Haji Shaabani *Journal of Inequalities and Applications* (2018) 2018:23
https://doi.org/10.1186/s13660-018-1615-0

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Fredholmness of multiplication of a weighted composition operator with its adjoint on H^2 and A^2_α

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Abstract

In this paper, we obtain that $C^*_{\psi,\varphi}$ is bounded below on H^2 or A^2_α if and only if $C_{\psi,\varphi}$ is invertible. Moreover, we investigate the Fredholm operators $C_{\psi_1,\varphi_1}C^*_{\psi_2,\varphi_2}$ and $C^*_{\psi_1,\varphi_1}C_{\psi_2,\varphi_2}$ on H^2 and A^2_α.

MSC: Primary 47B33; secondary 47A53

Keywords: Hardy space; weighted Bergman spaces; weighted composition operator; Fredholm operator

1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. The Hardy space, denoted $H^2(\mathbb{D}) = H^2$, is the set of all analytic functions f on $\mathbb{D}$ satisfying the norm condition

$$\|f\|_1^2 = \lim_{r\to 1}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} < \infty.$$

The space $H^\infty(\mathbb{D}) = H^\infty$ consists of all analytic and bounded functions on $\mathbb{D}$ with supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|$.

For $\alpha > -1$, the weighted Bergman space $A^2_\alpha(\mathbb{D}) = A^2_\alpha$ is the set of functions f analytic in $\mathbb{D}$ with

$$\|f\|^2_{\alpha+2} = (\alpha+1)\int_{\mathbb{D}} \left|f(z)\right|^2\left(1-|z|^2\right)^\alpha\, dA(z) < \infty,$$

where dA is the normalized area measure in $\mathbb{D}$. The case where $\alpha = 0$ is known as the (unweighted) Bergman space, often simply denoted by A^2.

Let φ be an analytic map from the open unit disk $\mathbb{D}$ into itself. The operator that takes the analytic map f to $f\circ\varphi$ is a composition operator and is denoted by C_φ. A natural generalization of a composition operator is an operator that takes f to $\psi\cdot f\circ\varphi$, where ψ is a fixed analytic map on $\mathbb{D}$. This operator is aptly named a weighted composition operator and is usually denoted by $C_{\psi,\varphi}$. More precisely, if z is in the unit disk, then $(C_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$. For some results on weighted composition and related operators on the weighted Bergman and Hardy spaces, see, for example, [1–14].

If ψ is a bounded analytic function on the open unit disk, then the multiplication operator M_ψ defined by $M_\psi(f)(z) = \psi(z)f(z)$ is a bounded operator on H^2 and A^2_α and $\|M_\psi(f)\|_\gamma \le \|\psi\|_\infty \|f\|_\gamma$ when $\gamma = 1$ for H^2 and $\gamma = \alpha + 2$ for A^2_α. Let P denote the orthogonal projection of $L^2(\partial\mathbb{D})$ onto H^2. For each $b \in L^\infty(\partial\mathbb{D})$, the Toeplitz operator T_b acts on H^2 by $T_b(f) = P(bf)$. Also suppose that P_α is the orthogonal projection of $L^2(\mathbb{D}, dA_\alpha)$ onto A^2_α. For each function $w \in L^\infty(\mathbb{D})$, the Toeplitz operator T_w on A^2_α is defined by $T_w(f) = P_\alpha(wf)$. Since P and P_α are bounded on H^2 and A^2_α, respectively, the Toeplitz operators are bounded.

Let $w \in \mathbb{D}$, and let H be a Hilbert space of analytic functions on $\mathbb{D}$. Let e_w be the point evaluation at w, that is, $e_w(f) = f(w)$ for $f \in H$. If e_w is a bounded linear functional on H, then the Riesz representation theorem implies that there is a function (usually denoted K_w) in H that induces this linear functional, that is, $e_w(f) = \langle f, K_w\rangle$. In this case, the functions K_w are called the reproducing kernels, and the functional Hilbert space is also called a reproducing kernel Hilbert space. Both the weighted Bergman spaces and the Hardy space are reproducing kernel Hilbert spaces, where the reproducing kernel for evaluation at w is given by $K_w(z) = (1 - \overline{w}z)^{-\gamma}$ for $z, w \in \mathbb{D}$, with $\gamma = 1$ for H^2 and $\gamma = \alpha + 2$ for A^2_α. Let k_w denote the normalized reproducing kernel given by $k_w(z) = K_w(z)/\|K_w\|_\gamma$, where $\|K_w\|^2_\gamma = (1 - |w|^2)^{-\gamma}$.

