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Bounds for the M-spectral radius of a fourth-order partially symmetric tensor

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Abstract

M-eigenvalues of fourth-order partially symmetric tensors play an important role in many real fields such as quantum entanglement and nonlinear elastic materials analysis. In this paper, we give two bounds for the maximal absolute value of all the M-eigenvalues (called the M-spectral radius) of a fourth-order partially symmetric tensor and discuss the relation of them. A numerical example is given to explain the proposed results.

Keywords: M-eigenvalues; M-spectral radius; partially symmetric; tensors

1 Introduction

A fourth-order real tensor $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{m \times n \times m \times n}$ is called partially symmetric [1] if it has the following symmetry:

$$a_{i_1 i_2 i_3 i_4} = a_{i_3 i_2 i_1 i_4} = a_{i_1 i_4 i_3 i_2}, \quad \forall i_1, i_3 \in [m], \forall i_2, i_4 \in [n],$$

where $[m] = \{1, 2, \dots, m\}$ and $[n] = \{1, 2, \dots, n\}$. Such a tensor often arises in nonlinear elastic materials analysis [2, 3] and entanglement studies in quantum physics [4–6]. For this tensor, there are many kinds of eigenvalues such as H-eigenvalues, Z-eigenvalues, and D-eigenvalues [7, 8]; here we only discuss its M-eigenvalues [1, 9].

Definition 1 ([9]) Let $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{m \times n \times m \times n}$ be a partially symmetric tensor, and let $\lambda \in \mathbb{R}$. Suppose that there are real vectors $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ such that

$$\begin{cases} \mathcal{A} \cdot yxy = \lambda x, \\ \mathcal{A}xyx = \lambda y, \\ x^T x = 1, \\ y^T y = 1, \end{cases} \quad (1)$$

where $\mathcal{A} \cdot yxy \in \mathbb{R}^m$ and $\mathcal{A}xyx \in \mathbb{R}^n$ with i th components

$$(\mathcal{A} \cdot yxy)_i = \sum_{i_3 \in [m]} \sum_{i_2, i_4 \in [n]} a_{ii_2 i_3 i_4} y_{i_2} x_{i_3} y_{i_4}$$

and

$$(\mathcal{A}xyx)_i = \sum_{i_1, i_3 \in [m]} \sum_{i_2 \in [n]} a_{i_1 i_2 i_3 i} x_{i_1} y_{i_2} x_{i_3}.$$

Then λ is called an M-eigenvalue of \mathcal{A} with left M-eigenvector x and right M-eigenvector y .

Note that M-eigenvalues of a fourth-order partially symmetric tensor always exist [1]. They have a close relation to many problems in the theory of elasticity and quantum physics [1, 9, 10]. For example, the largest M-eigenvalue of $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{m \times n \times m \times n}$, denoted by

$$\lambda^* = \max\{\lambda : \lambda \text{ is an M-eigenvalue of } \mathcal{A}\},$$

is the optimum solution of the problem (for details, see [9])

$$\begin{aligned} \max \quad & f(x, y) = \sum_{i_1, i_3=1}^m \sum_{i_2, i_4=1}^n a_{i_1 i_2 i_3 i_4} x_{i_1} y_{i_2} x_{i_3} y_{i_4}, \\ \text{s.t.} \quad & x^T x = 1, \quad y^T y = 1, \quad x \in \mathbb{R}^m, y \in \mathbb{R}^n. \end{aligned}$$

The outer product $\lambda x \circ y \circ x \circ y$, where

$$(\lambda x \circ y \circ x \circ y)_{i_1 i_2 i_3 i_4} = \lambda x_{i_1} y_{i_2} x_{i_3} y_{i_4}, \quad \forall i_1, i_3 \in [m], \forall i_2, i_4 \in [n]$$

and λ is an M-eigenvalue with the maximal absolute value of $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{m \times n \times m \times n}$ with left M-eigenvector $x \in \mathbb{R}^m$ and right M-eigenvector $y \in \mathbb{R}^n$, is a partially symmetric best rank-one approximation of \mathcal{A} [1], which has wide applications in signal and image processing, wireless communication systems, and independent component analysis [11–14]. The M-spectral radius of $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{m \times n \times m \times n}$, denoted by

$$\rho_M(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an M-eigenvalue of } \mathcal{A}\},$$

has significant impacts on identifying nonsingular \mathcal{M} -tensors, which satisfy the strong ellipticity condition [10].

