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The bounds estimate of sub-band operators for multi-band wavelets

Qingyun Zou[1*], Guoqiu Wang[2] and Qian Cao[1,2]

*Correspondence:
qyzou2000@sina.com
[1]Hunan Province Cooperative
Innovation Center for The
Construction Development of
Dongting Lake Ecological Economic
Zone, Hunan University of Arts and
Science, Changde, P.R. China
Full list of author information is
available at the end of the article

Abstract

A concept of the sub-band operator of multi-band wavelets is introduced, the theory of d-circular matrices is developed and the upper bound and the lower bound of the norm of the sub-band operator are obtained. Examples are provided to illustrate the results proposed in this paper.

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1 Introduction

During the last three decades, the theory of frames, which generalize the notion of bases by allowing redundancy yet still providing a reconstruction formula, has been growing rapidly, since several new applications such as nonlinear sparse approximation (e.g., image compression), coarse quantization, data transmission with erasures, and wireless communication, have been developed [1–7]. As a special class of frames, the multi-band wavelets have attracted considerable attention due to their richer parameter space, to give better energy compaction than 2-band wavelets [8–16].

Let H be a separable Hilbert space, and E an indexing set. A sequence $\{f_l\}_{l \in E}$ is called a frame in H if there exist constants $0 < A \leq B < +\infty$ such that

$$A\|f\|^2 \leq \sum_{l \in E} \left|\langle f, f_l \rangle\right|^2 \leq B\|f\|^2, \quad \forall f \in H, \tag{1.1}$$

where A and B are called lower and upper frame bounds, respectively. If $A = B$, the frame is called tight frame.

When $H = L^2(R)$, a wavelet $\psi(x) \in H$ gives rise to a classical wavelet frame $\{a^{-j/2}\psi(a^{-j}x - bk), j, k \in Z\}$ with parameters $a > 1$, $b > 0$. Chui and Shi [17] established the relationship between the parameters and the frame bounds,

$$A \leq \frac{1}{2b \ln a} \int_{-\infty}^{+\infty} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|} \, d\omega \leq B. \tag{1.2}$$

Equation (1.2) shows that the energy of a biorthogonal wavelet transform is controllable although it is not conservative. Moreover, $B - A$ or $\widetilde{B} - \widetilde{A}$ are smaller, the performance of

a biorthogonal wavelet transform may be better, i.e., the energy is amplified in some cases and decreased in other cases. The classical biorthogonal wavelets are not tight frames, so obtaining the exact values of their bounds is difficult. Instead of estimating the bounds in (1.2), we can try to obtain the upper bound and the lower bound of the norm of the sub-band operator.

This paper is organized as follows. In Sect. 2, we define the sub-band operator, and obtain the limit form of the norm of the sub-band operator. We present a method for computing the upper bound and the lower bound of the norm of the sub-band operator based on the theory of circular matrix. Section 3 gives some examples to illustrate the results proposed in this paper.

2 Sub-band operator and d-circular matrix

Recall the sub-band coding scheme or Mallat algorithm associated to a d-band real biorthogonal wavelets. There are $2d$ filters $h = (h_n)_{n \in Z}$, $g^i = (g_n^i)_{n \in Z}$ $(i = 1, 2, \ldots, d-1)$, $\widetilde{h} = (\widetilde{h}_n)_{n \in Z}$ and $\widetilde{g}^i = (\widetilde{g}_n^i)_{n \in Z}$ $(i = 1, 2, \ldots, d-1)$, $\{h, g^1, g^2, \ldots, g^{d-1}\}$ are used for decomposition and $\{\widetilde{h}, \widetilde{g}^1, \widetilde{g}^2, \ldots, \widetilde{g}^{d-1}\}$ for reconstruction. Starting from a data sequence $x = (x_n)_{n \in Z}$, we convolve with $\{h, g^1, g^2, \ldots, g^{d-1}\}$,

$$
\begin{aligned}
c_n &= \sum_k h_{dn-k} x_k, \\
p_n^i &= \sum_k g_{dn-k}^i x_k, \quad i = 1, 2, \ldots, d-1.
\end{aligned}
\tag{2.1}
$$

The reconstruction operation is

$$
\widetilde{x}_k = \sum_n \left(\widetilde{h}_{dn-k} c_n + \sum_{i=1}^{d-1} \widetilde{g}_{dn-k}^i p_n^i \right).
\tag{2.2}
$$

The constraint conditions for biorthogonal d-band filter banks with perfect reconstruction property are:

a. the low-pass and high-pass condition

$$
\sum h_k = \sum \widetilde{h}_k = \sqrt{d}, \qquad \sum g_k^i = \sum \widetilde{g}_k^i = 0, \quad i = 1, 2, \ldots, d-1;
$$

b. the biorthogonal condition

$$
\sum_k h_k \widetilde{h}_{k+dj} = \delta_j, \qquad h_k \widetilde{g}_{k+dj}^i = g_k^i \widetilde{h}_{k+dj} = 0, \qquad g_k^i \widetilde{g}_{k+dj}^l = \delta_{i-l} \delta_j,
$$

where δ_j denotes the *Dirac* sequence such that $\delta_j = 1$ for $j = 0$ otherwise $\delta_j = 0$;

c. the perfect reconstruction condition $x = \widetilde{x}$.

In order to define (2.1) and (2.2) as operators, namely, sub-band operators, we assume that the input signal $x \in l^2(-\infty, +\infty)$. Now consider the separable Hilbert space $l^2(-\infty, +\infty)$. Define

$$
(x, y) = \sum_{n=-\infty}^{+\infty} x_i \bar{y}_i
$$

as the usual inner product on $l^2(-\infty, +\infty)$, where $x, y \in l^2(-\infty, +\infty)$, and $\bar{y}_i$ denotes the conjugate of the complex number y_i.

Definition 2.1 The operator

$$T : Tx = y$$

is called a sub-band operator, where $x \in l^2(-\infty, +\infty)$, $y = (\ldots, c_0, p_0^1, p_0^2, \ldots, p_0^{d-1}, c_1, p_1^1, p_1^2, \ldots, p_1^{d-1}, \ldots)$, c_n and p_n^i, $i = 1, 2, \ldots, d-1$ are defined as in (2.1).