Suppose that H and H' are Hilbert spaces and $A : H \to H'$ is a bounded operator. The operator A is said to be left semi-Fredholm if there are a bounded operator $B : H' \to H$ and a compact operator K on H such that $BA = I + K$. Analogously, A is right semi-Fredholm if there are a bounded operator $B' : H' \to H$ and a compact operator K' on H' such that $AB' = I + K'$. An operator A is said to be Fredholm if it is both left and right semi-Fredholm. It is not hard to see that A is left semi-Fredholm if and only if A^* is right semi-Fredholm. Hence A is Fredholm if and only if A^* is Fredholm. Note that an invertible operator is Fredholm. By using the definition of a Fredholm operator it is not hard to see that if the operators A and B are Fredholm on a Hilbert space H, then AB is also Fredholm on H. The Fredholm composition operators on H^2 were first identified by Cima et al. [15] and later by a different and more general method by Bourdon [2]. Cima et al. [15] proved that only the invertible composition operators on H^2 are Fredholm. Moreover, MacCluer [16] characterized Fredholm composition operators on a variety of Hilbert spaces of analytic functions in both one and several variables. Recently, Fredholm composition operators on various spaces of analytic functions have been studied (see [13] and [14]).

The automorphisms of $\mathbb{D}$, that is, the one-to-one analytic maps of the disk onto itself, are just the functions $\varphi(z) = \lambda\frac{a-z}{1-\overline{a}z}$ with $|\lambda| = 1$ and $|a| < 1$. We denote the class of automorphisms of $\mathbb{D}$ by $\mathrm{Aut}(\mathbb{D})$. Automorphisms of $\mathbb{D}$ take $\partial\mathbb{D}$ onto $\partial\mathbb{D}$. It is known that C_φ is Fredholm on the Hardy space if and only if $\varphi \in \mathrm{Aut}(\mathbb{D})$ (see [2]).

An analytic map φ of the disk to itself is said to have a finite angular derivative at a point ζ on the boundary of the disk if there exists a point η, also on the boundary of the disk, such that the nontangential limit as $z \to \zeta$ of the difference quotient $(\eta - \varphi(z))/(\zeta - z)$ exists as a finite complex value. We write $\varphi'(\zeta) = \angle\lim_{z\to\zeta}\frac{\eta-\varphi(z)}{\zeta-z}$.

In the second section, we investigate Fredholm and invertible weighted composition operators. In Theorem 2.7, we show that the operator $C^*_{\psi,\varphi}$ is bounded below on H^2 or A^2_α if and only if $C_{\psi,\varphi}$ is invertible.

In the third section, we investigate the Fredholm operators $C_{\psi_1,\varphi_1}C^*_{\psi_2,\varphi_2}$ and $C^*_{\psi_1,\varphi_1}C_{\psi_2,\varphi_2}$ on H^2 and A^2_α.

2 Bounded below operators $C^*_{\psi,\varphi}$

Let H be a Hilbert space. The set of all bounded operators from H into itself is denoted by $B(H)$. We say that an operator $A \in B(H)$ is bounded below if there is a constant $c > 0$ such that $c\|h\| \leq \|A(h)\|$ for all $h \in H$.

If f is defined on a set V and if there is a positive constant m such that $|f(z)| \geq m$ for all z in V, then we say that f is bounded away from zero on V. In particular, we say that ψ is bounded away from zero near the unit circle if there are $\delta > 0$ and $\epsilon > 0$ such that

$$\left|\psi(z)\right| > \epsilon \quad \text{for } \delta < |z| < 1.$$

Suppose that T belongs to $B(H)$. We denote the spectrum of T, the essential spectrum of T, the approximate point spectrum of T, and the point spectrum of T by $\sigma(T)$, $\sigma_e(T)$, $\sigma_{\mathrm{ap}}(T)$ and $\sigma_p(T)$, respectively. Moreover, the left essential spectrum of T is denoted by $\sigma_{\mathrm{le}}(T)$.

Suppose that φ is an analytic self-map of $\mathbb{D}$. For almost all $\zeta \in \partial\mathbb{D}$, we define $\varphi(\zeta) = \lim_{r\to 1}\varphi(r\zeta)$ (the statement of the existence of this limit can be found in [17, Theorem 2.2]). If f is a bounded analytic function on the unit disk such that $|f(e^{i\theta})| = 1$ almost everywhere, then we call f an inner function. We know that if φ is inner, then C_φ is bounded below on H^2, and therefore C_φ has a closed range (see [17, Theorem 3.8]).

