\xi = \eta = 0$, $\lambda = 1$, $\lambda_1 = \lambda_2 = \frac{1}{2}$ in (33), setting

$$\theta(m) := \frac{1}{\pi}\int_0^{\frac{\ln 2}{\ln m}} \frac{u^{-1/2}}{1+u}\,du = O\left(\frac{1}{\ln^{1/2} m}\right) \in (0,1),$$

we have the following reverse Mulholland inequality (2) with the best possible constant π:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln mn} > \pi\left[\sum_{m=2}^{\infty}\bigl(1-\theta(m)\bigr)\frac{\ln^{\frac{p}{2}-1} m}{m^{1-p}}a_m^p\right]^{\frac{1}{p}} \times\left(\sum_{n=2}^{\infty}\frac{\ln^{\frac{q}{2}-1} n}{n^{1-q}}b_n^q\right)^{\frac{1}{q}}. \tag{34}$$

4 Conclusions

In this paper, by introducing multi-parameters, applying the weight coefficients, and using Hermite–Hadamard's inequality, we give a reverse of the extension of Mulholland's inequality in the whole plane with the best possible constant factor in Theorems 1–2. The equivalent forms and a few particular cases are considered. The technique of real analysis is very important as it is the key to proving the reverse equivalent inequalities with the best possible constant factor. The lemmas and theorems provide an extensive account of this type of inequalities.

Acknowledgements
This work is supported by the National Natural Science Foundation (Nos. 61370186, 61640222), and the Science and Technology Planning Project Item of Guangzhou City (No. 201707010229). We are grateful for this help.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. AW participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

Publisher's Note

Received: 23 December 2017 Accepted: 2 February 2018 Published online: 14 February 2018

References

1. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge University Press, Cambridge (1934)
2. Mitrinović, D.S., Pečarić, J.E., Fink, A.M.: Inequalities Involving Functions and Their Integrals and Derivatives. Kluwer Academic, Boston (1991)
3. Yang, B.: A new Hilbert's type integral inequality. Soochow J. Math. **33**(4), 849–859 (2007)
4. Gao, M., Yang, B.: On the extended Hilbert's inequality. Proc. Am. Math. Soc. **126**(3), 751–759 (1998)
5. Hong, Y.: All-sided generalization about Hardy–Hilbert integral inequalities. Acta Math. Sin. **44**(4), 619–626 (2001)
6. Laith, E.A.: On some extensions of Hardy–Hilbert's inequality and applications. J. Inequal. Appl. **2008**, Article ID 546828 (2008)
7. Krnić, M., Pečarić, J.E.: General Hilbert's and Hardy's inequalities. Math. Inequal. Appl. **8**(1), 29–51 (2005)
8. Perić, I., Vuković, P.: Multiple Hilbert's type inequalities with a homogeneous kernel. Banach J. Math. Anal. **5**(2), 33–43 (2011)
9. Rassias, M.T., Yang, B.: A Hilbert-type integral inequality in the whole plane related to the hyper geometric function and the beta function. J. Math. Anal. Appl. **428**(2), 1286–1308 (2015)
10. Vandanjav, A., Tserendorj, B., Krnić, M.: Multiple Hilbert-type inequalities involving some differential operators. Banach J. Math. Anal. **10**(2), 320–337 (2016)
11. Yang, B.: On a more accurate Hardy–Hilbert's type inequality and its applications. Acta Math. Sin. **49**(2), 363–368 (2006)
12. Li, Y., He, B.: On inequalities of Hilbert's type. Bull. Aust. Math. Soc. **76**(1), 1–13 (2007)

13. Zhong, W.: A Hilbert-type linear operator with the norm and its applications. J. Inequal. Appl. **2009**, Article ID 494257 (2009)
14. Huang, Q.: On a multiple Hilbert's inequality with parameters. J. Inequal. Appl. **2010**, Article ID 309319 (2010)
15. Krnić, M., Vukovic, P.: On a multidimensional version of the Hilbert type inequality. Anal. Math. **38**(4), 291–303 (2012)
16. Huang, Q.: A new extension of Hardy–Hilbert-type inequality. J. Inequal. Appl. **2015**, 397 (2015)
17. Huang, Q., Yang, B.: A more accurate half-discrete Hilbert inequality with a nonhomogeneous kernel. J. Funct. Spaces Appl. **2013**, Article ID 628250 (2013)
18. Huang, Q., Wang, A., Yang, B.: A more accurate half-discrete Hilbert-type inequality with a general nonhomogeneous kernel and operator expressions. Math. Inequal. Appl. **17**(1), 367–388 (2014)
19. Huang, Q., Wu, S., Yang, B.: Parameterized Hilbert-type integral inequalities in the whole plane. Sci. World J. **2014**, Article ID 169061 (2014)
20. Rassias, M.T., Yang, B.: On a multidimensional Hilbert-type integral inequality associated to the gamma function. Appl. Math. Comput. **249**, 408–418 (2014)
21. Yang, B., Chen, Q.: A more accurate multidimensional Hardy–Hilbert type inequality with a general homogeneous kernel. J. Math. Inequal. **12**(1), 113–128 (2018)
22. Batbold, T., Sawano, Y.: Sharp bounds for m-linear Hilbert-type operators on the weighted Morrey spaces. Math. Inequal. Appl. **20**, 263–283 (2017)
23. Garayev, M.T., Gurudal, M., Okudan, A.: Hardy–Hilbert's inequality and power inequalities for Berezin numbers of operators. Math. Inequal. Appl. **19**, 883–891 (2016)
24. He, B., Wang, Q.: A multiple Hilbert-type discrete inequality with a new kernel and best possible constant factor. J. Math. Anal. Appl. **431**(2), 889–902 (2015)
25. Yang, B., Chen, Q.: A new extension of Hardy–Hilbert's inequality in the whole plane. J. Funct. Spaces **2016**, Article ID 9197476 (2016)
26. Xin, D., Yang, B., Chen, Q.: A discrete Hilbert-type inequality in the whole plane. J. Inequal. Appl. **2016**, 133 (2016)
27. Chen, Q., Yang, B.: On a more accurate half-discrete Hardy–Hilbert-type inequality related to the kernel of arc tangent function. SpringerPlus **5**, 1317 (2016)
28. Kuang, J.: Applied Inequalities. Shangdong Science Technic Press, Jinan (2010) (in Chinese)