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On a boundary property of analytic functions

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Abstract
Let f be an analytic function in the unit disc $|z| < 1$ on the complex plane $\mathbb{C}$. This paper is devoted to obtaining the correspondence between $f(z)$ and $zf'(z)$ at the point w, $0 < |w| = R < 1$, such that $|f(w)| = \min\{|f(z)| : f(z) \in \partial f(|z| \le R)\}$. We present several applications of the main result. A part of them improve the previous results of this type.

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1 Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $|z| < 1$ on the complex plane $\mathbb{C}$. The following lemma is a particular case of the Julia-Wolf theorem. It is known as Jack's lemma.

Lemma 1.1 ([1]) *Let $\omega(z) \in \mathcal{H}$ with $\omega(0) = 0$. If for a certain z_0, $|z_0| < 1$, we have $|\omega(z)| \le |\omega(z_0)|$ for $|z| \le |z_0|$, then $z_0\omega'(z_0) = m\omega(z_0)$, $m \ge 1$.*

In this paper, we consider a related problem. We establish a relation between $w(z)$ and $zw'(z)$ at the point z_0 such that $|w(z_0)| = \min\{|w(z)| : |z| = |z_0|\}$ or at the point z_0 satisfying (1.1). We consider the p-valent functions.

Lemma 1.2 *Let $w(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ be analytic in $|z| < 1$. Assume that there exists a point z_0, $|z_0| = R < 1$, such that*

$$\min\left\{\left|w(z)\right| : w(z) \in \partial w\left(|z| \le R\right)\right\} = \left|w(z_0)\right| > 0. \tag{1.1}$$

If $w(z)/z^p \neq 0$ in $|z| < R$, then

$$\frac{z_0 w'(z_0)}{w(z_0)} = k_1 \le p. \tag{1.2}$$

If the function $w(z)/z^p$ has a zero in $|z| < R$ and $\partial w(|z| \leq R)$ is a smooth curve at $w(z_0)$, then

$$\frac{z_0 w'(z_0)}{w(z_0)} = k_2 \geq p, \tag{1.3}$$

where k_1, k_2 are real.

Proof If

$$\min\{|w(z)| : w(z) \in \partial w(|z| \leq R)\} = |w(z_0)| > 0,$$

then

$$|w(z)| \geq |w(z_0)| \quad \text{for } w(z) \in \partial w(|z| \leq R). \tag{1.4}$$

Then, we also have

$$\left|\frac{w(z)}{z^p}\right| \geq \left|\frac{w(z_0)}{z_0^p}\right| \quad \text{for } w(z) \in \partial w(|z| \leq R). \tag{1.5}$$

Let

$$\Phi(z) = w(z)/z^p, \quad |z| < 1. \tag{1.6}$$

Then, from (1.5) and from hypothesis (1.1) we have

$$\min\{|\Phi(z)| : \Phi(z) \in \partial\Phi(|z| \leq R)\} = |\Phi(z_0)|. \tag{1.7}$$

There are two cases: $\Phi(|z| < R)$ contains the origin (see Figure 2); and $\Phi(|z| < R)$ does not (see Figure 1).

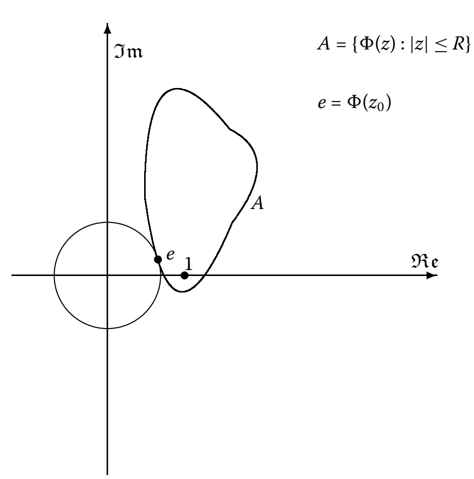

Figure 1 **$\Phi(z)$ in the case $\Phi(z) \neq 0$.**

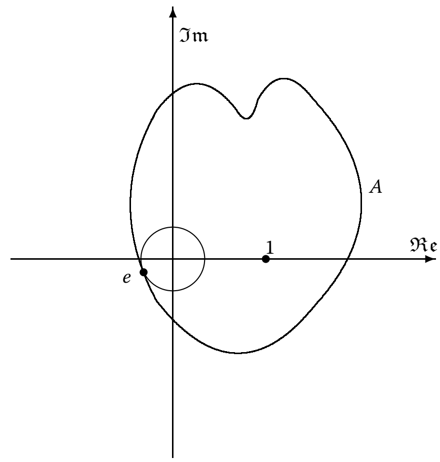

Figure 2 $\Phi(z)$ **in the case** $0 \in \Phi(|z| \leq R)$.