Now we state the following simple and well-known lemma, and we frequently use it in this paper.

Lemma 2.1 *Let $C_{\psi,\varphi}$ be a bounded operator on H^2 or A^2_α. Then $C^*_{\psi,\varphi}K_w = \overline{\psi(w)}K_{\varphi(w)}$ for all $w \in \mathbb{D}$.*

In this paper, for convenience, we assume that $\gamma = 1$ for H^2 and $\gamma = \alpha + 2$ for A^2_α.

Lemma 2.2 *Suppose that A and B are two bounded operators on a Hilbert space H. If AB is a Fredholm operator, then B is left semi-Fredholm.*

Proof Suppose that AB is a Fredholm operator. Then there are a bounded operator C and a compact operator K such that $CAB = I + K$. Therefore B is left semi-Fredholm. □

Zhao [13] characterized Fredholm weighted composition operators on H^2. Also, Zhao [14] found necessary conditions of φ and ψ for a weighted composition operator $C_{\psi,\varphi}$ on A^2_α to be Fredholm. In the following proposition, we obtain a necessary and sufficient condition for $C_{\psi,\varphi}$ to be Fredholm on H^2 and A^2_α. Then we use it to find when $C^*_{\psi_1,\varphi_1}C_{\psi_2,\varphi_2}$ and $C_{\psi_1,\varphi_1}C^*_{\psi_2,\varphi_2}$ are Fredholm. The idea of the proof of the next proposition is different from [13] and [14].

Proposition 2.3 *The operator $C^*_{\psi,\varphi}$ is left semi-Fredholm on H^2 or A^2_α if and only if $\varphi \in \mathrm{Aut}(\mathbb{D})$ and $\psi \in H^\infty$ is bounded away from zero near the unit circle. Under these conditions, $C_{\psi,\varphi}$ is a Fredholm operator.*

Proof Let $C_{\psi,\varphi}$ be Fredholm on H^2 or A^2_α. Assume that ψ is not bounded away from zero near the unit circle. Then for each positive integer n, there is $x_n \in \mathbb{D}$ such that $1 - 1/n <$

$|x_n| < 1$ and $|\psi(x_n)| < 1/n$. Then there exist a subsequence $\{x_{n_m}\}$ and $\zeta \in \partial\mathbb{D}$ such that $x_{n_m} \to \zeta$ as $m \to \infty$. Since $\psi(x_{n_m}) \to 0$ as $m \to \infty$, by Lemma 2.1 we see that

$$\begin{aligned}\lim_{m\to\infty}\|C^*_{\psi,\varphi}k_{x_{n_m}}\|_\gamma &= \lim_{m\to\infty}|\psi(x_{n_m})|\frac{\|K_{\varphi(x_{n_m})}\|_\gamma}{\|K_{x_{n_m}}\|_\gamma}\\ &\le \limsup_{m\to\infty}|\psi(x_{n_m})|\left(\frac{(1+|x_{n_m}|)(1-|x_{n_m}|)}{(1+|\varphi(x_{n_m})|)(1-|\varphi(x_{n_m})|)}\right)^{\gamma/2}\\ &\le 2^{\gamma/2}\lim_{m\to\infty}|\psi(x_{n_m})|\limsup_{m\to\infty}\left(\frac{1-|x_{n_m}|}{1-|\varphi(x_{n_m})|}\right)^{\gamma/2}\\ &= 0,\end{aligned}$$

where the last equality follows from the fact that $\liminf \frac{1-|\varphi(x_{n_k})|}{1-|x_{n_k}|} \neq 0$ (see [17, Corollary 2.40]). Since $k_{x_{n_m}}$ tends to zero weakly as $m \to \infty$ (see [17, Theorem 2.17]), by [18, Theorem 2.3, p. 350], $C^*_{\psi,\varphi}$ is not left semi-Fredholm. This is a contradiction. Hence ψ is bounded away from zero near the unit circle. Denote the inner product on H^2 or A^2_α by by $\langle\cdot,\cdot\rangle_\gamma$, where $\gamma = 1$ for H^2 and $\gamma = \alpha + 2$ for A^2_α. Define the bounded linear functional F_ψ by $F_\psi(f) = \langle f, \psi\rangle_\gamma$ for each f that belongs to H^2 or A^2_α, where $\gamma = 1$ for H^2 and $\gamma = \alpha + 2$ for A^2_α. We know that, for each $\zeta \in \partial\mathbb{D}$, $K_{r\zeta}/\|K_{r\zeta}\|_\gamma$ tends to zero weakly as $r \to 1$. Then

