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Generalized (p,q)-Bleimann-Butzer-Hahn operators and some approximation results

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Abstract
The aim of this paper is to introduce a new generalization of Bleimann-Butzer-Hahn operators by using (p,q)-integers which is based on a continuously differentiable function μ on $[0,\infty)=\mathbb{R}_+$. We establish the Korovkin type approximation results and compute the degree of approximation by using the modulus of continuity. Moreover, we investigate the shape preserving properties of these operators.

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1 Introduction and preliminaries

The q-generalization of Bernstein polynomials [1] was introduced by Lupaş [2] as follows:

$$L_{n,q}(f;x)=\frac{1}{\prod_{j=1}^{n}\{(1-x)+q^{j-1}x\}}\sum_{i=0}^{n}f\left(\frac{[i]_q}{[n]_q}\right)\begin{bmatrix} n\\ i\end{bmatrix}_q q^{\frac{i(i-1)}{2}}x^i(1-x)^{n-i}.$$

In 1997, Phillips [3] introduced another modification of Bernstein polynomials, obtained the rate of convergence and the Voronovskaja type asymptotic expansion for these polynomials.

The (p,q)-integer was introduced in order to generalize or unify several forms of q-oscillator algebras well known in the early physics literature related to the representation theory of single parameter quantum algebras [4].

In the recent years, the first (p,q)-analogue of Bernstein operators was introduced by Mursaleen et al. (see [5]), and some approximation properties were studied (see [6–9]). Moreover, the (p,q)-calculus in computer-aided geometric design (CAGD) given by Khalid et al. (see [10]) will help readers to understand the applications. Besides this, we also refer the reader to some recent papers on (p,q)-calculus in approximation theory [11–20] and [21].

We recall some definitions and notations of (p,q)-calculus.

The (p,q) integers $[n]_{p,q}$ are defined by

$$\begin{aligned}
[n]_{p,q} &= p^{n-1} + qp^{n-2} + \cdots + q^{n-1} = \begin{cases} \frac{p^n-q^n}{p-q} & (p \neq q \neq 1), \\ \frac{1-q^n}{1-q} & (p = 1), \\ n & (p = q = 1), \end{cases} \\
(au+bv)^n_{p,q} &:= \sum_{i=0}^{n} p^{\frac{(n-i)(n-i-1)}{2}} q^{\frac{i(i-1)}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_{p,q} a^{n-i} b^i u^{n-i} v^i, \\
(u+v)^n_{p,q} &= (u+v)(pu+qv)\bigl(p^2u+q^2v\bigr)\cdots\bigl(p^{n-1}u+q^{n-1}v\bigr), \\
(1-u)^n_{p,q} &= (1-u)(p-qu)\bigl(p^2-q^2u\bigr)\cdots\bigl(p^{n-1}-q^{n-1}u\bigr)
\end{aligned} \tag{1.1}$$

and the (p,q)-binomial coefficients are defined by

$$\begin{bmatrix} n \\ i \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[i]_{p,q}![n-i]_{p,q}!}.$$

By a simple calculation [5], we have the following relation:

$$q^i[n-i+1]_{p,q} = [n+1]_{p,q} - p^{n-i+1}[i]_{p,q}.$$

For details on q-calculus and (p,q)-calculus, one can refer to [22–25].

Totik [26] studied the uniform approximation properties of Bleimann-Butzer-Hahn operators [27] when f belongs to the class $C(\mathbb{R}_+)$ of continuous functions on $\mathbb{R}_+$ that have finite limits at infinity.

The Bleimann-Butzer-Hahn operators (BBH) based on q-integers are defined as follows:

$$L_n^q(f;x) = \frac{1}{\ell_{n,q}(x)} \sum_{i=0}^{n} f\left(\frac{[i]_q}{[n-i+1]_q q^i}\right) q^{\frac{i(i-1)}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q x^i,$$

where $\ell_{n,q}(x) = \prod_{i=0}^{n-1}(1+q^s x)$. For $q = 1$, these operators reduce to the classical BBH operators [27].

For $f \in C[0,1]$, $x \in [0,1]$, Morales et al. [28] introduced a new generalization of Bernstein polynomials denoted by B_n^μ

$$B_n^\mu(f;x) = B_n\bigl(f \circ \mu^{-1}; \mu(x)\bigr) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \mu(x)^i \bigl(1-\mu(x)\bigr)^{n-i} \bigl(f \circ \mu^{-1}\bigr)\left(\frac{i}{n}\right),$$

where μ is a continuously differentiable function of infinite order on $[0,1]$ such that $\mu(0) = 0$, $\mu(1) = 1$, and $\mu'(x) > 0$ for $x \in [0,1]$. They have also studied some shape preserving and convergence properties on approximation concerning the generalized Bernstein operators $B_n^\mu(f;x)$.

For $0 < q < p \leq 1$ and f defined on semiaxis $\mathbb{R}_+$, we give a generalization of (p,q)-Bleimann-Butzer-Hahn type operators (see [21]) as follows:

$$L_{n,\mu}^{p,q}(f;x) = \frac{1}{\ell_{n,\mu}^{p,q}(x)} \sum_{i=0}^{n} \bigl(f \circ \mu^{-1}\bigr)\left(\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q} q^i}\right) p^{\frac{(n-i)(n-i-1)}{2}} q^{\frac{i(i-1)}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_{p,q} \mu(x)^i, \tag{1.2}$$

where

$$\ell_{n,\mu}^{p,q}(x)=\prod_{s=0}^{n-1}\big(p^s+q^s\mu(x)\big),$$

and μ is a continuously differentiable function defined on $\mathbb{R}_+$ having the property

$$\mu(0)=0 \quad \text{and} \quad \inf_{x\in[0,\infty)}\mu'(x)\geq 1. \tag{1.3}$$

We can easily see that

$$L_{n,\mu}^{p,q}f=L_{n,p,q}\big(f\circ\mu^{-1}\big)\mu,$$

where $L_{n,p,q}$ is defined in [21] as

$$L_{n,p,q}(f;x)=\frac{1}{\ell_{n,p,q}(x)}\sum_{i=0}^{n}f\left(\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q}q^i}\right)p^{\frac{(n-i)(n-i-1)}{2}}q^{\frac{i(i-1)}{2}}\begin{bmatrix} n\\ i\end{bmatrix}_{p,q}x^i.$$

The operators defined by (1.2) are more flexible and sensitive to the rate of convergence than the (p,q)-BBH operators. Our results show that the new operators are sensitive to the rate of convergence to f, depending on the selection of μ. For the particular case $\mu(x)=x$, the previous results for (p,q)-Bleimann-Butzer-Hahn operators are obtained (see [21]).