First, suppose that $\Phi(z)$ does not become 0 in $|z| < R$. If there exists a point $z_0 = R\exp(i\varphi_0)$, $0 \leq \varphi_0 < 2\pi$, $0 < R < 1$, such that

$$\min\{|\Phi(z)| : \Phi(z) \in \partial\Phi(|z| \leq R)\} = |\Phi(z_0)|, \tag{1.8}$$

then the function

$$F(z) = \frac{z}{\Phi(z)} = \frac{z^{p+1}}{w(z)}, \quad |z| \leq R,$$

satisfies the assumptions of Jack's lemma (Lemma 1.1),

$$F(z_0) = \max_{\theta \in [0,2\pi)} \{|F(z)| : z = Re^{i\theta}\},$$

and hence

$$\frac{z_0 F'(z_0)}{F(z_0)} = p + 1 - \frac{z_0 w'(z_0)}{w(z_0)} \geq m \geq 1.$$

This gives (1.2).

For the case $0 \in \Phi(|z| < R)$ (see Figure 2), for $\Phi(z)$ given in (1.6), we have that $|\Phi(z)|$ has an extremum at z_0, and so

$$\left.\frac{\mathrm{d}|\Phi(z)|}{\mathrm{d}\varphi}\right|_{z=z_0} = 0. \tag{1.9}$$

Furthermore, $\arg\{\Phi(z)\}$ is increasing at z_0, and so

$$\left.\frac{\mathrm{d}\arg\{\Phi(z)\}}{\mathrm{d}\varphi}\right|_{z=z_0} \geq 0. \tag{1.10}$$

Then we have

$$\begin{aligned}\frac{z_0\Phi'(z_0)}{\Phi(z_0)} &= \frac{\mathrm{d}\log\Phi(z)}{\mathrm{d}\log z}\bigg|_{z=z_0}\\ &= \frac{\mathrm{d}\log|\Phi(z)| + i\mathrm{d}\arg\{\Phi(z)\}}{i\mathrm{d}\varphi}\bigg|_{z=z_0}\\ &= \frac{\mathrm{d}\arg\{\Phi(z)\}}{\mathrm{d}\varphi} - \frac{i}{|\Phi(z)|}\frac{\mathrm{d}|\Phi(z)|}{\mathrm{d}\varphi}\bigg|_{z=z_0}\\ &= \frac{\mathrm{d}\arg\{\Phi(z)\}}{\mathrm{d}\varphi}\bigg|_{z=z_0}\\ &\geq 0,\end{aligned}\tag{1.11}$$

because of (1.9). On the other hand, by (1.6) we have $w'(z) = z^p\Phi'(z) + pz^{p-1}\Phi(z)$, and hence

$$\frac{z_0w'(z_0)}{w(z_0)} = \frac{z_0\Phi'(z_0)}{\Phi(z_0)} + p.\tag{1.12}$$

Relations (1.11) and (1.12) imply that

$$\frac{z_0w'(z_0)}{w(z_0)} \geq p.$$

Therefore, by (1.11) we obtain (1.3). □

If we additionally assume that $w(z)/z^p$ is univalent in the unit disc, then we have the following result.

Remark 1.3 Let $w(z) = z^p + \sum_{n=p+1}^{\infty} a_nz^n$ be analytic in $|z| < 1$. Assume that there exists a point z_0, $|z_0| = R < 1$, such that

$$\min_{\theta\in[0,2\pi)}\left\{\left|w(z)\right| : z = Re^{i\theta}\right\} = \left|w(z_0)\right| > 0.\tag{1.13}$$

If $w(z)/z^p$ is univalent and $w(z)/z^p \neq 0$ in $|z| \leq R$, then

$$\frac{z_0w'(z_0)}{w(z_0)} = k_1 \leq p,\tag{1.14}$$

where k_1 is real. If $w(z)/z^p$ is univalent and $w(z)/z^p$ vanishes in $|z| \leq R$, then

$$\frac{z_0w'(z_0)}{w(z_0)} = k_2 \geq p,\tag{1.15}$$

where k_2 is real.

2 Applications

For $p = 0$, then Lemma 1.2 becomes the following corollary.

Corollary 2.1 *Let $w(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ be analytic in $|z| < 1$. Assume that there exists a point z_0, $|z_0| = R < 1$, such that*

$$\min\{|w(z)| : w(z) \in \partial w(|z| \le R)\} = |w(z_0)| > 0. \tag{2.1}$$

If $w(z) \neq 0$ in $|z| < R$, then

$$\frac{z_0 w'(z_0)}{w(z_0)} = k_1 \le 0. \tag{2.2}$$

If the function $w(z)$ has a zero in $|z| < R$ and $\partial w(|z| \le R)$ is a smooth curve at $w(z_0)$, then

$$\frac{z_0 w'(z_0)}{w(z_0)} = k_2 \ge 0. \tag{2.3}$$

A simple contraposition of Lemma 1.2 provides the following two corollaries.