$$\lim_{r\to1}F_\psi\left(\frac{K_{r\zeta}}{\|K_{r\zeta}\|_\gamma}\right) = \lim_{r\to1}\left\langle\frac{K_{r\zeta}}{\|K_{r\zeta}\|_\gamma}, \psi\right\rangle_\gamma = 0,$$

and so $|\psi(r\zeta)|/\|K_{r\zeta}\|_\gamma \to 0$ as $r \to 1$. Now, we show that φ is inner. For each $\zeta \in \partial\mathbb{D}$ such that $\varphi(\zeta) := \lim_{r\to1}\varphi(r\zeta)$ exists, by Lemma 2.1 we have

$$\begin{aligned}\lim_{r\to1}\|C^*_{\psi,\varphi}k_{r\zeta}\|_\gamma &= \lim_{r\to1}\frac{|\psi(r\zeta)|}{\|K_{r\zeta}\|_\gamma}\left(\frac{1}{1-|\varphi(r\zeta)|^2}\right)^{\gamma/2}\\ &\le \lim_{r\to1}\frac{|\psi(r\zeta)|}{\|K_{r\zeta}\|_\gamma}\left(\frac{1}{1-|\varphi(r\zeta)|}\right)^{\gamma/2}.\end{aligned}$$

Since $|\psi(r\zeta)|/\|K_{r\zeta}\|_\gamma \to 0$ as $r \to 1$, if $\varphi(\zeta) \notin \partial\mathbb{D}$, then $\lim_{r\to1}\|C^*_{\psi,\varphi}k_{r\zeta}\|_\gamma = 0$, which is a contradiction (see [18, Theorem 2.3, p. 350]). Hence φ is inner. Since $C^*_{\psi,\varphi}$ is left semi-Fredholm, by Lemma 2.2, T^*_ψ is left semi-Fredholm. Then $\dim\ker T^*_\psi$ is finite. It follows from Lemma 2.1 that ψ has only finite zeroes in $\mathbb{D}$. If φ is constant on $\mathbb{D}$, then $\dim\ker C^*_{\psi,\varphi} = \dim(\operatorname{ran} C_{\psi,\varphi})^\perp = \infty$, a contradiction. If $\varphi(a) = \varphi(b)$ for some $a, b \in \mathbb{D}$ with $a \neq b$, then by using the idea similar to that used in [2, Lemma] there exist infinite sets $\{a_n\}$ and $\{b_n\}$ in $\mathbb{D}$ which are disjoint and such that $\varphi(a_n) = \varphi(b_n)$. We can assume that $\psi(a_n)\psi(b_n) \neq 0$ because ψ has only finite zeroes in $\mathbb{D}$. By Lemma 2.1 we see that

$$C^*_{\psi,\varphi}\left(\frac{K_{a_n}}{\overline{\psi(a_n)}} - \frac{K_{b_n}}{\overline{\psi(b_n)}}\right) = K_{\varphi(a_n)} - K_{\varphi(b_n)} \equiv 0.$$

Therefore, $K_{a_n}/\overline{\psi(a_n)} - K_{b_n}/\overline{\psi(b_n)} \in \ker C^*_{\psi,\varphi}$. It is not hard to see that $\{K_{a_n}/\overline{\psi(a_n)} - K_{b_n}/\overline{\psi(b_n)}\}$ is a linearly independent set in the kernel of $C^*_{\psi,\varphi}$, and so we have our desired contradiction. Hence φ must be univalent. Then [17, Corollary 3.28] implies that φ is an automorphism of $\mathbb{D}$. Since $C_{\psi,\varphi}C^{-1}_\varphi = M_\psi$ is a bounded multiplication operator on H^2 and A^2_α, by [19, p. 215], $\psi \in H^\infty$.