Throughout this paper, we assume $h, g^1, g^2, \ldots, g^{d-1}, \widetilde{h}, \widetilde{g}^1, \widetilde{g}^2, \ldots, \widetilde{g}^{d-1}$ have only finitely many nonzero elements.

Theorem 2.1 *If the d-band biorthogonal wavelets determined by the filters $h, g^1, g^2, \ldots, g^{d-1}, \widetilde{h}, \widetilde{g}^1, \widetilde{g}^2, \ldots, \widetilde{g}^{d-1}$ have the perfect reconstruction property, then the sub-band operator T is a bounded linear operator and reversible on $l^2(-\infty, +\infty)$.*

The proof of Theorem 2.1 is trivial.

As is well known, a bounded linear operator on $l^2(-\infty, +\infty)$ can be expressed by an infinite-dimensional matrix. Using matrix notations, we have a more helpful expression of the operator T. Let

$$s_{n,k} = \begin{bmatrix} h_{k-dn} \\ g^1_{k-dn} \\ g^2_{k-dn} \\ \vdots \\ g^{d-1}_{k-dn} \end{bmatrix}, \qquad \widetilde{s}_{n,k} = \begin{bmatrix} \widetilde{h}_{k-dn} \\ \widetilde{g}^1_{k-dn} \\ \widetilde{g}^2_{k-dn} \\ \vdots \\ \widetilde{g}^{d-1}_{k-dn} \end{bmatrix}.$$

Then $A = [s_{n,k}]$ and $\widetilde{A} = [\widetilde{s}_{n,k}]$ $(-\infty < n, k < +\infty)$ are infinite block circular matrices along four directions (up, down, left and right). At this time,

$$y = Ax,$$

where x and y are doubly infinite column vectors to fit the matrix operation. Hence T can be viewed as an infinite matrix A, i.e., $T = A$. If the sub-band decomposition has the perfect reconstruction condition, then it is obvious that A and $\widetilde{A}$ should satisfy

$$A\widetilde{A}^* = \widetilde{A}^*A = I, \tag{2.3}$$

where I denotes the infinite identity matrix and A^* denotes the transpose of complex conjugate of A. Thus, $T^{-1} = \widetilde{A}^*$.

Since $(Ax, y) = (x, A^*y)$, the adjoint operator of T is $T^* = A^*$. Let $Q = T^*T$, we have the following lemma.

Lemma 2.1 *Retaining the definitions and notations as above, we have*

(1) $\|Q\| = \|T\|^2;$

(2) $T^{-1} = \widetilde{T}^* = \widetilde{A}^*, \qquad \widetilde{T}^{-1} = T^* = A^*;$

(3) $\|Q\| = \sup\limits_{\|x\|=1} \{|(Qx, x)|\}.$

Proof Items (1) and (2) are trivial according to the operator theory [18]. Item (3) follows the fact that Q is a self-adjoint operator due to $Q^* = Q$. □

$Q = T^*T$ is called a frame operator in general [2]. Let v_n and $\widetilde{v}_n$ denote the nth rows of A and $\widetilde{A}$, respectively. Clearly, $(v_j, \widetilde{v}_k) = \delta_{j-k}$, where δ_j is the Dirac sequence, i.e., $\delta_j = 1$ for $j = 0$ otherwise $\delta_j = 0$. Therefore, $\{v_n\}$ and $\{\widetilde{v}_n\}$ are dual biorthogonal bases in $l^2(-\infty, +\infty)$. Let $e_n \in l^2(-\infty, +\infty)$, its nth component be 1 and otherwise be 0. Then $T\widetilde{v}_n = e_n = \widetilde{T}v_n$. For an arbitrary $x \in l^2(-\infty, +\infty)$,

$$x = \sum_{n=-\infty}^{+\infty} (x, v_n)\widetilde{v}_n = \sum_{n=-\infty}^{+\infty} (x, \widetilde{v}_n)v_n,$$

$$Qx = \sum_{n=-\infty}^{+\infty} (x, v_n)Q\widetilde{v}_n = \sum_{n=-\infty}^{+\infty} (x, v_n)T^*e_n = \sum_{n=-\infty}^{+\infty} (x, v_n)T^*\widetilde{T}v_n = \sum_{n=-\infty}^{+\infty} (x, v_n)v_n.$$

Let m and M denote the lower bound and the upper bound of $\|T\|$, respectively.

$$Tx = \sum_{n=-\infty}^{+\infty} (x, v_n)T\widetilde{v}_n = \sum_{n=-\infty}^{+\infty} (x, v_n)e_n,$$

$$\|Tx\|^2 = \sum_{n=-\infty}^{+\infty} |(x, v_n)|^2.$$

Then

$$m^2\|x\|^2 \leq \sum_{n=-\infty}^{+\infty} |(x, v_n)|^2 \leq M^2\|x\|^2. \tag{2.4}$$

Similarly,

$$\widetilde{m}^2\|x\|^2 \leq \sum_{n=-\infty}^{+\infty} |(x, \widetilde{v}_n)|^2 \leq \widetilde{M}^2\|x\|^2, \tag{2.5}$$

where $\widetilde{m}$ and $\widetilde{M}$ are the lower bound and the upper bound of $\|\widetilde{T}\|$, respectively. Therefore, (2.4) and (2.5) are the counterparts of (1.1) in $l^2(-\infty, +\infty)$.