To our knowledge, there are few results about bounds for the M-spectral radius of a fourth-order partially symmetric tensor. In this paper, we present two bounds for the M-spectral radius and discuss their relation. A numerical example is also given to explain the proposed results.

2 Two bounds for the M-spectral radius

In this section, we give two bounds for the M-spectral radius of fourth-order partially symmetric tensors and discuss their relation.

Theorem 1 *Let $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{m \times n \times m \times n}$ be a partially symmetric tensor. Then*

$$\rho_M(\mathcal{A}) \leq \sqrt{\max_{i \in [m]} \{R_i(\mathcal{A})\} \cdot \max_{l \in [n]} \{C_l(\mathcal{A})\}}, \quad (2)$$

where

$$R_i(\mathcal{A}) = \sum_{i_3 \in [m]} \sum_{i_2, i_4 \in [n]} |a_{ii_2i_3i_4}|, \quad C_l(\mathcal{A}) = \sum_{i_1, i_3 \in [m]} \sum_{i_2 \in [n]} |a_{i_1i_2i_3l}|.$$

Proof Suppose that λ is an M-eigenvalue of \mathcal{A} and that $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ are associated left M-eigenvector and right M-eigenvector. Then (1) holds. Let

$$x_p = \max_{k \in [m]} \{|x_k|\}, \quad y_q = \max_{k \in [n]} \{|y_k|\}.$$

Since $x^T x = 1$ and $y^T y = 1$, we have

$$0 < |x_p| \leq 1, \quad 0 < |y_q| \leq 1. \quad (3)$$

The p th equation of $\mathcal{A} \cdot yxy = \lambda x$ is

$$\lambda x_p = \sum_{i_3 \in [m]} \sum_{i_2, i_4 \in [n]} a_{pi_2i_3i_4} y_{i_2} x_{i_3} y_{i_4}. \quad (4)$$

Taking the absolute values on both sides of (4) and using the triangle inequality give

$$\begin{aligned} |\lambda| |x_p| &\leq \sum_{i_3 \in [m]} \sum_{i_2, i_4 \in [n]} |a_{pi_2i_3i_4}| |y_{i_2}| |x_{i_3}| |y_{i_4}| \\ &\leq \sum_{i_3 \in [m]} \sum_{i_2, i_4 \in [n]} |a_{pi_2i_3i_4}| |y_q| \\ &= R_p(\mathcal{A}) |y_q|. \end{aligned} \quad (5)$$

Similarly, by the q th equation of $\mathcal{A}xyx = \lambda y$ we have

$$\begin{aligned} |\lambda| |y_q| &\leq \sum_{i_1, i_3 \in [m]} \sum_{i_2 \in [n]} |a_{i_1i_2i_3q}| |x_{i_1}| |y_{i_2}| |x_{i_3}| \\ &\leq \sum_{i_1, i_3 \in [m]} \sum_{i_2 \in [n]} |a_{i_1i_2i_3q}| |x_p| \\ &= C_q(\mathcal{A}) |x_p|. \end{aligned} \quad (6)$$

Multiplying (5) and (6) gives

$$|\lambda|^2 |x_p| |y_q| \leq R_p(\mathcal{A}) C_q(\mathcal{A}) |x_p| |y_q|,$$

which, together with (3), yields

$$|\lambda|^2 \leq R_p(\mathcal{A}) C_q(\mathcal{A}) \leq \max_{i \in [m], l \in [n]} \{R_i(\mathcal{A}) C_l(\mathcal{A})\}. \quad (7)$$