Conversely, suppose that $\varphi \in \mathrm{Aut}(\mathbb{D})$ and $\psi \in H^\infty$ is bounded away from zero near the unit circle. Since C_φ is invertible, C_φ has a closed range. Since $\psi \not\equiv 0$, $\ker T_\psi = (0)$. We infer that T_ψ has a closed range by [18, Corollary 2.4, p. 352], [20, Theorem 3], and [12, Theorem 8], so by [18, Proposition 6.4, p. 208], T_ψ is bounded below. We claim that $C_{\psi,\varphi}$ has a closed range. This can be seen as follows. Suppose that $\{h_n\}$ is a sequence such that $\{C_{\psi,\varphi}(h_n)\}$ converges to f as $n \to \infty$. Since T_ψ has a closed range, $\{C_{\psi,\varphi}(h_n)\}$ converges to $T_\psi g$ for some g as $n \to \infty$. Since T_ψ is bounded below, there is a constant $c > 0$ such that $\|T_\psi(C_\varphi(h_n) - g)\| \geq c\|C_\varphi(h_n) - g\|$. Therefore $C_\varphi(h_n) \to g$ as $n \to \infty$. There exists h such that $C_\varphi(h) = g$ because C_φ has a closed range. Hence $f = C_{\psi,\varphi}(h)$, as desired. Hence $\mathrm{ran}\, C_{\psi,\varphi}$ is closed and $\ker C_{\psi,\varphi} = (0)$. [20, Theorem 3] and [12, Theorem 10] imply that T_ψ is Fredholm, and so $\ker T_\psi^*$ is finite dimensional. Since $\varphi \in \mathrm{Aut}(\mathbb{D})$, it is not hard to see that

$$\ker C_{\psi,\varphi}^* = (\mathrm{ran}\, C_{\psi,\varphi})^\perp = (\mathrm{ran}\, T_\psi)^\perp = \ker T_\psi^*.$$

Therefore, $\dim\ker C_{\psi,\varphi}^* < \infty$, and the conclusion follows from [18, Corollary 2.4, p. 352]. □

In the next proposition, we give a necessary condition of ψ for an operator $C_{\psi,\varphi}^*$ to be bounded below on H^2 and A_α^2. Then we use Proposition 2.4 to obtain all invertible weighted composition operators on H^2 and A_α^2.

Proposition 2.4 *Let ψ be an analytic map of $\mathbb{D}$, and let φ be an analytic self-map of $\mathbb{D}$. If $C_{\psi,\varphi}^*$ is bounded below on H^2 or A_α^2, then $\psi \in H^\infty$ is bounded away from zero on $\mathbb{D}$, and $\varphi \in \mathrm{Aut}(\mathbb{D})$.*

Proof Let $\varphi \equiv d$ for some $d \in \mathbb{D}$. Since $C_{\psi,\varphi}^*$ is bounded below, there is a constant $c > 0$ such that $\|C_{\psi,\varphi}^* f\|_\gamma \geq c\|f\|_\gamma$ for all f. Then for each $w \in \mathbb{D}$, by Lemma 2.1, $\|C_{\psi,\varphi}^* K_w\|_\gamma = |\psi(w)|\|K_d\|_\gamma \geq c\|K_w\|_\gamma$. Therefore, for each $w \in \mathbb{D}$,

$$\bigl|\psi(w)\bigr| \geq \frac{c}{\|K_d\|_\gamma}\frac{1}{(1-|w|^2)^{\gamma/2}}.$$

It is easy to see that ψ is bounded away from zero on $\mathbb{D}$. Now assume that φ is not a constant function. Suppose that ψ is not bounded away from zero on $\mathbb{D}$. Therefore, there exist a sequence $\{x_n\}$ in $\mathbb{D}$ and $a \in \overline{\mathbb{D}}$ such that $x_n \to a$ and $|\psi(x_n)| \to 0$ as $n \to \infty$. First, suppose that $a \in \mathbb{D}$. By Lemma 2.1 we have

$$\bigl\|C_{\psi,\varphi}^* k_a\bigr\|_\gamma = \bigl|\psi(a)\bigr|\left(\frac{1-|a|^2}{1-|\varphi(a)|^2}\right)^{\gamma/2} = 0.$$

Since $C_{\psi,\varphi}^*$ is bounded below, $0 \geq c\|k_a\|_\gamma = c$, a contradiction. Now assume that $a \in \partial\mathbb{D}$. It is not hard to see that there is a subsequence $\{x_{n_m}\}$ such that $\{\varphi(x_{n_m})\}$ converges. By Lemma 2.1 we see that

$$\limsup_{m\to\infty}\bigl\|C_{\psi,\varphi}^* k_{x_{n_m}}\bigr\|_\gamma = \limsup_{m\to\infty}\bigl|\psi(x_{n_m})\bigr|\left(\frac{1-|x_{n_m}|^2}{1-|\varphi(x_{n_m})|^2}\right)^{\gamma/2}. \tag{1}$$

If $\{\varphi(x_{n_m})\}$ converges to a point in $\mathbb{D}$, then (1) is equal to zero. Now assume that $\{\varphi(x_{n_m})\}$ converges to a point in $\partial\mathbb{D}$. If φ has a finite angular derivative at a, then by the Julia-