Now we define a finite matrix A_n as the partial matrix of the infinite matrix A, its row index and column index are finite with the following form:

$$
A_n =
\begin{bmatrix}
h_0 & h_1 & h_2 & \dots & h_{d-1} & \cdots & 0 & 0 & 0 & 0 \\
g_0^1 & g_1^1 & g_2^1 & \dots & g_{d-1}^1 & \cdots & 0 & 0 & 0 & 0 \\
g_0^2 & g_1^2 & g_2^2 & \dots & g_{d-1}^2 & \cdots & 0 & 0 & 0 & 0 \\
g_0^{d-1} & g_1^{d-1} & g_2^{d-1} & \dots & g_{d-1}^{d-1} & \cdots & 0 & 0 & 0 & 0 \\
h_{-d} & \dots & h_{-3} & h_{-2} & h_{-1} & \cdots & 0 & 0 & 0 & 0 \\
g_{-d}^1 & \dots & g_{-3}^1 & g_{-2}^1 & g_{-1}^1 & \cdots & 0 & 0 & 0 & 0 \\
g_{-d}^2 & \dots & g_{-3}^2 & g_{-2}^2 & g_{-1}^2 & \cdots & 0 & 0 & 0 & 0 \\
g_{-d}^{d-1} & \dots & g_{-3}^{d-1} & g_{-2}^{d-1} & g_{-1}^{d-1} & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & h_0 & h_1 & h_2 & \dots & h_{d-1} \\
0 & 0 & 0 & 0 & \cdots & g_0^1 & g_1^1 & g_2^1 & \dots & g_{d-1}^1 \\
0 & 0 & 0 & 0 & \cdots & g_0^2 & g_1^2 & g_2^2 & \dots & g_{d-1}^2 \\
0 & 0 & 0 & 0 & \cdots & g_0^{d-1} & g_1^{d-1} & g_2^{d-1} & \dots & g_{d-1}^{d-1} \\
0 & 0 & 0 & 0 & \cdots & h_{-d} & \dots & h_{-3} & h_{-2} & h_{-1} \\
0 & 0 & 0 & 0 & \cdots & g_{-d}^1 & \dots & g_{-3}^1 & g_{-2}^1 & g_{-1}^1 \\
0 & 0 & 0 & 0 & \cdots & g_{-d}^2 & \dots & g_{-3}^2 & g_{-2}^2 & g_{-1}^2 \\
0 & 0 & 0 & 0 & \cdots & g_{-d}^{d-1} & \dots & g_{-3}^{d-1} & g_{-2}^{d-1} & g_{-1}^{d-1}
\end{bmatrix}.
$$

Theorem 2.2 *Let $Q_n = A_n^* A_n$ and $\{\tau_1^{(n)}, \tau_2^{(n)}, \dots, \tau_{dn}^{(n)}\}$ be all eigenvalues of Q_n. Then*

$$
\|Q\| = \lim_{n \to \infty} \max\{\tau_1^{(n)}, \tau_2^{(n)}, \dots, \tau_{dn}^{(n)}\}.
$$

Proof Since Q_n is a positive operator, it has dn positive real eigenvalues, and $(Qx, x) \geq 0$. From Lemma 2.1, we have

$$
\|Q\| = \sup_{\|x\|=1}\{|(Qx, x)|\} = \sup_{\|x\|=1}\{(Qx, x)\}.
$$

Hence, there exists an $x^{(m)} \in l^2(-\infty, +\infty)$, such that $\|x^{(m)}\| = 1$ and $\|Q\| = \lim_{m \to \infty}(Qx^{(m)}, x^{(m)})$. Let dn-dimensional vector $[x]_n^{(m)}$ be the finite part of $x^{(m)}$.

On one hand, clearly, $\|[x]_n^{(m)}\| \leq \|x^{(m)}\|$ for an arbitrary n. Let $[y]_n$ be a dn-dimensional vector. Then

$$
\|Q\| = \lim_{m \to \infty}\left(Qx^{(m)}, x^{(m)}\right) = \lim_{m \to \infty}\lim_{n \to \infty}\left(Q_n[x]_n^{(m)}, [x]_n^{(m)}\right).
$$

Since Q_n is a finite-dimensional self-adjoint compact operator, we have

$$
\sup_{\|[y]_n\| \leq 1}\left\{\left(Q_n[y]_n, [y]_n\right)\right\} = \max\{\tau_1^{(n)}, \tau_2^{(n)}, \dots, \tau_{dn}^{(n)}\}.
$$

Hence,

$$
\|Q\| \leq \lim_{n \to \infty} \max\{\tau_1^{(n)}, \tau_2^{(n)}, \dots, \tau_{dn}^{(n)}\}.
$$

On the other hand, for every sufficiently large n, there exists a dn-dimensional vector $[y]_n$, such that $\|[y]_n\| = 1$, and

$$\sup_{\|y\|=1}\left\{\left(Q_n[y],[y]\right)\right\} = \left(Q_n[y]_n,[y]_n\right) = \max\left\{\tau_1^{(n)},\tau_2^{(n)},\ldots,\tau_{dn}^{(n)}\right\}.$$

Extend $[y]_n$ to $y^{(n)} \in l^2(-\infty,+\infty)$ by appending 0s to the tuples which are not defined by $[y]_n$. Clearly, $\|[y]_n\| = \|y^{(n)}\| = 1$, and

$$\|Q\| = \sup_{\|x\|=1}\left\{(Qx,x)\right\} \geq \left(Qy^{(n)},y^{(n)}\right) = \left(Q[y]_n,[y]_n\right) = \max\left\{\tau_1^{(n)},\tau_2^{(n)},\ldots,\tau_{dn}^{(n)}\right\}.$$

It implies that

$$\|Q\| \geq \lim_{n\to\infty}\max\left\{\tau_1^{(n)},\tau_2^{(n)},\ldots,\tau_{dn}^{(n)}\right\}.$$

Hence, $\|Q\| = \lim_{n\to\infty}\max\{\tau_1^{(n)},\tau_2^{(n)},\ldots,\tau_{dn}^{(n)}\}$. The proof is complete. $\square$

Theoretically, Theorem 2.2 gives an exact value for $\|Q\|$ and $\|T\| = \sqrt{\|Q\|}$ is used to compute the norm of T. However, the eigenvalues are not easy to compute for generalized block Toeplitz matrices. We shall use the theory of circular matrix to compute the norm of Q. The block circular matrix is defined as follows [19]:

$$B_n = \begin{bmatrix}
h_0 & h_1 & h_2 & \cdots & h_{d-1} & \cdots & h_{-d} & \cdots & h_{-3} & h_{-2} & h_{-1} \\
g_0^1 & g_1^1 & g_2^1 & \cdots & g_{d-1}^1 & \cdots & g_{-d}^1 & \cdots & g_{-3}^1 & g_{-2}^1 & g_{-1}^1 \\
g_0^2 & g_1^2 & g_2^2 & \cdots & g_{d-1}^2 & \cdots & g_{-d}^2 & \cdots & g_{-3}^2 & g_{-2}^2 & g_{-1}^2 \\
g_0^{d-1} & g_1^{d-1} & \cdots & g_2^{d-1} & g_{d-1}^{d-1} & \cdots & g_{-d}^{d-1} & \cdots & g_{-3}^{d-1} & g_{-2}^{d-1} & g_{-1}^{d-1} \\
h_{-d} & \cdots & h_{-3} & h_{-2} & h_{-1} & \cdots & h_{-2d} & \cdots & h_{-7} & h_{-6} & h_{-5} \\
g_{-d}^1 & \cdots & g_{-3}^1 & g_{-2}^1 & g_{-1}^1 & \cdots & g_{-2d}^1 & \cdots & g_{-7}^1 & g_{-6}^1 & g_{-5}^1 \\
g_{-d}^2 & \cdots & g_{-3}^2 & g_{-2}^2 & g_{-1}^2 & \cdots & g_{-2d}^2 & \cdots & g_{-7}^2 & g_{-6}^2 & g_{-5}^2 \\
g_{-d}^{d-1} & \cdots & g_{-3}^{d-1} & g_{-2}^{d-1} & g_{-1}^{d-1} & \cdots & g_{-2d}^{d-1} & \cdots & g_{-7}^{d-1} & g_{-6}^{d-1} & g_{-5}^{d-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
h_{2d} & h_{2d+1} & h_{2d+2} & \cdots & h_{3d-1} & \cdots & h_d & h_{d+1} & h_{d+2} & \cdots & h_{2d-1} \\
g_{2d}^1 & g_{2d+1}^1 & g_{2d+2}^1 & \cdots & g_{3d-1}^1 & \cdots & g_d^1 & g_{d+1}^1 & g_{d+2}^1 & \cdots & g_{2d-1}^1 \\
g_{2d}^2 & g_{2d+1}^2 & g_{2d+2}^2 & \cdots & g_{3d-1}^2 & \cdots & g_d^2 & g_{d+1}^2 & g_{d+2}^2 & \cdots & g_{2d-1}^2 \\
g_{2d}^{d-1} & g_{2d+1}^{d-1} & g_{2d+2}^{d-1} & \cdots & g_{3d-1}^{d-1} & \cdots & g_d^{d-1} & g_{d+1}^{d-1} & g_{d+2}^{d-1} & \cdots & g_{2d-1}^{d-1} \\
h_d & h_{d+1} & h_{d+2} & \cdots & h_{2d-1} & \cdots & h_0 & h_1 & h_2 & \cdots & h_{d-1} \\
g_d^1 & g_{d+1}^1 & g_{d+2}^1 & \cdots & g_{2d-1}^1 & \cdots & g_0^1 & g_1^1 & g_2^1 & \cdots & g_{d-1}^1 \\
g_d^2 & g_{d+1}^2 & g_{d+2}^2 & \cdots & g_{2d-1}^2 & \cdots & g_0^2 & g_1^2 & g_2^2 & \cdots & g_{d-1}^2 \\
g_d^{d-1} & g_{d+1}^{d-1} & g_{d+2}^{d-1} & \cdots & g_{2d-1}^{d-1} & \cdots & g_0^{d-1} & g_1^{d-1} & g_2^{d-1} & \cdots & g_{d-1}^{d-1}
\end{bmatrix}.$$

Clearly, A_n is different from B_n. Let $C_n = B_n - A_n$, then only finitely many (fixed) elements in C_n are not 0 no matter how large the dimension $4n$ is. We have

$$A_n^* A_n = B_n^* B_n - C_n^* B_n - B_n^* C_n + C_n^* C_n. \tag{2.6}$$

We have Theorem 2.3.

Theorem 2.3 *Let $P_n = B_n^* B_n$ and $\{\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_{dn}^{(n)}\}$ be all eigenvalues of P_n. Then*

$$\|Q\| = \lim_{n \to \infty} \max\{\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_{dn}^{(n)}\}.$$

Proof Let $[x]_n$ be a *dn*-dimensional vector. We first prove that for $\|[x]_n\| = 1$, $\lim_{n \to \infty} |(C_n^* B_n[x]_n, [x]_n)| = 0$, $\lim_{n \to \infty} |(B_n^* C_n[x]_n, [x]_n)| = 0$, $\lim_{n \to \infty} |(C_n^* C_n[x]_n, [x]_n)| = 0$.

In fact, the operator B_n is bounded. For $\|[x]_n\| = 1$, there exists a positive real number M such that

$$\begin{aligned}
\left|\left(C_n^* B_n[x]_n, [x]_n\right)\right| &= \left|\left(B_n[x]_n, C_n[x]_n\right)\right| \\
&\leq \|B_n[x]_n\| \|C_n[x]_n\| \leq M\|C_n[x]_n\|.
\end{aligned}$$

Note that $C_n[x]_n$ only contains *dn* nonzero components.

Hence $\lim_{n \to \infty} \|C_n[x]_n\| = 0$, i.e., $\lim_{n \to \infty} |(C_n^* C_n[x]_n, [x]_n)| = 0$.

Similarly, we can verify $\lim_{n \to \infty} |(B_n^* C_n[x]_n, [x]_n)| = 0$ and $\lim_{n \to \infty} |(C_n^* C_n[x]_n, [x]_n)| = 0$.

It follows from (2.6) that

$$\left(Q_n[x]_n, [x]_n\right) = \left(P_n[x]_n, [x]_n\right) + \left(R_n[x]_n, [x]_n\right),$$

where $R_n = -C_n^* B_n - B_n^* C_n + C_n^* C_n$ and $\lim_{n \to \infty} |(R_n, [x]_n)| = 0$.