Since (7) holds for all M-eigenvalues of \mathcal{A} , we have

$$\rho_M(\mathcal{A}) \leq \sqrt{\max_{i \in [m], l \in [n]} \{R_i(\mathcal{A}) C_l(\mathcal{A})\}} = \sqrt{\max_{i \in [m]} \{R_i(\mathcal{A})\} \cdot \max_{l \in [n]} \{C_l(\mathcal{A})\}},$$

and the conclusion follows. \square

Theorem 2 Let $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{m \times n \times m \times n}$ be a partially symmetric tensor, and let α be any subset of $[m]$ and β be any subset of $[n]$. Then

$$\rho_M(\mathcal{A}) \leq \min\{\mu_1, \mu_2\}, \quad (8)$$

where

$$\begin{aligned} \mu_1 &= \min_{\alpha \subseteq [m]} \left\{ \max_{p \in [m], q \in [n]} \left\{ \frac{1}{2} (R_p^\alpha(\mathcal{A}) + \sqrt{R_p^\alpha(\mathcal{A})^2 + 4(R_p(\mathcal{A}) - R_p^\alpha(\mathcal{A}))C_q(\mathcal{A})}) \right\} \right\}, \\ \mu_2 &= \min_{\beta \subseteq [n]} \left\{ \max_{p \in [m], q \in [n]} \left\{ \frac{1}{2} (C_q^\beta(\mathcal{A}) + \sqrt{C_q^\beta(\mathcal{A})^2 + 4(C_q(\mathcal{A}) - C_q^\beta(\mathcal{A}))R_p(\mathcal{A})}) \right\} \right\}, \end{aligned}$$

and

$$R_p^\alpha(\mathcal{A}) = \sum_{i_3 \in \alpha} \sum_{i_2, i_4 \in [n]} |a_{pi_2 i_3 i_4}|, \quad C_q^\beta(\mathcal{A}) = \sum_{i_2 \in \beta} \sum_{i_1, i_3 \in [m]} |a_{i_1 i_2 i_3 q}|.$$

Proof Assume that λ is an M-eigenvalue of \mathcal{A} and that $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ are the corresponding left M-eigenvector and right M-eigenvector. Then (1) holds. Let

$$|x_p| = \max_{k \in [m]} \{|x_k|\}, \quad |y_q| = \max_{k \in [n]} \{|y_k|\}.$$

Then (3) holds. The p th equation of $\mathcal{A} \cdot yxy = \lambda x$ can be rewritten as

$$\lambda x_p = \sum_{i_3 \in \alpha} \sum_{i_2, i_4 \in [n]} a_{pi_2 i_3 i_4} y_{i_2} x_{i_3} y_{i_4} + \sum_{i_3 \notin \alpha} \sum_{i_2, i_4 \in [n]} a_{pi_2 i_3 i_4} y_{i_2} x_{i_3} y_{i_4}. \quad (9)$$

By the technique for the inequality in Theorem 1, we obtain from (9) that

$$\begin{aligned} |\lambda| |x_p| &\leq \sum_{i_3 \in \alpha} \sum_{i_2, i_4 \in [n]} |a_{pi_2 i_3 i_4}| |y_{i_2}| |x_p| |y_{i_4}| + \sum_{i_3 \notin \alpha} \sum_{i_2, i_4 \in [n]} |a_{pi_2 i_3 i_4}| |y_{i_2}| |x_{i_3}| |y_q| \\ &\leq \sum_{i_3 \in \alpha} \sum_{i_2, i_4 \in [n]} |a_{pi_2 i_3 i_4}| |x_p| + \sum_{i_3 \notin \alpha} \sum_{i_2, i_4 \in [n]} |a_{pi_2 i_3 i_4}| |y_q| \\ &= R_p^\alpha(\mathcal{A}) |x_p| + (R_p(\mathcal{A}) - R_p^\alpha(\mathcal{A})) |y_q|, \end{aligned}$$

that is,

$$(|\lambda| - R_p^\alpha(\mathcal{A})) |x_p| \leq (R_p(\mathcal{A}) - R_p^\alpha(\mathcal{A})) |y_q|. \quad (10)$$