Lemma 1.1 *Let $L_{n,\mu}^{p,q}$ be operators defined by* (1.2). *Then, for a continuously differentiable function $\mu(x)$ on $\mathbb{R}_+$ defined by* (1.3), *we have*

$$L_{n,\mu}^{p,q}(f;x)=\begin{cases}1, & \textit{for } f(t)=1,\\ \frac{p[n]_{p,q}}{[n+1]_{p,q}}(\frac{\mu(x)}{1+\mu(x)}), & \textit{for } f(t)=\frac{\mu(t)}{1+\mu(t)},\\ \frac{pq^2[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2}\frac{\mu(x)^2}{(1+\mu(x))(p+q\mu(x))}+\frac{p^{n+1}[n]_{p,q}}{[n+1]_{p,q}^2}(\frac{\mu(x)}{1+\mu(x)}), & \textit{for } f(t)=(\frac{\mu(t)}{1+\mu(t)})^2.\end{cases} \tag{1.4}$$

Proof For the proof of this lemma, we refer to [21]. □

2 Korovkin type approximation result

Here we propose to obtain a Korovkin type approximation theorem for operators $L_{n,\mu}^{p,q}$.

Let $C_B(\mathbb{R}_+)$ denote the set of all bounded and continuous functions defined on $\mathbb{R}_+$. $C_B(\mathbb{R}_+)$ is a normed linear space with

$$\|f\|_{C_B}=\sup_{u\geq 0}\big|f(u)\big|.$$

The modulus of continuity ω is a non-negative and non-decreasing function defined on $\mathbb{R}_+$ such that it is sub-additive and $\lim_{\delta\to 0}\omega(\delta)=0$.

One can easily see that

$$\omega(n\delta)\leq n\omega(\delta), \quad n\in\mathbb{N}, \tag{2.1}$$

$$\omega(\lambda\delta)\leq\omega\big(1+\big[|\lambda|\big]\big)\leq 1+\lambda\omega(\delta), \quad \lambda>0, \tag{2.2}$$

where $[|\lambda|]$ denotes the greatest integer which is not greater than λ.

Let H_ω denote the space of all real-valued functions f defined on $\mathbb{R}_+$ satisfying

$$\left|f(u)-f(v)\right| \leq \omega\left(\left|\frac{\mu(u)}{1+\mu(u)}-\frac{\mu(v)}{1+\mu(v)}\right|\right) \tag{2.3}$$

for any $u, v \in \mathbb{R}_+$.

Theorem 2.1 ([29]) *Let $P_n : H_\omega \to C_B(\mathbb{R}_+)$ be a sequence of positive linear operators such that*

$$\lim_{n\to\infty}\left\|P_n\left(\left(\frac{\mu(t)}{1+\mu(t)}\right)^\nu;x\right)-\left(\frac{\mu(x)}{1+\mu(x)}\right)^\nu\right\|_{C_B}=0$$

for $\nu = 0, 1, 2$. Then, for any function $f \in H_\omega$,

$$\lim_{n\to\infty}\left\|P_n(f)-f\right\|_{C_B}=0.$$

To compute the convergence results for the operators $L_{n,\mu}^{p,q}$ defined by (1.2), we take $q = q_n$, $p = p_n$, where $0 < q_n < p_n \leq 1$ satisfying

$$\lim_n p_n = 1, \qquad \lim_n q_n = 1, \tag{2.4}$$

$$\lim_n p_n^n = a, \qquad \lim_n q_n^n = b \quad (0 < a, b \leq 1). \tag{2.5}$$

Theorem 2.2 *Let $L_{n,\mu}^{p,q}$ be operators defined by* (1.2) *and take $p = p_n$, $q = q_n$ satisfying* (2.5). *Then, for $0 < q_n < p_n \leq 1$ and any function $f \in H_\omega$, we have*

$$\lim_n\left\|L_{n,\mu}^{p_n,q_n}(f)-f\right\|_{C_B}=0. \tag{2.6}$$

Proof Here we use Theorem 2.1. For $\nu = 0, 1, 2$, it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty}\left\|L_{n,\mu}^{p_n,q_n}\left(\left(\frac{\mu(t)}{1+\mu(t)}\right)^\nu;x\right)-\left(\frac{\mu(x)}{1+\mu(x)}\right)^\nu\right\|_{C_B}=0.$$

For $\nu = 0$, applying Lemma 1.1, (2.6) is fulfilled. Now, we observe that

$$\begin{aligned}&\left\|L_{n,\mu}^{p_n,q_n}\left(\left(\frac{\mu(t)}{1+\mu(t)}\right);x\right)-\left(\frac{\mu(x)}{1+\mu(x)}\right)\right\|_{C_B}\\&\quad\leq\left|\frac{p_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}-1\right|\\&\quad\leq\left|\left(\frac{p_n}{q_n}\right)\left(1-p_n^n\frac{1}{[n+1]_{p_n,q_n}}\right)-1\right|.\end{aligned}$$

Here we have used $q_n[n]_{p_n,q_n} = [n+1]_{p_n,q_n} - p_n^n$, $[n+1]_{p_n,q_n} \to \infty$ as $n \to \infty$, equation (2.6) holds for $\nu = 1$. Now, to verify for $\nu = 2$, we see that

$$\begin{aligned}&\left\| L_{n,\mu}^{p_n,q_n}\left(\left(\frac{\mu(t)}{1+\mu(t)}\right)^2;x\right) - \left(\frac{\mu(x)}{1+\mu(x)}\right)^2\right\|_{C_B} \\ &\quad = \sup_{x\geq 0}\left\{\frac{\mu(x)^2}{(1+\mu(x))^2}\left(\frac{p_n q_n^2[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2}\cdot\frac{1+\mu(x)}{p_n+q_n\mu(x)} - 1\right)\right. \\ &\qquad \left. + \frac{p_n^{n+1}[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2}\cdot\frac{\mu(x)}{1+\mu(x)}\right\}.\end{aligned}$$