It implies that

$$\lim_{n \to \infty} \sup_{\|[x]_n\|=1} \left\{\left(Q_n[x]_n, [x]_n\right)\right\} = \lim_{n \to \infty} \sup_{\|[x]_n\|=1} \left\{\left(P_n[x]_n, [x]_n\right)\right\}.$$

By Theorem 2.2, we have

$$\|Q\| = \lim_{n \to \infty} \max\{\tau_1^{(n)}, \tau_2^{(n)}, \ldots, \tau_{dn}^{(n)}\} = \lim_{n \to \infty} \max\{\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_{dn}^{(n)}\}.$$

The proof is complete. □

A so-called *d*-circular matrix [20], which is generated by the filters $h, g^1, g^2, \ldots, g^{d-1}$, is denoted as M_n. For $d = 4$, M_3 is as follows:

$$M_3 = \begin{bmatrix}
h_0 & h_1 & h_2 & h_3 & 0 & 0 & 0 & 0 & h_{-4} & h_{-3} & h_{-2} & h_{-1} \\
h_{-4} & h_{-3} & h_{-2} & h_{-1} & h_0 & h_1 & h_2 & h_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & h_{-4} & h_{-3} & h_{-2} & h_{-1} & h_0 & h_1 & h_2 & h_3 \\
g_0^1 & g_1^1 & g_2^1 & g_3^1 & 0 & 0 & 0 & 0 & g_{-4}^1 & g_{-3}^1 & g_{-2}^1 & g_{-1}^1 \\
g_{-4}^1 & g_{-3}^1 & g_{-2}^1 & g_{-1}^1 & g_0^1 & g_1^1 & g_2^1 & g_3^1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g_{-4}^1 & g_{-3}^1 & g_{-2}^1 & g_{-1}^1 & g_0^1 & g_1^1 & g_2^1 & g_3^1 \\
g_0^2 & g_1^2 & g_2^2 & g_3^2 & 0 & 0 & 0 & 0 & g_{-4}^2 & g_{-3}^2 & g_{-2}^2 & g_{-1}^2 \\
g_{-4}^2 & g_{-3}^2 & g_{-2}^2 & g_{-1}^2 & g_0^2 & g_1^2 & g_2^2 & g_3^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g_{-4}^2 & g_{-3}^2 & g_{-2}^2 & g_{-1}^2 & g_0^2 & g_1^2 & g_2^2 & g_3^2 \\
g_0^3 & g_1^3 & g_2^3 & g_3^3 & 0 & 0 & 0 & 0 & g_{-4}^3 & g_{-3}^3 & g_{-2}^3 & g_{-1}^3 \\
g_{-4}^3 & g_{-3}^3 & g_{-2}^3 & g_{-1}^3 & g_0^3 & g_1^3 & g_2^3 & g_3^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g_{-4}^3 & g_{-3}^3 & g_{-2}^3 & g_{-1}^3 & g_0^3 & g_1^3 & g_2^3 & g_3^3
\end{bmatrix}.$$

Clearly, B_n and M_n are not so different. One can be obtained by exchanging the places of some rows of another, i.e., there exists an orthonormal matrix E_n such that $M_n = E_n B_n$. The purpose is to facilitate the calculation of the eigenvalues.

Similarly, let $\widetilde{M}_n$ be a d-circular matrix generated by $\widetilde{h}, \widetilde{g}^1, \widetilde{g}^2, \dots, \widetilde{g}^{d-1}$. The perfect reconstruction condition is that there exists an integer N_0, such that, for all $n \geq N_0$ (in what follows, n sufficiently large is in this sense),

$$M_n \widetilde{M}_n^* = I_{dn}, \tag{2.7}$$

where I_{dn} is a $dn \times dn$ identity matrix.

Theorem 2.4 *Let M_n be a d-circular matrix generated by $h, g^1, g^2, \dots, g^{d-1}$. For sufficiently large n, then*

$$\max\{\lambda_1, \lambda_2, \dots, \lambda_{dn}\} \leq \max\{C_0, C_1, \dots, C_{d-1}\}, \tag{2.8}$$

where

$$C_0 = \sum_j \left| \sum_i h_i h_{i+dj} \right| + \sum_{n=1}^{d-1} \sum_j \left| \sum_i h_i g_{i+dj}^n \right|,$$

$$C_k = \sum_j \left| \sum_i g_i^k h_{i+dj} \right| + \sum_{n=1}^{d-1} \sum_j \left| \sum_i g_i^k g_{i+dj}^n \right|, \quad 1 \leq k \leq d-1.$$

Proof Define

$$\|A_{dn}\|_\infty = \max_{1 \leq i \leq dn} \sum_{j=1}^{dn} |a_{i,j}|.$$

Since $\|A_{dn} B_{dn}\|_\infty \leq \|A_{dn}\|_\infty \times \|B_{dn}\|_\infty$, $\| \cdot \|_\infty$ is a compatible matrix norm. Note that $A_{dn} = M_n M_n^T$ positive definite matrices, and all of the eigenvalues of $M_n M_n^T$ are positive. According to the theory of matrices [19], we have

$$|\lambda_i| \leq \|A_{dn}\|_\infty = \max_{1 \leq i \leq dn} \sum_{j=1}^{dn} |a_{i,j}|, \quad i = 1, 2, \dots, dn.$$

Note that the sub-matrices of $M_n M_n^T$ such as $HH^T, HG_1^T, HG_2^T, \dots, HG_{d-1}^T, \dots$ are all 1-circular matrices. Then

$$\sum_{j=kn+1}^{kn+n} |a_{kn+1,j}| = \sum_{j=kn+1}^{kn+n} |a_{kn+2,j}| = \dots = \sum_{j=kn+1}^{kn+n} |a_{kn+n,j}|, \quad 0 \leq k \leq d-1.$$

We have

$$C_0 = \sum_{j=1}^{dn} |a_{1,j}| = \sum_j \left| \sum_i h_i h_{i+dj} \right| + \sum_{n=1}^{d-1} \sum_j \left| \sum_i h_i g_{i+dj}^n \right|.$$

Firstly, we verify that, for $n \geq N_0$,

$$\sum_{j=1}^{n} |a_{1,j}| \leq \sum_{j} \left| \sum_{i} h_i h_{i+dj} \right|.$$

1. There exists N_0^*: when $n \geq N_0^* \geq N_0$, we have

$$HH^T = \left[\sum h_i h_i, \sum h_i h_{i+d}, \sum h_i h_{i+2d}, \ldots, 0, \ldots, \sum h_i h_{i+d} \right].$$

Note that when the dimension increases, the nonzero elements of HH^T are the same. Therefore

$$\sum_{j=1}^{n} |a_{1,j}| = \sum_{j} \left| \sum_{i} h_i h_{i+dj} \right|.$$

2. When $N_0 \leq n \leq N_0^*$, the nonzero elements of HH^T decrease.
    It follows from $|a + b| \leq |a| + |b|$ that

$$\sum_{j=1}^{n} |a_{1,j}| \leq \sum_{j} \left| \sum_{i} h_i h_{i+dj} \right|.$$

We obtain

$$\sum_{j=1}^{n} |a_{1,j}| \leq \sum_{j} \left| \sum_{i} h_i h_{i+dj} \right|.$$

Similarly, (2.8) holds true. □

Remark The right-hand side of (2.8) is only determined by the filters.