In addition, by the q th equation of $\mathcal{A}xyx = \lambda y$ we have

$$|\lambda| |y_q| \leq \sum_{i_1, i_3 \in [m]} \sum_{i_2 \in [n]} |a_{i_1 i_2 i_3 q}| |x_p| = C_q(\mathcal{A}) |x_p|. \quad (11)$$

Multiplying (10) with (11) and using (3) yield

$$(|\lambda| - R_p^\alpha(\mathcal{A})) |\lambda| \leq (R_p(\mathcal{A}) - R_p^\alpha(\mathcal{A})) C_q(\mathcal{A}). \quad (12)$$

Then

$$\begin{aligned} |\lambda| &\leq \frac{1}{2} (R_p^\alpha(\mathcal{A}) + \sqrt{R_p^\alpha(\mathcal{A})^2 + 4(R_p(\mathcal{A}) - R_p^\alpha(\mathcal{A}))C_q(\mathcal{A})}) \\ &\leq \max_{p \in [m], q \in [n]} \left\{ \frac{1}{2} (R_p^\alpha(\mathcal{A}) + \sqrt{R_p^\alpha(\mathcal{A})^2 + 4(R_p(\mathcal{A}) - R_p^\alpha(\mathcal{A}))C_q(\mathcal{A})}) \right\}. \end{aligned} \quad (13)$$

Note that (13) holds for all M-eigenvalues of \mathcal{A} and any $\alpha \subseteq [m]$. Hence

$$\rho_M(\mathcal{A}) \leq \mu_1. \quad (14)$$

On the other hand, for the q th equation of $\mathcal{A}xyx = \lambda y$, we have

$$\lambda y_q = \sum_{i_2 \in \beta} \sum_{i_1, i_3 \in [m]} a_{i_1 i_2 i_3 q} x_{i_1} y_{i_2} x_{i_3} + \sum_{i_2 \notin \beta} \sum_{i_1, i_3 \in [m]} a_{i_1 i_2 i_3 q} x_{i_1} y_{i_2} x_{i_3}. \quad (15)$$

Then

$$\begin{aligned} |\lambda| |y_q| &\leq \sum_{i_2 \in \beta} \sum_{i_1, i_3 \in [m]} |a_{i_1 i_2 i_3 q}| |y_q| + \sum_{i_2 \notin \beta} \sum_{i_1, i_3 \in [m]} |a_{i_1 i_2 i_3 q}| |x_p| \\ &= C_q^\beta(\mathcal{A}) |y_q| + (C_q(\mathcal{A}) - C_q^\beta(\mathcal{A})) |x_p|, \end{aligned}$$

that is,

$$(|\lambda| - C_q^\beta(\mathcal{A})) |y_q| \leq (C_q(\mathcal{A}) - C_q^\beta(\mathcal{A})) |x_p|. \quad (16)$$

By the p th equation of $\mathcal{A} \cdot yxy = \lambda x$ we have

$$|\lambda| |x_p| \leq \sum_{i_3 \in [m]} \sum_{i_2, i_4 \in [n]} |a_{p i_2 i_3 i_4}| |y_q| = R_p(\mathcal{A}) |y_q|. \quad (17)$$

Multiplying (16) with (17) and using (3), we derive

$$(|\lambda| - C_q^\beta(\mathcal{A})) |\lambda| \leq (C_q(\mathcal{A}) - C_q^\beta(\mathcal{A})) R_p(\mathcal{A}). \quad (18)$$