By a simple calculation, we have

$$\frac{[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2} = \frac{1}{q_n^3}\left\{1 - p_n^n\left(2+\frac{q_n}{p_n}\right)\frac{1}{[n+1]_{p_n,q_n}} + \left(p_n^n\right)^2\left(1+\frac{q_n}{p_n}\right)\frac{1}{[n+1]_{p_n,q_n}^2}\right\},$$

and

$$\frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2} = \frac{1}{q_n}\left(\frac{1}{[n+1]_{p_n,q_n}} - p_n^n\frac{1}{[n+1]_{p_n,q_n}^2}\right).$$

Thus, we have

$$\begin{aligned}&\left\| L_{n,\mu}^{p_n,q_n}\left(\left(\frac{\mu(t)}{1+\mu(t)}\right)^2;x\right) - \left(\frac{\mu(x)}{1+\mu(x)}\right)^2\right\|_{C_B} \\ &\quad \leq \frac{p_n}{q_n}\left\{1 - p_n^n\left(2+\frac{q_n}{p_n}\right)\frac{1}{[n+1]_{p_n,q_n}} + \left(p_n^n\right)^2\left(1+\frac{q_n}{p_n}\right)\frac{1}{[n+1]_{p_n,q_n}^2} - 1\right\} \\ &\qquad + p_n^n\frac{p_n}{q_n}\left(\frac{1}{[n+1]_{p_n,q_n}} - p_n^n\frac{1}{[n+1]_{p_n,q_n}^2}\right).\end{aligned}$$

Hence (2.6) holds for $\nu = 2$, and the proof is completed by Theorem 2.1. □

3 Rate of convergence

In this section, we determine the rate of convergence of operators $L_{n,\mu}^{p,q}$.

For $f \in H_\omega$, the modulus of continuity is defined by

$$\widetilde{\omega}(f;\delta) = \sum_{\substack{|\frac{\mu(u)}{1+\mu(u)} - \frac{\mu(v)}{1+\mu(v)}|\leq\delta, \\ u,v\geq 0}} \left|f(u) - f(v)\right|$$

which satisfies the following conditions:

(1) $\widetilde{\omega}(f;\delta) \to 0$ $(\delta \to 0)$;

(2) $|f(u) - f(v)| \leq \widetilde{\omega}(f;\delta)(\frac{|\frac{\mu(u)}{1+\mu(u)} - \frac{\mu(v)}{1+\mu(v)}|}{\delta} + 1)$.

Theorem 3.1 *Let $p = p_n$, $q = q_n$, $0 < q_n < p_n \leq 1$ satisfying* (2.5). *Then, for each μ defined by* (1.3) *on $\mathbb{R}_+$ and for any function $f \in H_\omega$, we have*

$$\left|L_{n,\mu}^{p_n,q_n}(f;x) - f(x)\right| \leq 2\widetilde{\omega}\left(f;\sqrt{\delta_n^\mu(x)}\right),$$

where

$$\delta_n^{\mu}(x) = \frac{\mu(x)^2}{(1+\mu(x))^2}\left(\frac{p_n q_n^2[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2}\frac{1+\mu(x)}{p_n+q_n\mu(x)} - 2\frac{p_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} + 1\right) + \frac{p_n^{n+1}[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2}\frac{\mu(x)}{1+\mu(x)}.$$

Proof For $L_{n,\mu}^{p_n,q_n}$, we have

$$\begin{aligned}\left|L_{n,\mu}^{p_n,q_n}(f;x) - f(x)\right| &\le L_{n,\mu}^{p_n,q_n}\left(\left|f(t)-f(x)\right|;x\right)\\ &\le \widetilde{\omega}(f;\delta)\left\{1 + \frac{1}{\delta}L_{n,\mu}^{p_n,q_n}\left(\left|\frac{\mu(t)}{1+\mu(t)} - \frac{\mu(x)}{1+\mu(x)}\right|;x\right)\right\}.\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\left|L_{n,\mu}^{p_n,q_n}(f;x) - f(x)\right|\\ &\quad\le \widetilde{\omega}(f;\delta_n)\left\{1 + \frac{1}{\delta_n}\left[\left(L_{n,\mu}^{p_n,q_n}\left(\left(\frac{\mu(t)}{1+\mu(t)} - \frac{\mu(x)}{1+\mu(x)}\right)^2;x\right)\right]^{\frac{1}{2}}\left(L_{n,\mu}^{p_n,q_n}(1;x)\right)^{\frac{1}{2}}\right\}\\ &\quad\le \widetilde{\omega}(f;\delta_n)\left\{1 + \frac{1}{\delta_n}\left[\frac{\mu(x)^2}{(1+\mu(x))^2}\left(\frac{p_n q_n^2[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2}\frac{1+\mu(x)}{p_n+q_n\mu(x)}\right.\right.\right.\\ &\qquad\left.\left.\left. - 2\frac{p_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} + 1\right) + \frac{p_n^{n+1}[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2}\frac{\mu(x)}{1+\mu(x)}\right]^{\frac{1}{2}}\right\}.\end{aligned}$$

This completes the proof. □

4 Pointwise estimation of the operators $L_{n,\mu}^{p,q}$

The aim of this section is to give an estimate concerning the rate of convergence. Here, we take the Lipschitz type maximal function space defined on $F \subset \mathbb{R}_+$ (see [30])

$$\widetilde{E}_{\beta,F} = \left\{\tilde{f} : \sup(1+u)^{\beta}\tilde{f}_{\beta}(u) \le C\frac{1}{(1+v)^{\beta}} : u \le 0, \text{and } v \in F\right\},$$

where $\tilde{f}$ is a bounded and continuous function on $\mathbb{R}_+$, $0 < \beta \le 1$ and C is a positive constant.

Lenze [31] introduced a Lipschitz type maximal function f_{β} as follows:

$$f_{\beta}(u,v) = \sum_{\substack{u>0\\ u\ne v}}\frac{|f(u)-f(v)|}{|u-v|^{\beta}}.$$

Theorem 4.1 *Let $L_{n,\mu}^{p,q}$ be operators defined by* (1.2). *Then, for all $f \in \widetilde{E}_{\beta,F}$, we have*

$$\left|L_{n,\mu}^{p,q}(f;x) - f(x)\right| \le C\left(\sqrt{\left(\delta_n^{\mu}(x)\right)^{\beta}} + 2\left(\inf\left\{|x-y|; y \in F\right\}\right)^{\beta}\right),$$

where $\delta_n^{\mu}(x)$ is defined in Theorem 3.1.