Theorem 2.5 *Let T be the sub-band operator of d-band wavelets. Then*

$$\frac{1}{\sqrt{\max\{\widetilde{C}_0, \widetilde{C}_1, \ldots, \widetilde{C}_{d-1}\}}} \leq \|T\| \leq \sqrt{\max\{C_0, C_1, \ldots, C_{d-1}\}}, \tag{2.9}$$

$$\frac{1}{\sqrt{\max\{C_0, C_1, \ldots, C_{d-1}\}}} \leq \|T^{-1}\| \leq \sqrt{\max\{\widetilde{C}_0, \widetilde{C}_1, \ldots, \widetilde{C}_{d-1}\}}, \tag{2.10}$$

where the filter bands are $\{h, g^1, g^2, \ldots, g^{d-1}\}$ and $\{\widetilde{h}, \widetilde{g}^1, \widetilde{g}^2, \ldots, \widetilde{g}^{d-1}\}$, respectively,

$$C_0 = \sum_{j} \left| \sum_{i} h_i h_{i+dj} \right| + \sum_{n=1}^{d-1} \sum_{j} \left| \sum_{i} h_i g^n_{i+dj} \right|,$$

$$C_k = \sum_{j} \left| \sum_{i} g^k_i h_{i+dj} \right| + \sum_{n=1}^{d-1} \sum_{j} \left| \sum_{i} g^k_i g^n_{i+dj} \right|, \quad 1 \leq k \leq d-1,$$

$$\widetilde{C}_0 = \sum_j \left| \sum_i \widetilde{h}_i \widetilde{h}_{i+dj} \right| + \sum_{n=1}^{d-1} \sum_j \left| \sum_i \widetilde{h}_i \widetilde{g}_{i+dj}^n \right|,$$

$$\widetilde{C}_k = \sum_j \left| \sum_i \widetilde{g}_i^k \widetilde{h}_{i+dj} \right| + \sum_{n=1}^{d-1} \sum_j \left| \sum_i \widetilde{g}_i^k \widetilde{g}_{i+dj}^n \right|, \quad 1 \le k \le d-1.$$

Proof By Theorem 2.3, we have

$$\|T\|^2 = \|Q\| = \lim_{n \to \infty} \max\{\lambda_1, \lambda_2, \dots, \lambda_{dn}\},$$

where $\{\lambda_1, \lambda_2, \dots, \lambda_{dn}\}$ are all eigenvalues of $M_n M_n^T$.

According to Theorem 2.4, we have

$$\|T\| = \sqrt{\|Q\|} \le \sqrt{\max\{C_0, C_1, \dots, C_{d-1}\}}.$$

Similarly,

$$\left\| T^{-1} \right\| \le \sqrt{\max\{\widetilde{C}_0, \widetilde{C}_1, \dots, \widetilde{C}_{d-1}\}}.$$

The proof is complete. □

3 Examples

In this section, we present two examples to illustrate the proposed results.

Example 3.1 Let the lengths of scaling filters be $(15, 9)$. Assume that the scaling symbols $(H_0(z), \widetilde{H}_0(z))$ have the following form:

$$H_0(z) = \left(\frac{1 + z + z^2}{3} \right)^5 Q(z),$$

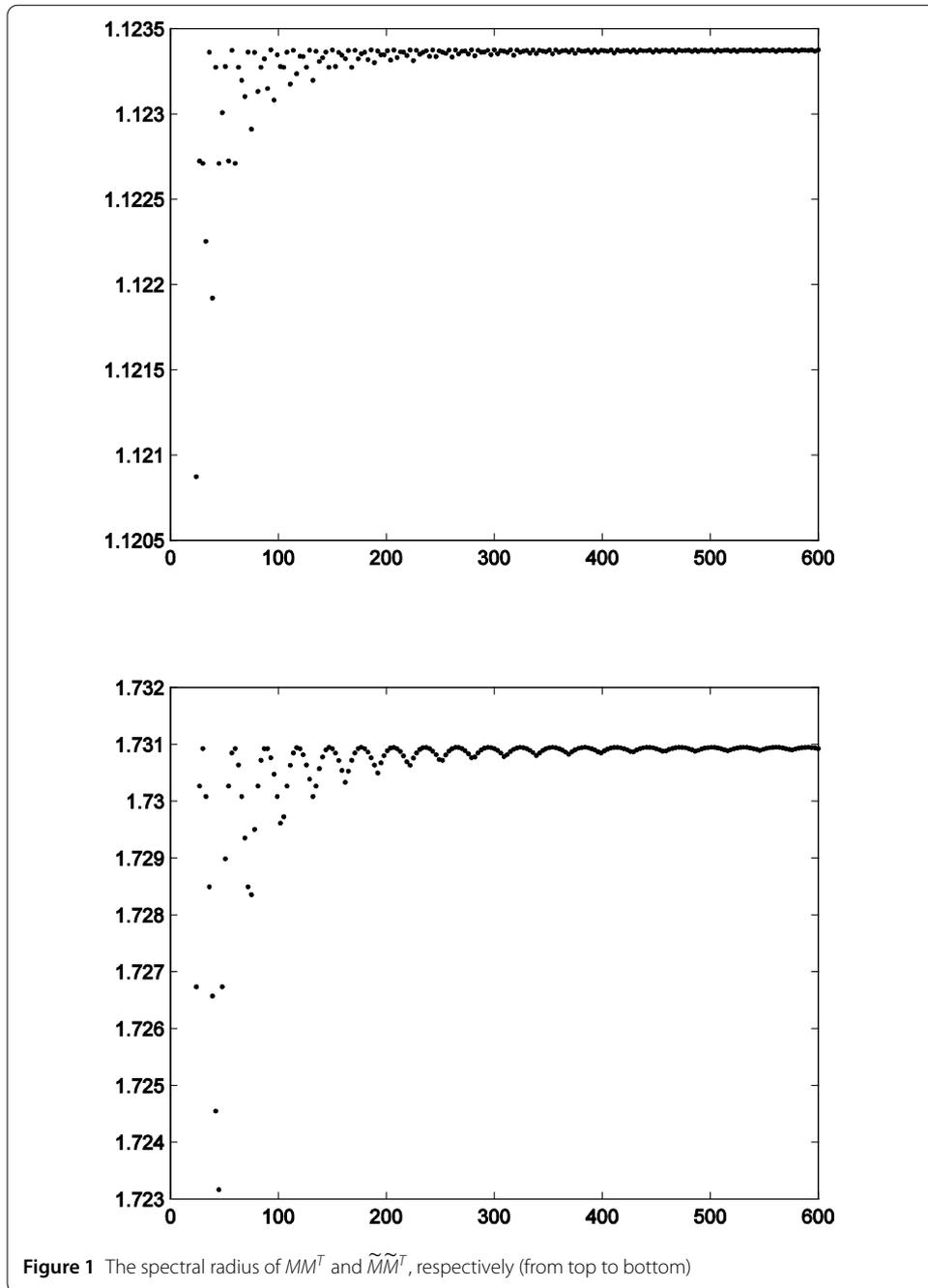$$\widetilde{H}_0(z) = \left(\frac{1 + z + z^2}{3} \right)^3 \widetilde{Q}(z),$$