Hence

$$\begin{aligned} |\lambda| &\leq \frac{1}{2} (C_q^\beta(\mathcal{A}) + \sqrt{C_q^\beta(\mathcal{A})^2 + 4(C_q(\mathcal{A}) - C_q^\beta(\mathcal{A}))R_p(\mathcal{A})}) \\ &\leq \max_{p \in [m], q \in [n]} \left\{ \frac{1}{2} (C_q^\beta(\mathcal{A}) + \sqrt{C_q^\beta(\mathcal{A})^2 + 4(C_q(\mathcal{A}) - C_q^\beta(\mathcal{A}))R_p(\mathcal{A})}) \right\}. \end{aligned} \quad (19)$$

Since (19) holds for all M-eigenvalues of \mathcal{A} and any $\beta \subseteq [n]$, we have

$$\rho_M(\mathcal{A}) \leq \mu_2. \quad (20)$$

From (14) and (20) we have

$$\rho_M(\mathcal{A}) \leq \min\{\mu_1, \mu_2\}.$$

The proof is completed. \square

Remark 1 Since

$$\begin{aligned} & \max_{p \in [m], q \in [n]} \left\{ \frac{1}{2} (R_p^\alpha(\mathcal{A}) + \sqrt{R_p^\alpha(\mathcal{A})^2 + 4(R_p(\mathcal{A}) - R_p^\alpha(\mathcal{A}))C_q(\mathcal{A})}) \right\} \\ &= \max_{p \in [m], q \in [n]} \left\{ \frac{1}{2} (C_q^\beta(\mathcal{A}) + \sqrt{C_q^\beta(\mathcal{A})^2 + 4(C_q(\mathcal{A}) - C_q^\beta(\mathcal{A}))R_p(\mathcal{A})}) \right\} \\ &= \sqrt{\max_{p \in [m]} \{R_p(\mathcal{A})\} \cdot \max_{q \in [n]} \{C_q(\mathcal{A})\}} \end{aligned}$$

when $\alpha = \emptyset$ and $\beta = \emptyset$, we have

$$\min\{\mu_1, \mu_2\} \leq \sqrt{\max_{p \in [m]} \{R_p(\mathcal{A})\} \cdot \max_{q \in [n]} \{C_q(\mathcal{A})\}}.$$

Therefore, the bound in (8) is tighter than the bound in (2) for the M-spectral radius $\rho_M(\mathcal{A})$ of a given tensor \mathcal{A} .

Remark 2 Although the bound in (8) is tighter than the bound in (2), it is easier to compute the bound in (2) for the M-spectral radius of a given tensor.

Next, we use a numerical example to show the effectiveness of the bounds in Theorems 1 and 2.

Example 1 Consider the partially symmetric tensor $\mathcal{A}_1 = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{3 \times 3 \times 3 \times 3}$ with

$$\begin{aligned} a_{1111} &= 1.1112, & a_{1311} &= 6.1096, & a_{3111} &= 0.3032, & a_{2121} &= 1.4125, \\ a_{3131} &= 1, & a_{1212} &= 0.0788, & a_{2222} &= 1, & a_{3222} &= 0.6032, \\ a_{3232} &= 0.3657, & a_{1313} &= 2, & a_{2323} &= 0.6226, & a_{3333} &= 0.3, \end{aligned}$$

and the remaining zero elements. By Theorem 1 we have

$$\rho_M(\mathcal{A}_1) \leq 12.6843.$$

By Theorem 2 we have

$$\rho_M(\mathcal{A}_1) \leq 10.2397.$$

In fact, $\rho_M(\mathcal{A}_1) \approx 7.6841$.

3 Conclusions

In this paper, we have presented two bounds for the M-spectral radius of a fourth-order partially symmetric tensor and have indicated their relation. To show the effectiveness of the proposed results, a numerical example is also given.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally to this work. Both authors read and approved the final manuscript.

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