Proof Let $\overline{F}$ be the closure of F. Then there exists $x_0 \in \overline{F}$ such that $|x - x_0| = d(x,F) = \inf\{|x-y|; y \in F\}$, where $x \in \mathbb{R}_+$. Thus we can write

$$\left|f - f(x)\right| \leq \left|f - f(x_0)\right| + \left|f(x_0) - f(x)\right|.$$

For $f \in \widetilde{E}_{\beta,F}$, we have

$$\begin{aligned}
&\left|L_{n,\mu}^{p,q}(f;x) - f(x)\right| \\
&\quad\leq L_{n,\mu}^{p,q}\left(\left|f - f(x_0)\right|;x\right) + \left|f(x_0) - f(x)\right| L_{n,\mu}^{p,q}(1;x) \\
&\quad\leq C\left(L_{n,\mu}^{p,q}\left(\left|\frac{\mu(t)}{1+\mu(t)} - \frac{\mu(x_0)}{1+\mu(x_0)}\right|^{\beta};x\right) + \frac{|\mu(x)-\mu(x_0)|^{\beta}}{(1+\mu(x))^{\beta}(1+\mu(x_0))^{\beta}} L_{n,\mu}^{p,q}(1;x)\right).
\end{aligned}$$

Using the inequality $(u+v)^{\beta} \leq u^{\beta} + v^{\beta}$, we obtain

$$\begin{aligned}
&L_{n,\mu}^{p,q}\left(\left|\frac{\mu(t)}{1+\mu(t)} - \frac{\mu(x_0)}{1+\mu(x_0)}\right|^{\beta};x\right) \\
&\quad\leq L_{n,\mu}^{p,q}\left(\left|\frac{\mu(t)}{1+\mu(t)} - \frac{\mu(x)}{1+\mu(x)}\right|^{\beta};x\right) + L_{n,\mu}^{p,q}\left(\left|\frac{\mu(x)}{1+\mu(x)} - \frac{\mu(x_0)}{1+\mu(x_0)}\right|^{\beta};x\right) \\
&\quad\leq L_{n,\mu}^{p,q}\left(\left|\frac{\mu(t)}{1+\mu(t)} - \frac{\mu(x)}{1+\mu(x)}\right|^{\beta};x\right) + \frac{|\mu(x)-\mu(x_0)|^{\beta}}{(1+\mu(x))^{\beta}(1+\mu(x_0))^{\beta}} L_{n,\mu}^{p,q}(1;x).
\end{aligned}$$

Applying Hölder's inequality, we have

$$\begin{aligned}
&L_{n,\mu}^{p,q}\left(\left|\frac{\mu(t)}{1+\mu(t)} - \frac{\mu(x_0)}{1+\mu(x_0)}\right|^{\beta};x\right) \\
&\quad\leq L_{n,\mu}^{p,q}\left(\left(\frac{\mu(t)}{1+\mu(t)} - \frac{\mu(x)}{1+\mu(x)}\right)^{2};x\right)^{\frac{\beta}{2}} \left(L_{n,\mu}^{p,q}(1;x)\right)^{\frac{2-\beta}{2}} \\
&\qquad+ \frac{|\mu(x)-\mu(x_0)|^{\beta}}{(1+\mu(x))^{\beta}(1+\mu(x_0))^{\beta}} L_{n,\mu}^{p,q}(1;x) \\
&\quad\leq \sqrt{\left(\delta_n^{\mu}\right)^{\beta}} + \frac{|\mu(x)-\mu(x_0)|^{\beta}}{(1+\mu(x))^{\beta}(1+\mu(x_0))^{\beta}}.
\end{aligned}$$

This completes the proof. □

Corollary 4.2 *For $F = \mathbb{R}_+$, we have*

$$\left|L_{n,\mu}^{p,q}(f;x) - f(x)\right| \leq C\sqrt{\left(\delta_n^{\mu}(x)\right)^{\beta}},$$

where δ_n^{μ} is defined in Theorem 3.1.

5 Other results

Theorem 5.1 *If $x \in (0,\infty) \setminus \{p^{n-i+1}\frac{[i]_{p,q}}{[n-i+1]_{p,q}q^i} | i = 0,1,2,\dots,n\}$, then*

$$\begin{aligned}
&L_{n,\mu}^{p,q}(f;x) - f\left(\frac{px}{q}\right) \\
&\quad= -\frac{\mu(x)^{n+1}}{\ell_{n,\mu}^{p,q}(x)} pq^{\frac{n(n-1)}{2}-1}\left[\frac{p\mu(x)}{q}; \frac{p[n]_{p,q}}{q^n}; \left(f \circ \mu^{-1}\right)\right]
\end{aligned}$$

$$+\frac{\mu(x)}{\ell_{n,\mu}^{p,q}(x)}\sum_{i=0}^{n-1}\left[\frac{p\mu(x)}{q};p^{n-i+1}\frac{[i]_{p,q}}{[n-i+1]_{p,q}q^{i}};(f\circ\mu^{-1})\right]$$
$$\times\frac{1}{[n-i]_{p,q}}p^{\frac{(n-i)(n-i+1)}{2}+1}q^{\frac{i(i-3)}{2}-2}\begin{bmatrix}n\\i\end{bmatrix}_{p,q}\mu(x)^{i}.$$

Proof We have

$$\begin{aligned}
&L_{n,\mu}^{p,q}(f;x)-f\left(\frac{px}{q}\right)\\
&=\frac{1}{\ell_{n,\mu}^{p,q}(x)}\sum_{i=0}^{n}\left[(f\circ\mu^{-1})\left(\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q}q^{i}}\right)-f\left(\frac{px}{q}\right)\right]p^{\frac{(n-i)(n-i-1)}{2}}q^{\frac{i(i-1)}{2}}\begin{bmatrix}n\\i\end{bmatrix}_{p,q}\mu(x)^{i}\\
&=-\frac{1}{\ell_{n,\mu}^{p,q}(x)}\sum_{i=0}^{n}\left(\frac{p\mu(x)}{q}-\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q}q^{i}}\right)\left[\frac{p\mu(x)}{q};\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q}q^{i}};(f\circ\mu^{-1})\right]\\
&\quad\times p^{\frac{(n-i)(n-i-1)}{2}}q^{\frac{i(i-1)}{2}}\begin{bmatrix}n\\i\end{bmatrix}_{p,q}\mu(x)^{i}.
\end{aligned}$$