where $(Q(z), \widetilde{Q}(z))$ are symmetric Laurent polynomials with degree $(4, 2)$.

We can obtain the associated scaling filters as follows [21]:

$$h_0 \approx [0.0302708750, 0.0197271260, 0.0109853080, -0.12261759, 0.011382944,$$

$$0.24928687, 0.83207454, 0.93777986],$$

$$\widetilde{h}_0 \approx [-0.20140256, -0.090291448, 0.13193077, 1.0694718, 1.1805829]$$

(whereas the other half is symmetric and so is skipped). Thus, the wavelet filters h_1, h_2 and $\widetilde{h}_1$, $\widetilde{h}_2$ can be obtained as follows:

$$h_1 \approx [0.032348717, 0.021081228, 0.011739358, -0.75426796, 1.3781973],$$

$$h_2 \approx 4[-0.044296947, -0.028867731, -0.016075373, 0.29594173, 0],$$

Figure 1 The spectral radius of MM^T and $\widetilde{M}\widetilde{M}^T$, respectively (from top to bottom)

$$\widetilde{h}_1 \approx [-0.0047182543, -0.0021152562, 0.0030907400, 0.038680809, 0.033766352,$$
$$0.016128444, -0.75722871, 1.3447918],$$
$$\widetilde{h}_2 \approx [0.14107656, 0.063246501, -0.092413623, -0.45441064, -0.69483538,$$
$$-0.94219467, 4.8815912, 0]/4$$

(whereas the other half is symmetric/antisymmetric and so skipped). See Fig. 1 for the spectral radius of $M_n M_n^T$ and $\widetilde{M}_n \widetilde{M}_n^T$. From Theorem 2.3, we can obtain $\|T\| \approx 1.06$, which

Figure 2 The spectral radius of MM^T ($\widetilde{M}\widetilde{M}^T$)
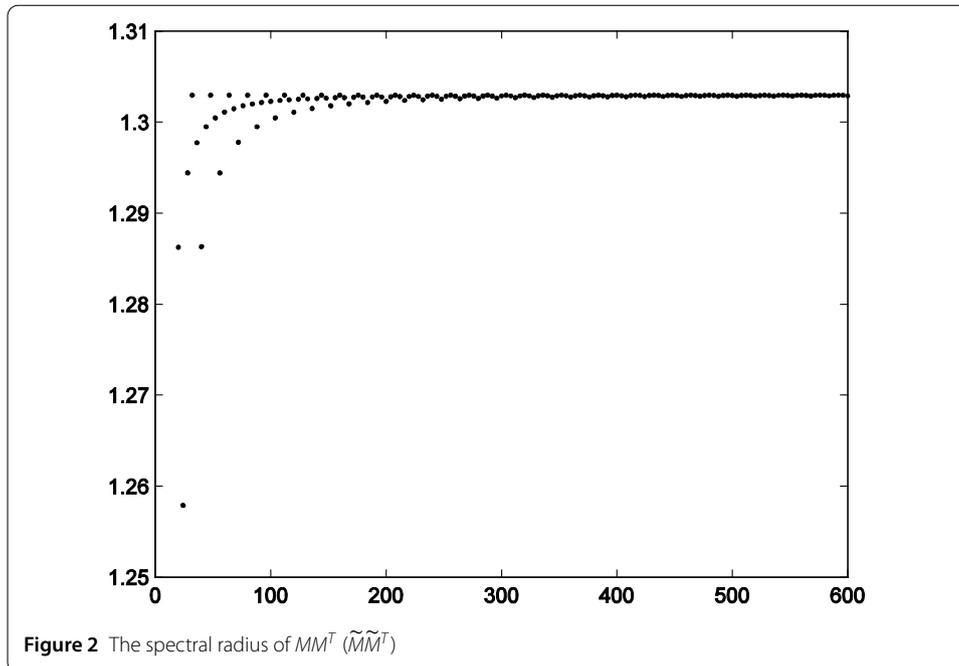
is an approximation and not an exact value. Similarly, $\|T^{-1}\| \approx 1.32$. By Theorem 2.5, we can calculate that $0.66 \le \|T\| \le 1.15$ and $0.94 \le \|T^{-1}\| \le 1.51$.