Corollary 2.2 *Let $w(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and suppose that there exists a point z_0, $|z_0| = R < 1$, such that*

$$\min\{|w(z)| : w(z) \in \partial w(|z| \le R)\} = |w(z_0)| > 0. \tag{2.4}$$

If

$$\frac{z_0 w'(z_0)}{w(z_0)} = k_1 < p \tag{2.5}$$

and $\partial w(|z| \le R)$ is a smooth curve at $w(z_0)$, then $w(z)/z^p$ has no zero in $|z| \le R$. If

$$\frac{z_0 w'(z_0)}{w(z_0)} = k_2 > p, \tag{2.6}$$

then the function $w(z)/z^p$ has a zero in $|z| \le R$.

Corollary 2.3 *Let $q(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ be analytic in $|z| \le 1$. Assume that $q(z)/z^p$ has a zero in $|z| < 1$. If for given $c \in [0,1)$,*

$$|zq'(z)| < \frac{p}{c}|q(z)|^2, \quad |z| < 1, \tag{2.7}$$

then the image domain $q(|z| < 1)$ covers the disc $|w| < c$.

Proof If

$$\min\{|q(z)| : q(z) \in \partial q(|z| \le 1)\} = |q(z_0)| < c, \tag{2.8}$$

then by (1.2) in Lemma 1.2 we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = k \ge p \quad \Rightarrow \quad |z_0 q'(z_0)| \ge p|q(z_0)|. \tag{2.9}$$

Therefore, by (2.8) and (2.9) we have

$$\left|z_0 q'(z_0)\right| \geq \frac{p}{c}\left|q(z_0)\right|^2,$$

which contradicts hypothesis (2.7) and therefore completes the proof. □

Theorem 2.4 *Let $p(z)$ be analytic in $|z| < 1$ with $p(z) \neq 0$, $|p(0)| > c$, in $|z| < 1$ and suppose that*

$$\left|p(z) + zp'(z)\right| > c, \quad |z| < 1, \tag{2.10}$$

where $c > 0$, and that

$$\mathfrak{Re}\left\{\frac{zp'(z)}{p(z)}\right\} > -2, \quad |z| < 1. \tag{2.11}$$

Then we have

$$\left|p(z)\right| > c, \quad |z| < 1. \tag{2.12}$$

Proof If there exists a point z_0, $|z_0| < 1$, such that

$$\left|p(z)\right| > c \quad \text{for } |z| < |z_0| \tag{2.13}$$

and $|p(z_0)| = c$, then $p(|z| \leq |z_0|)$ has the shape as in Figure 1 and $\mathrm{d}|p(z)|/\mathrm{d}\varphi$, $z = re^{i\varphi}$, vanishes at the point $z = z_0$. Therefore, we have

$$\begin{aligned}
\frac{z_0 p'(z_0)}{p(z_0)} &= \left.\frac{\mathrm{d}\log p(z)}{\mathrm{d}\log z}\right|_{z=z_0} \\
&= \left.\frac{\mathrm{d}\log|p(z)| + i\mathrm{d}\arg\{p(z)\}}{i\mathrm{d}\varphi}\right|_{z=z_0} \\
&= \left.\frac{\mathrm{d}\arg\{p(z)\}}{\mathrm{d}\varphi} - \frac{i}{|p(z)|}\frac{\mathrm{d}|p(z)|}{\mathrm{d}\varphi}\right|_{z=z_0} \\
&= \left.\frac{\mathrm{d}\arg\{p(z)\}}{\mathrm{d}\varphi}\right|_{z=z_0} \\
&\leq 0.
\end{aligned} \tag{2.14}$$

From (2.11) and (2.14) we have

$$-2 < \frac{z_0 p'(z_0)}{p(z_0)} \leq 0,$$

and hence

$$0 \leq \left|1 + \frac{z_0 p'(z_0)}{p(z_0)}\right| \leq 1.$$

Then it follows that

$$\left|p(z_0) + z_0 p'(z_0)\right| = \left|p(z_0)\right| \left|1 + \frac{z_0 p'(z_0)}{p(z_0)}\right| \leq \left|p(z_0)\right| = c, \tag{2.15}$$

which contradicts hypothesis (2.10) and therefore completes the proof. □

For some other geometrical properties of analytic functions, we refer to the papers [2–4].

3 Conclusion

In this paper, we have presented a correspondence between an analytic function $f(z)$ and $zf'(z)$ at the point w, $0 < |w| = R < 1$, in the unit disc $|z| < 1$ on the complex plane such that $|f(w)| = \min\{|f(z)| : f(z) \in \partial f(|z| \leq R)\}$.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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