Using $\frac{[i]_{p,q}}{[n-i+1]_{p,q}}\left[\begin{smallmatrix}n\\i\end{smallmatrix}\right]_{p,q}=\left[\begin{smallmatrix}n\\i-1\end{smallmatrix}\right]_{p,q}$, we have

$$\begin{aligned}
&L_{n,\mu}^{p,q}(f;x)-f\left(\frac{px}{q}\right)\\
&=-\frac{\mu(x)}{\ell_{n,\mu}^{p,q}(x)}\sum_{i=0}^{n}\left[\frac{p\mu(x)}{q};\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q}q^{i}};(f\circ\mu^{-1})\right]\\
&\quad\times p^{\frac{(n-i)(n-i-1)}{2}+1}q^{\frac{i(i-1)}{2}-1}\begin{bmatrix}n\\i\end{bmatrix}_{p,q}\mu(x)^{i}\\
&\quad+\frac{1}{\ell_{n,\mu}^{p,q}(x)}\sum_{i=1}^{n}\left[\frac{p\mu(x)}{q};\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q}q^{i}};(f\circ\mu^{-1})\right]\\
&\quad\times p^{\frac{(n-i)(n-i-1)}{2}-(i-n-1)}q^{\frac{i(i-1)}{2}-i}\begin{bmatrix}n\\i-1\end{bmatrix}_{p,q}\mu(x)^{i}\\
&=-\frac{\mu(x)}{\ell_{n,\mu}^{p,q}(x)}\sum_{i=0}^{n}\left[\frac{p\mu(x)}{q};\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q}q^{i}};(f\circ\mu^{-1})\right]\\
&\quad\times p^{\frac{(n-i)(n-i-1)}{2}+1}q^{\frac{i(i-1)}{2}-1}\begin{bmatrix}n\\i\end{bmatrix}_{p,q}\mu(x)^{i}\\
&\quad+\frac{\mu(x)}{\ell_{n,\mu}^{p,q}(x)}\sum_{i=0}^{n-1}\left[\frac{p\mu(x)}{q};\frac{p^{n-i}[i+1]_{p,q}}{[n-i]_{p,q}q^{i+1}};(f\circ\mu^{-1})\right]\\
&\quad\times p^{\frac{(n-i-1)(n-i-2)}{2}-(i-n)}q^{\frac{i(i+1)}{2}-(i+1)}\begin{bmatrix}n\\i\end{bmatrix}_{p,q}\mu(x)^{i}\\
&=-\frac{\mu(x)^{n+1}}{\ell_{n,\mu}^{p,q}(x)}\left[\frac{p\mu(x)}{q};\frac{p[n]_{p,q}}{q^{n}};(f\circ\mu^{-1})\right]pq^{\frac{n(n-1)}{2}-1}
\end{aligned}$$

$$+\frac{\mu(x)}{\ell^{p,q}_{n,\mu}(x)}\sum_{i=0}^{n-1}\left\{\left[\frac{p\mu(x)}{q};\frac{p^{n-i}[i+1]_{p,q}}{[n-i]_{p,q}q^{i+1}};(f\circ\mu^{-1})\right]\right.$$
$$\left.-\left[\frac{p\mu(x)}{q};\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q}q^{i}};(f\circ\mu^{-1})\right]\right\}p^{\frac{(n-i)(n-i-1)}{2}+1}q^{\frac{i(i-1)}{2}-1}\begin{bmatrix}n\\i\end{bmatrix}_{p,q}\mu(x)^i.$$

By a simple calculation, we have

$$\left[\frac{p\mu(x)}{q};\frac{p^{n-i}[i+1]_{p,q}}{[n-i]_{p,q}q^{i+1}};(f\circ\mu^{-1})\right]-\left[\frac{p\mu(x)}{q};\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q}q^{i}};(f\circ\mu^{-1})\right]$$
$$=\left(\frac{p^{n-i}[i+1]_{p,q}}{[n-i]_{p,q}q^{i+1}}-\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q}q^{i}}\right)(f\circ\mu^{-1})$$
$$\times\left[\frac{p\mu(x)}{q};\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q}q^{i}};\frac{p^{n-i}[i+1]_{p,q}}{[n-i]_{p,q}q^{i+1}};(f\circ\mu^{-1})\right]$$

and

$$\frac{p^{n-i}[i+1]_{p,q}}{[n-i]_{p,q}q^{i+1}}-\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q}q^{i}}=[n+1]_{p,q},$$

we have

$$L^{p,q}_{n,\mu}(f;x)-f\left(\frac{px}{q}\right)=-\frac{\mu(x)^{n+1}}{\ell^{p,q}_{n,\mu}(x)}\left[\frac{p\mu(x)}{q};\frac{p[n]_{p,q}}{q^n};(f\circ\mu^{-1})\right]pq^{\frac{n(n-1)}{2}-1}$$
$$+\frac{\mu(x)}{\ell^{p,q}_{n,\mu}(x)}\sum_{i=0}^{n-1}\left\{\left[\frac{p\mu(x)}{q};\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q}q^{i}};(f\circ\mu^{-1})\right]\right.$$
$$\left.\times\frac{p^{n-i}[n+1]_{p,q}}{[n-i]_{p,q}[n-i+1]_{p,q}q^{i+1}}\right\}p^{\frac{(n-i)(n-i-1)}{2}+1}q^{\frac{i(i-1)}{2}-1}\begin{bmatrix}n\\i\end{bmatrix}_{p,q}\mu(x)^i.$$