Carathéodory theorem we have

$$\limsup_{m\to\infty} \frac{1-|x_{n_m}|^2}{1-|\varphi(x_{n_m})|^2} = \frac{1}{|\varphi'(a)|},$$

which shows that (1) is equal to zero. If φ does not have a finite angular derivative at a, then

$$\limsup_{m\to\infty} \frac{1-|x_{n_m}|}{1-|\varphi(x_{n_m})|} = 0,$$

so again (1) is equal to zero. Since $C^*_{\psi,\varphi}$ is bounded below and $\|k_{x_{n_m}}\|_\gamma = 1$, we have $c = 0$, is a contradiction. Therefore, ψ is bounded away from zero on $\mathbb{D}$. Since by [18, Proposition 6.4, p. 208], $0 \notin \sigma_{\text{ap}}(C^*_{\psi,\varphi})$, we have that $\lim_{r\to 1} \|C^*_{\psi,\varphi} k_{r\zeta}\|_\gamma \neq 0$ for all $\zeta \in \partial\mathbb{D}$. We employ the idea of the proof of Proposition 2.3 to see that φ is a univalent inner function. Thus $\varphi \in \operatorname{Aut}(\mathbb{D})$ (see [17, Corollary 3.28]). Moreover, since $C_{\psi,\varphi}$ is a bounded operator, as we saw in the proof of Proposition 2.3, we conclude that $\psi \in H^\infty$, and the proposition follows. □

Bourdon [21, Theorem 3.4] obtained the following corollary; we give another proof (see also [22, Theorem 2.0.1]).

Corollary 2.5 *Let ψ be an analytic map of $\mathbb{D}$, and let φ be an analytic self-map of $\mathbb{D}$. The weighted composition operator $C_{\psi,\varphi}$ is invertible on H^2 or A^2_α if and only if $\varphi \in \operatorname{Aut}(\mathbb{D})$ and $\psi \in H^\infty$ is bounded away from zero on $\mathbb{D}$.*

Proof Let $C_{\psi,\varphi}$ be invertible. Then $C^*_{\psi,\varphi}$ is bounded below. The conclusion follows from Proposition 2.4. The reverse direction is trivial since C_φ and T_ψ are invertible. □

Note that if $C_{\psi,\varphi}$ is invertible, then $C^*_{\psi,\varphi}$ is bounded below. Hence by Proposition 2.4 and Corollary 2.5 we can see that $C^*_{\psi,\varphi}$ is bounded below if and only if $C_{\psi,\varphi}$ is invertible.

The algebra $A(\mathbb{D})$ consists of all continuous functions on the closure of $\mathbb{D}$ that are analytic on $\mathbb{D}$. In the next corollary, we find some Fredholm weighted composition operators that are not invertible.

Corollary 2.6 *Suppose that $\varphi \in \operatorname{Aut}(\mathbb{D})$ and $\psi \in A(\mathbb{D})$. Assume that $\{z \in \mathbb{D} : \psi(z) = 0\}$ is a nonempty finite set and $\psi(z) \neq 0$ for all $z \in \partial\mathbb{D}$. Then $C_{\psi,\varphi}$ is Fredholm, but it is not invertible.*

Proof It is easy to see that ψ is bounded away from zero near the unit circle. Therefore the result follows from Proposition 2.3 and Corollary 2.5. □

Theorem 2.7 *Suppose that ψ is an analytic map of $\mathbb{D}$ and φ is an analytic self-map of $\mathbb{D}$. The operator $C^*_{\psi,\varphi}$ is bounded below on H^2 or A^2_α if and only if $\varphi \in \operatorname{Aut}(\mathbb{D})$ and $\psi \in H^\infty$ is bounded away from zero on $\mathbb{D}$.*

3 The operators $C_{\psi_1,\varphi_1}C^*_{\psi_2,\varphi_2}$ and $C^*_{\psi_1,\varphi_1}C_{\psi_2,\varphi_2}$

In this section, we find all Fredholm operators $C_{\psi_1,\varphi_1}C^*_{\psi_2,\varphi_2}$ and $C^*_{\psi_1,\varphi_1}C_{\psi_2,\varphi_2}$.