Example 3.2 In [20], a 4-band symmetric biorthogonal wavelets which is denoted as Op(12-12) is designed. The corresponding wavelet filter banks $\{h, g^1, g^2, g^3\}$ are as follows:

$$
\begin{bmatrix}
t_0 & t_1 & t_2 & t_3 & t_4 & t_5 & t_5 & t_4 & t_3 & t_2 & t_1 & t_0 \\
t_1 & -t_0 & -t_3 & t_2 & t_5 & -t_4 & -t_4 & t_5 & t_2 & -t_3 & -t_0 & t_1 \\
\widetilde{t}_0 & -\widetilde{t}_1 & \widetilde{t}_2 & -\widetilde{t}_3 & \widetilde{t}_4 & -\widetilde{t}_5 & \widetilde{t}_5 & -\widetilde{t}_4 & \widetilde{t}_3 & -\widetilde{t}_2 & \widetilde{t}_1 & -\widetilde{t}_0 \\
\widetilde{t}_1 & \widetilde{t}_0 & -\widetilde{t}_3 & -\widetilde{t}_2 & \widetilde{t}_5 & \widetilde{t}_4 & -\widetilde{t}_4 & -\widetilde{t}_5 & \widetilde{t}_2 & \widetilde{t}_3 & -\widetilde{t}_0 & -\widetilde{t}_1
\end{bmatrix},
$$

where $t_0 = 0.01129264$, $t_1 = -0.01660958$, $t_2 = -0.01418315$, $t_3 = 0.02102888$, $t_4 = 0.4676785$, $t_5 = 0.5307927$, $\widetilde{t}_0 = -0.07653$, $\widetilde{t}_1 = -0.04528$, $\widetilde{t}_2 = 0.01722$, $\widetilde{t}_3 = 0.11097$, $\widetilde{t}_4 = 0.46556$, $\widetilde{t}_5 = 0.52806$.

It has been shown that the eigenvalues of $M_{4n}^T M_{4n}$ appear in pairs of reciprocal, $M_{4n}^T M_{4n}$ and $\widetilde{M}_{4n}^T \widetilde{M}_{4n}$ have the same eigenvalues [20]. It is obvious that $\max\{\lambda_i\} = \max\{\widetilde{\lambda}_i\} = \frac{1}{\min\{\lambda_i\}} = \frac{1}{\min\{\widetilde{\lambda}_i\}}$. See Fig. 2 for the spectral radius of $M_n M_n^T$ and $\widetilde{M}_n \widetilde{M}_n^T$. From Theorem 2.3, we can obtain $\|T\| \approx 1.14$, which is an approximation and not an exact value. According to Theorem 2.5, we have $\|T\| = \|T^{-1}\| \le 1.18$.

4 Conclusions and future work

We can obtain the upper bound and the lower bound or an approximation of the sub-band operator's norm based on the theory of circular matrix which plays an important role. We will calculate their norm for some special symmetric wavelets and design a family of biorthogonal wavelets based on the size of the norm.

Author details
[1]Hunan Province Cooperative Innovation Center for The Construction Development of Dongting Lake Ecological Economic Zone, Hunan University of Arts and Science, Changde, P.R. China. [2]College of Mathematics and Computer Science, Hunan Normal University, Changsha, P.R. China.

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References
1. Daubechies, I., Han, B., Ron, A., Shen, Z.: Framelets, MRA-based constructions of wavelet frames. Appl. Comput. Harmon. Anal. **14**, 1–46 (2003)
2. Gavruta, L.: Frames for operators. Appl. Comput. Harmon. Anal. **32**, 139–144 (2012)
3. Sun, W., Zhou, X.: Density and stability of wavelet frames. Appl. Comput. Harmon. Anal. **15**, 117–133 (2003)
4. Heil, C., Kutyniok, G.: Density of wavelet frames. J. Geom. Anal. **13**, 479–493 (2003)
5. Kim, H., Kim, R., Lee, Y., Yoon, J.: Quasi-interpolatory refinable functions and construction of biorthogonal wavelet systems. Adv. Comput. Math. **33**, 255–283 (2010)
6. Cohen, A., Daubechies, I., Feauveau, J.: Biorthogonal bases of compactly supported wavelets. Commun. Pure Appl. Math. **45**, 485–560 (1992)
7. Didenko, V.: Spectral radii of refinement and subdivision operators. Proc. Am. Math. Soc. **33**, 2335–2346 (2005)
8. Han, B.: Symmetric orthonormal scaling functions and wavelets with dilation factor 4. Adv. Comput. Math. **8**, 221–247 (1998)
9. Bi, N., Dai, X., Sun, Q.: Construction of compactly supported M-band wavelets. Appl. Comput. Harmon. Anal. **6**, 113–131 (1999)
10. Chui, C., Lian, J.: Construction of compactly supported symmetric and antisymmetric orthonormal wavelets with scale = 3. Appl. Comput. Harmon. Anal. **2**, 68–84 (1995)
11. Zou, Q., Wang, G., Yang, M.: Spectral radius of biorthogonal wavelets with its application. J. Korean Math. Soc. **51**, 941–953 (2014)
12. Wang, G., Zou, Q., Yang, M.: Sub-band operators and saddle point wavelets. Appl. Math. Comput. **227**, 27–42 (2014)
13. Cui, L., Zhai, B., Zhang, T.: Existence and design of biorthogonal matrix-valued wavelets. Nonlinear Anal., Real World Appl. **10**, 2679–2687 (2009)
14. Jiang, Q.: Biorthogonal wavelets with 4-fold axial symmetry for quadrilateral surface multiresolution processing. Adv. Comput. Math. **34**, 127–165 (2011)
15. He, T.: Biorthogonal wavelets with certain regularities. Appl. Comput. Harmon. Anal. **11**, 227–242 (2001)
16. Zhuang, X.: Matrix extension with symmetry and construction of biorthogonal multiwavelets with any integer dilation. Appl. Comput. Harmon. Anal. **33**, 159–181 (2012)
17. Chui, C., Shi, X.: Inequalities of Littlewood–Paley type for frames and wavelets. SIAM J. Math. Anal. **24**, 263–277 (1993)
18. Naylor, A., Sell, G.: Linear Operator Theory in Engineering and Science. Springer, New York (2000)
19. Wang, G.: Four-bank compactly supported bi-symmetric orthonormal wavelets bases. Opt. Eng. **43**, 2362–2368 (2004)
20. Zou, Q., Wang, G.: Optimal model for 4-band biorthogonal wavelets bases for fast calculation. J. Inequal. Appl. **2017**, Article ID 222 (2017)
21. Zou, Q., Wang, G.: Construction of biorthogonal 3-band wavelets with symmetry and high vanishing moments. J. Math. **35**, 12–22 (2015)