This completes the proof. □

6 Shape preserving properties

Theorem 6.1 *Let $f\in\widetilde{E}_{\beta,\mathbb{R}_+}$, which is a μ-convex function non-increasing on $\mathbb{R}_+$. Then we have*

$$L^{p,q}_{n,\mu}(f;x)\geq L^{p,q}_{n+1,\mu}(f;x),\quad n\in\mathbb{N}.$$

Proof We have

$$L^{p,q}_{n,\mu}(f;x)-L^{p,q}_{n+1,\mu}(f;x)$$
$$=\frac{1}{\ell^{p,q}_{n+1,\mu}(x)}\sum_{i=0}^{n}(f\circ\mu^{-1})\left(\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q}q^{i}};\right)p^{\frac{(n-i)(n-i-1)}{2}}q^{\frac{i(i-1)}{2}}\begin{bmatrix}n\\i\end{bmatrix}_{p,q}$$
$$\times\mu(x)^i\left(p^n+q^n\mu(x)\right)$$
$$+\frac{1}{\ell^{p,q}_{n+1,\mu}(x)}\sum_{i=0}^{n+1}(f\circ\mu^{-1})\left(\frac{p^{n-i+2}[i]_{p,q}}{[n-i+2]_{p,q}q^{i}};\right)p^{\frac{(n-i+1)(n-i+2)}{2}}q^{\frac{i(i-1)}{2}}\begin{bmatrix}n+1\\i\end{bmatrix}_{p,q}\mu(x)^i$$

$$\begin{aligned}
&= \frac{1}{\ell_{n+1,\mu}^{p,q}(x)} \sum_{i=0}^{n} (f \circ \mu^{-1}) \left(\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q} q^{i}}; \right) p^{\frac{(n-i)(n-i-1)}{2}+n} q^{\frac{i(i-1)}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_{p,q} \mu(x)^{i} \\
&\quad + \frac{1}{\ell_{n+1,\mu}^{p,q}(x)} \sum_{i=0}^{n} (f \circ \mu^{-1}) \left(\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q} q^{i}}; \right) p^{\frac{(n-i)(n-i-1)}{2}} q^{\frac{i(i-1)}{2}+n} \begin{bmatrix} n \\ i \end{bmatrix}_{p,q} \mu(x)^{i+1} \\
&\quad - \frac{1}{\ell_{n+1,\mu}^{p,q}(x)} \sum_{i=0}^{n+1} (f \circ \mu^{-1}) \left(\frac{p^{n-i+2}[i]_{p,q}}{[n-i+2]_{p,q} q^{i}}; \right) p^{\frac{(n-i+1)(n-i+2)}{2}} q^{\frac{i(i-1)}{2}} \begin{bmatrix} n+1 \\ i \end{bmatrix}_{p,q} \mu(x)^{i} \\
&= \frac{\mu(x)^{n+1}}{\ell_{n+1,\mu}^{p,q}(x)} q^{\frac{n(n+1)}{2}} \left[(f \circ \mu^{-1}) \left(\frac{p[n]_{p,q}}{q^{n}} \right) - (f \circ \mu^{-1}) \left(\frac{p[n+1]_{p,q}}{q^{n+1}} \right) \right] \\
&\quad + \frac{1}{\ell_{n+1,\mu}^{p,q}(x)} \sum_{i=1}^{n} (f \circ \mu^{-1}) \left(\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q} q^{i}}; \right) p^{\frac{(n-i)(n-i-1)}{2}+n} q^{\frac{i(i-1)}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_{p,q} \mu(x)^{i} \\
&\quad + \frac{1}{\ell_{n+1,\mu}^{p,q}(x)} \sum_{i=0}^{n-1} (f \circ \mu^{-1}) \left(\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q} q^{i}}; \right) p^{\frac{(n-i)(n-i-1)}{2}} q^{\frac{i(i-1)}{2}+n} \begin{bmatrix} n \\ i \end{bmatrix}_{p,q} \mu(x)^{i+1} \\
&\quad - \frac{1}{\ell_{n+1,\mu}^{p,q}(x)} \sum_{i=1}^{n} (f \circ \mu^{-1}) \left(\frac{p^{n-i+2}[i]_{p,q}}{[n-i+2]_{p,q} q^{i}}; \right) p^{\frac{(n-i+1)(n-i+2)}{2}} q^{\frac{i(i-1)}{2}} \begin{bmatrix} n+1 \\ i \end{bmatrix}_{p,q} \mu(x)^{i} \\
&= \frac{\mu(x)^{n+1}}{\ell_{n+1,\mu}^{p,q}(x)} q^{\frac{n(n+1)}{2}} \left[(f \circ \mu^{-1}) \left(\frac{p[n]_{p,q}}{q^{n}} \right) - (f \circ \mu^{-1}) \left(\frac{p[n+1]_{p,q}}{q^{n+1}} \right) \right] \\
&\quad + \frac{1}{\ell_{n+1,\mu}^{p,q}(x)} \sum_{i=0}^{n-1} (f \circ \mu^{-1}) \left(\frac{p^{n-i}[i+1]_{p,q}}{[n-i]_{p,q} q^{i+1}}; \right) p^{\frac{(n-i)(n-i-1)}{2}+n} q^{\frac{i(i+1)}{2}} \begin{bmatrix} n \\ i+1 \end{bmatrix}_{p,q} \mu(x)^{i+1} \\
&\quad + \frac{1}{\ell_{n+1,\mu}^{p,q}(x)} \sum_{i=0}^{n-1} (f \circ \mu^{-1}) \left(\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q} q^{i}}; \right) p^{\frac{(n-i)(n-i-1)}{2}} q^{\frac{i(i-1)}{2}+n} \begin{bmatrix} n \\ i \end{bmatrix}_{p,q} \mu(x)^{i+1} \\
&\quad - \frac{1}{\ell_{n+1,\mu}^{p,q}(x)} \sum_{i=0}^{n-1} (f \circ \mu^{-1}) \left(\frac{p^{n-i+1}[i+1]_{p,q}}{[n-i+1]_{p,q} q^{i+1}}; \right) p^{\frac{(n-i)(n-i+1)}{2}} q^{\frac{i(i+1)}{2}} \begin{bmatrix} n+1 \\ i+1 \end{bmatrix}_{p,q} \mu(x)^{i+1}.
\end{aligned}$$