A linear-fractional self-map of $\mathbb{D}$ is a mapping of the form $\varphi(z) = (az + b)/(cz + d)$ with $ad - bc \neq 0$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. We denote the set of those maps by $\mathrm{LFT}(\mathbb{D})$. Suppose $\varphi(z) = (az + b)/(cz + d)$. It is well known that the adjoint of C_φ acting on H^2 and A^2_α is given by $C^*_\varphi = T_g C_\sigma T^*_h$, where $\sigma(z) = (\overline{a}z - \overline{c})/(-\overline{b}z + \overline{d})$ is a self-map of $\mathbb{D}$, $g(z) = (-\overline{b}z + \overline{d})^{-\gamma}$, and $h(z) = (cz + d)^\gamma$. Note that g and h are in H^∞ ([17, Theorem 9.2]). If $\varphi(\zeta) = \eta$ for $\zeta, \eta \in \partial\mathbb{D}$, then $\sigma(\eta) = \zeta$. We know that φ is an automorphism if and only if σ is, and in this case, $\sigma = \varphi^{-1}$. The map σ is called the Krein adjoint of φ. We denote by $F(\varphi)$ the set of all points in $\partial\mathbb{D}$ at which φ has a finite angular derivative.

Example 3.1 Suppose that $\varphi \in \mathrm{LFT}(\mathbb{D})$ is not an automorphism of $\mathbb{D}$. Assume that $\psi \in H^\infty$ is continuously extendable to $\mathbb{D} \cup F(\varphi)$. Assume that $C_{\psi,\varphi}C^*_{\psi,\varphi}$ is considered as an operator on H^2 or A^2_α. Since φ is not an automorphism of $\mathbb{D}$, $\overline{\varphi(\mathbb{D})} \subseteq \mathbb{D}$ or there is only one point $\zeta \in \partial\mathbb{D}$ such that $\varphi(\zeta) \in \partial\mathbb{D}$. If $\overline{\varphi(\mathbb{D})} \subseteq \mathbb{D}$, then by [17, p. 129], C_φ is compact. It is easy to see that $C_{\psi,\varphi}C^*_{\psi,\varphi}$ is a compact operator. Since compact operators are not Fredholm, we can see that $C_{\psi,\varphi}C^*_{\psi,\varphi}$ is not Fredholm.

In the other case, assume that $F(\varphi) = \{\zeta\}$. Because for each $w \in \partial\mathbb{D}$ such that $w \neq \zeta$, $\sigma(\varphi(w)) \notin \partial\mathbb{D}$, we obtain $\sigma \circ \varphi \notin \mathrm{Aut}(\mathbb{D})$. Since $C_{\sigma\circ\varphi}$ is not Fredholm (see e.g. [2] and [16]), $0 \in \sigma_e(C_{\sigma\circ\varphi})$. By [23, Corollary 2.2] and [4, Proposition 2.3] there is a compact operator K such that

$$C_{\psi,\varphi}C^*_{\psi,\varphi} = \left|\psi(\zeta)\right|^2 C_\varphi C^*_\varphi + K.$$

Also, [23, Theorem 3.1], [23, Proposition 3.6], and [24, Theorem 3.2] imply that there is a compact operator K' such that

$$C_{\psi,\varphi}C^*_{\psi,\varphi} = \left|\psi(\zeta)\right|^2 \left|\varphi'(\zeta)\right|^{-\gamma} C_{\sigma\circ\varphi} + K'. \tag{2}$$

From the fact that $0 \in \sigma_e(C_{\sigma\circ\varphi})$ and equation (2) we can infer that $0 \in \sigma_e(C_{\psi,\varphi}C^*_{\psi,\varphi})$. Then $C_{\psi,\varphi}C^*_{\psi,\varphi}$ is not Fredholm.

By the preceding example it seems natural to conjecture that if $C_{\psi,\varphi}C^*_{\psi,\varphi}$ is Fredholm, then $\varphi \in \mathrm{Aut}(\mathbb{D})$. We will prove our conjecture in Theorem 3.2 and show that if $C_{\psi_1,\varphi_1}C^*_{\psi_2,\varphi_2}$ is Fredholm on H^2 or A^2_α, then C_{ψ_1,φ_1} and C_{ψ_2,φ_2} are Fredholm.

Theorem 3.2 *The operator $C_{\psi_1,\varphi_1}C^*_{\psi_2,\varphi_2}$ is Fredholm on H^2 or A^2_α if and only if $\varphi_1, \varphi_2 \in \mathrm{Aut}(\mathbb{D})$, $\psi_1, \psi_2 \in H^\infty$, and ψ_1 and ψ_2 are bounded away from zero near the unit circle.*

Proof Let $C_{\psi_1,\varphi_1}C^*_{\psi_2,\varphi_2}$ be Fredholm. Therefore $C^*_{\psi_2,\varphi_2}$ is left semi-Fredholm. By Proposition 2.3 we see that $\varphi_2 \in \mathrm{Aut}(\mathbb{D})$ and $\psi_2 \in H^\infty$ is bounded away from zero near the unit circle. Since $C_{\psi_2,\varphi_2}C^*_{\psi_1,\varphi_1}$ is Fredholm, again we can see that φ_1 is an automorphism of $\mathbb{D}$ and $\psi_1 \in H^\infty$ is bounded away from zero near the unit circle.