By a simple calculation, we have

$$\begin{aligned}
&\begin{bmatrix} n+1 \\ i+1 \end{bmatrix}_{p,q} = \frac{[n]_{p,q}[n+1]_{p,q}}{[n-i]_{p,q}[i+1]_{p,q}} \begin{bmatrix} n-1 \\ i \end{bmatrix}_{p,q}, \\
&\begin{bmatrix} n \\ i \end{bmatrix}_{p,q} = \frac{[n]_{p,q}}{[n-i]_{p,q}} \begin{bmatrix} n-1 \\ i \end{bmatrix}_{p,q}, \\
&\begin{bmatrix} n \\ i+1 \end{bmatrix}_{p,q} = \frac{[n]_{p,q}}{[i+1]_{p,q}} \begin{bmatrix} n-1 \\ i \end{bmatrix}_{p,q},
\end{aligned}$$

we get

$$\begin{aligned}
&L_{n,\mu}^{p,q}(f;x) - L_{n+1,\mu}^{p,q}(f;x) \\
&\quad = \frac{\mu(x)^{n+1}}{\ell_{n+1,\mu}^{p,q}(x)} q^{\frac{n(n+1)}{2}} \left[(f \circ \mu^{-1}) \left(\frac{p[n]_{p,q}}{q^{n}} \right) - (f \circ \mu^{-1}) \left(\frac{p[n+1]_{p,q}}{q^{n+1}} \right) \right]
\end{aligned}$$

$$+\frac{1}{\ell^{p,q}_{n+1,\mu}(x)}\sum_{i=0}^{n-1}p^{\frac{(n-i)(n-i-1)}{2}}q^{\frac{i(i+1)}{2}}\frac{[n]_{p,q}[n+1]_{p,q}}{[n-i]_{p,q}[i+1]_{p,q}}\begin{bmatrix}n-1\\i\end{bmatrix}_{p,q}\mu(x)^{i+1}$$

$$\times\left\{(f\circ\mu^{-1})\left(\frac{p^{n-i}[i+1]_{p,q}}{[n-i]_{p,q}q^{i+1}};\right)p^{n}\frac{[n-i]_{p,q}}{[n+1]_{p,q}}\right.$$

$$+(f\circ\mu^{-1})\left(\frac{p^{n-i+1}[i]_{p,q}}{[n-i+1]_{p,q}q^{i}};\right)q^{n-i}\frac{[i+1]_{p,q}}{[n+1]_{p,q}}$$

$$\left.-(f\circ\mu^{-1})\left(\frac{p^{n-i+1}[i+1]_{p,q}}{[n-i+1]_{p,q}q^{i+1}};\right)p^{n-i}\right\}.$$

By the calculation, $\frac{p[n+1]_{p,q}}{q^{n+1}}-\frac{p[n]_{p,q}}{q^{n}}=(\frac{p}{q})^{n+1}$, hence we have

$$(f\circ\mu^{-1})\left(\frac{p[n]_{p,q}}{q^{n}}\right)-(f\circ\mu^{-1})\left(\frac{p[n+1]_{p,q}}{q^{n+1}}\right)>0.$$

Since f is μ-convex, by using [32], we obtain

$$L^{p,q}_{n,\mu}(f;x)-L^{p,q}_{n+1,\mu}(f;x)>0,$$

where $x\in[0,\infty)$ and $n\in\mathbb{N}$.

This completes the proof. □

7 Generalization of $L^{p,q}_{n,\mu}$

In this section, we give a generalization of the operators $L^{p,q}_{n,\mu}$ based on (p,q)-integers similar to the work done in [30, 33].

Consider

$$L^{p,q}_{n,\mu,\gamma}(f;x)=\frac{1}{\ell^{p,q}_{n,\mu}(x)}\sum_{i=0}^{n}(f\circ\mu^{-1})\left(\frac{p^{n-i+1}[i]_{p,q}+\gamma}{\theta_{n,i}}\right)$$
$$\times p^{\frac{(n-i)(n-i-1)}{2}}q^{\frac{i(i-1)}{2}}\begin{bmatrix}n\\i\end{bmatrix}_{p,q}\mu(x)^{i}\quad(\gamma\in\mathbb{R}),$$

where $\theta_{n,i}$ satisfies the following conditions:

$$p^{n-i+1}[i]_{p,q}+\theta_{n,i}=b_n\quad\text{and}\quad\frac{[n]_{p,q}}{b_n}\to1\quad\text{for } n\to\infty.$$

Note that for $\mu=t$, these operators reduce to [21]. If we choose $\gamma=0$, $q=1$, $p=1$ and $\mu=t$, then we get the Balázs type generalization of q-BBH operators [30] given in [33].

Theorem 7.1 *Let $p=p_n$, $q=q_n$, $0<q_n<p_n\leq1$ satisfying* (2.5). *Then, for any function $f\in\widetilde{E}_{\beta,\mathbb{R}_+}$, we have*

$$\lim_{n}\left\|L^{p_n,q_n}_{n,\mu,\gamma}(f;x)-f(x)\right\|_{C_B}\leq3C\max\left\{\left(\frac{[n]_{p_n,q_n}}{b_n+\gamma}\right)^{\beta}\left(\frac{\gamma}{[n]_{p_n,q_n}}\right)^{\beta},\right.$$
$$\left|1-\frac{[n+1]_{p_n,q_n}}{b_n+\gamma}\right|^{\beta}\left(\frac{p_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\right)^{\beta},$$
$$\left.1-2\frac{p_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}+\frac{q_n[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}}\right\}.$$

Proof We have

$$
\begin{aligned}
&\left|L_{n,\mu,\gamma}^{p_n,q_n}(f;x)-f(x)\right| \\
&\quad\le \frac{1}{\ell_{n,\mu}^{p_n,q_n}(x)}\sum_{i=0}^{n}\left|(f\circ\mu^{-1})\left(\frac{p_n^{n-i+1}[i]_{p_n,q_n}+\gamma}{\theta_{n,i}}\right)-(f\circ\mu^{-1})\left(\frac{p_n^{n-i+1}[i]_{p_n,q_n}}{\gamma+\theta_{n,i}}\right)\right| \\
&\qquad\times p_n^{\frac{(n-i)(n-i-1)}{2}}q_n^{\frac{i(i-1)}{2}}\begin{bmatrix} n\\ i\end{bmatrix}_{p_n,q_n}\mu(x)^i \\
&\qquad+\frac{1}{\ell_{n,\mu}^{p_n,q_n}(x)}\sum_{i=0}^{n}\left|(f\circ\mu^{-1})\left(\frac{p_n^{n-i+1}[i]_{p_n,q_n}}{\gamma+\theta_{n,i}}\right)-(f\circ\mu^{-1})\left(\frac{p_n^{n-i+1}[i]_{p_n,q_n}}{[n-i+1]_{p_n,q_n}q_n^i}\right)\right| \\
&\qquad\times p_n^{\frac{(n-i)(n-i-1)}{2}}q_n^{\frac{i(i-1)}{2}}\begin{bmatrix} n\\ i\end{bmatrix}_{p_n,q_n}\mu(x)^i \\
&\qquad+\left|L_{n,\mu,\gamma}^{p_n,q_n}(f;x)-f(x)\right|.
\end{aligned}
$$

Since $f\in\widetilde{E}_{\beta,\mathbb{R}_+}$ and by using Corollary 4.2, we can write

$$
\begin{aligned}
&\left|L_{n,\mu,\gamma}^{p_n,q_n}(f;x)-f(x)\right| \\
&\quad\le \frac{C}{\ell_{n,\mu}^{p_n,q_n}(x)}\sum_{i=0}^{n}\left|\frac{p_n^{n-i+1}[i]_{p_n,q_n}+\gamma}{p_n^{n-i+1}[i]_{p_n,q_n}+\gamma+\theta_{n,i}}-\frac{p_n^{n-i+1}[i]_{p_n,q_n}}{\gamma+p_n^{n-i+1}[i]_{p_n,q_n}+\theta_{n,i}}\right|^{\beta} \\
&\qquad\times p_n^{\frac{(n-i)(n-i-1)}{2}}q_n^{\frac{i(i-1)}{2}}\begin{bmatrix} n\\ i\end{bmatrix}_{p_n,q_n}\mu(x)^i \\
&\qquad+\frac{C}{\ell_{n,\mu}^{p_n,q_n}(x)}\sum_{i=0}^{n}\left|\frac{p_n^{n-i+1}[i]_{p_n,q_n}}{p_n^{n-i+1}[i]_{p_n,q_n}+\gamma+\theta_{n,i}}-\frac{p_n^{n-i+1}[i]_{p_n,q_n}}{p_n^{n-i+1}[i]_{p_n,q_n}+[n-i+1]_{p_n,q_n}q_n^i}\right| \\
&\qquad\times p_n^{\frac{(n-i)(n-i-1)}{2}}q_n^{\frac{i(i-1)}{2}}\begin{bmatrix} n\\ i\end{bmatrix}_{p_n,q_n}\mu(x)^i+C\left(\delta_n^{\mu}\right)^{\frac{\beta}{2}}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\left|L_{n,\mu,\gamma}^{p_n,q_n}(f;x)-f(x)\right| \\
&\quad\le C\left(\frac{[n]_{p_n,q_n}}{b_n+\gamma}\right)^{\beta}\left(\frac{\gamma}{[n]_{p_n,q_n}}\right)^{\beta} \\
&\qquad+\frac{C}{\ell_{n,\mu}^{p_n,q_n}(x)}\left|1-\frac{[n+1]_{p_n,q_n}}{b_n+\gamma}\right|^{\beta}\sum_{i=0}^{n}\left(\frac{p_n^{n-i+1}[i]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\right)^{\beta}p_n^{\frac{(n-i)(n-i-1)}{2}}q_n^{\frac{i(i-1)}{2}}\begin{bmatrix} n\\ i\end{bmatrix}_{p_n,q_n}\mu(x)^i \\
&\qquad+C\left(\delta_n^{\mu}\right)^{\frac{\beta}{2}} \\
&\quad=C\left(\frac{[n]_{p_n,q_n}}{b_n+\gamma}\right)^{\beta}\left(\frac{\gamma}{[n]_{p_n,q_n}}\right)^{\beta}+C\left|1-\frac{[n+1]_{p_n,q_n}}{b_n+\gamma}\right|^{\beta}L_{n,\mu,\gamma}^{p_n,q_n}\left(\left(\frac{\mu(t)}{1+\mu(t)}\right)^{\beta};x\right) \\
&\qquad+C\left(\delta_n^{\mu}\right)^{\frac{\beta}{2}}.
\end{aligned}
$$

Applying Hölder's inequality, we get

$$\begin{aligned}
&\left|L_{n,\mu,\gamma}^{p_n,q_n}(f;x)-f(x)\right| \\
&\quad\leq C\left(\frac{[n]_{p_n,q_n}}{b_n+\gamma}\right)^{\beta}\left(\frac{\gamma}{[n]_{p_n,q_n}}\right)^{\beta} \\
&\qquad + C\left|1-\frac{[n+1]_{p_n,q_n}}{b_n+\gamma}\right|^{\beta} L_{n,\mu,\gamma}^{p_n,q_n}\left(\frac{\mu(t)}{1+\mu(t)};x\right)^{\beta}\left(L_{n,\mu,\gamma}^{p_n,q_n}(1;x)\right)^{1-\beta} + C\left(\delta_n^{\mu}\right)^{\frac{\beta}{2}} \\
&\quad\leq C\left(\frac{[n]_{p_n,q_n}}{b_n+\gamma}\right)^{\beta}\left(\frac{\gamma}{[n]_{p_n,q_n}}\right)^{\beta} + C\left|1-\frac{[n+1]_{p_n,q_n}}{b_n+\gamma}\right|^{\beta}\left(\frac{p_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\frac{\mu(x)}{1+\mu(x)}\right)^{\beta} \\
&\qquad + C\left(\delta_n^{\mu}\right)^{\frac{\beta}{2}}.
\end{aligned}$$

This completes the proof. □

8 Conclusion

In this paper we have used the (p,q)-integers to the Bleimann-Butzer-Hahn operators based on a continuously differentiable function μ on $\mathbb{R}_+ = [0,\infty)$. We have obtained some approximation results on the Korovkin type theorem and computed the rate of convergence by using the modulus of continuity as well as Lipschitz type maximal functions. Further, we investigated the shape preserving properties of these operators.

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Authors' contributions
All authors of the manuscript have read and agreed to its content and are accountable for all aspects of the accuracy and integrity of the manuscript. All authors read and approved the final manuscript.

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