Conversely, let $\varphi_1, \varphi_2 \in \mathrm{Aut}(\mathbb{D})$ and $\psi_1, \psi_2 \in H^\infty$ be bounded away from zero near the unit circle. By Proposition 2.3, C_{ψ_1,φ_1} and $C^*_{\psi_2,\varphi_2}$ are Fredholm, so the result follows. □

In the following theorem for functions $\psi_1, \psi_2 \in A(\mathbb{D})$, we find all Fredholm operators $C^*_{\psi_1,\varphi_1}C_{\psi_2,\varphi_2}$ when φ_1 and φ_2 are univalent self-maps of $\mathbb{D}$.

Theorem 3.3 *Suppose that $\psi_1, \psi_2 \in A(\mathbb{D})$. Let φ_1 and φ_2 be univalent self-maps of $\mathbb{D}$. The operator $C^*_{\psi_1,\varphi_1} C_{\psi_2,\varphi_2}$ is Fredholm on H^2 or A^2_α if and only if C_{ψ_1,φ_1} and C_{ψ_2,φ_2} are Fredholm on H^2 or A^2_α, respectively.*

Proof Let $C^*_{\psi_1,\varphi_1} C_{\psi_2,\varphi_2}$ be Fredholm on H^2 or A^2_α. Then $C^*_{\psi_2,\varphi_2} C_{\psi_1,\varphi_1}$ is also Fredholm. It is easy to see that C_{φ_2} and C_{φ_1} are left semi-Fredholm. Therefore, $0 \notin \sigma_{\mathrm{le}}(C_{\varphi_1})$ and $0 \notin \sigma_{\mathrm{le}}(C_{\varphi_2})$. Since $\dim\ker C^*_{\psi_1,\varphi_1} C_{\psi_2,\varphi_2} < \infty$ and $\dim\ker C^*_{\psi_2,\varphi_2} C_{\psi_1,\varphi_1} < \infty$, $\psi_1 \not\equiv 0$, $\psi_2 \not\equiv 0$, and φ_1 and φ_2 are not constant functions. By the open mapping theorem we know that $0 \notin \sigma_p(C_{\varphi_1})$ and $0 \notin \sigma_p(C_{\varphi_2})$. Now [18, Proposition 4.4, p. 359] implies that $0 \notin \sigma_{\mathrm{ap}}(C_{\varphi_1})$ and $0 \notin \sigma_{\mathrm{ap}}(C_{\varphi_2})$. Hence by [18, Proposition 6.4, p. 208], $\operatorname{ran} C_{\varphi_1}$ and $\operatorname{ran} C_{\varphi_2}$ are closed. By [1, Theorem 5.1], $\varphi_1, \varphi_2 \in \operatorname{Aut}(\mathbb{D})$. Since $C^*_{\varphi_1}$ and C_{φ_2} are invertible, $(C^*_{\varphi_1})^{-1}$ and $C^{-1}_{\varphi_2}$ are Fredholm. Then $T^*_{\psi_1} T_{\psi_2}$ is Fredholm, and so $0 \notin \sigma_e(T_{\overline{\psi_1}\psi_2})$. We get from [25] and [20, Theorem 2] that ψ_1 and ψ_2 never vanish on $\partial\mathbb{D}$. Since $\psi_1 \not\equiv 0$ and $\psi_2 \not\equiv 0$, ψ_1 and ψ_2 have only finite zeroes on $\mathbb{D}$. This implies that there is $r < 1$ such that for each w with $r < |w| < 1$, $\psi_1(w) \neq 0$ and $\psi_2(w) \neq 0$. Therefore, ψ_1 and ψ_2 are bounded away from zero near the unit circle. Therefore, by Proposition 2.3, C_{ψ_1,φ_1} and C_{ψ_2,φ_2} are Fredholm on H^2 or A^2_α. The reverse implication follows from the fact stated before Theorem 3.2. □

Acknowledgements
The author would like to thank the referees for their valuable comments and suggestions.

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author read and approved the final manuscript.

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Received: 28 August 2017 Accepted: 16 January 2018 Published online: 22 January 2018

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