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Oscillation and variation inequalities for the multilinear singular integrals related to Lipschitz functions

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Abstract

The main purpose of this paper is to establish the weighted (L^p, L^q) inequalities of the oscillation and variation operators for the multilinear Calderón-Zygmund singular integral with a Lipschitz function.

MSC: 42B25; 42B20; 47G10

Keywords: oscillation operator; variation operator; multilinear operator; Lipschitz function

1 Introduction and results

Let K be a kernel on $\mathbb{R} \times \mathbb{R} \setminus \{(x,x) : x \in \mathbb{R}\}$. Suppose that there exist two constants δ and C such that

$$\left|K(x,y)\right| \le \frac{C}{|x-y|} \quad \text{for } x \ne y; \tag{1.1}$$

$$\left|K(x,y) - K\left(x',y\right)\right| \le \frac{C|x-x'|^\delta}{|x-y|^{1+\delta}} \quad \text{for } |x-y| \ge 2\left|x-x'\right|; \tag{1.2}$$

$$\left|K(x,y) - K\left(x,y'\right)\right| \le \frac{C|y-y'|^\delta}{|x-y|^{1+\delta}} \quad \text{for } |x-y| \ge 2\left|y-y'\right|. \tag{1.3}$$

We consider the family of operators $T = \{T_\epsilon\}_{\epsilon>0}$ given by

$$T_\epsilon f(x) = \int_{|x-y|>\epsilon} K(x,y) f(y)\, dy. \tag{1.4}$$

A common method of measuring the speed of convergence of the family T_ϵ is to consider the square functions

$$\left(\sum_{i=1}^{\infty} |T_{\epsilon_i} f - T_{\epsilon_{i+1}} f|^2\right)^{1/2},$$

where ϵ_i is a monotonically decreasing sequence which approaches 0. For convenience, other expressions have also been considered. Let $\{t_i\}$ be a fixed sequence which decreases to zero. Following [1], the oscillation operator is defined as

$$\mathcal{O}(Tf)(x) = \left(\sum_{i=1}^{\infty} \sup_{t_{i+1} \le \epsilon_{i+1} < \epsilon_i \le t_i} \left| T_{\epsilon_{i+1}} f(x) - T_{\epsilon_i} f(x) \right|^2 \right)^{1/2}$$

and the ρ-variation operator is defined as

$$\mathcal{V}_\rho(Tf)(x) = \sup_{\epsilon_i \searrow 0} \left(\sum_{i=1}^{\infty} \left| T_{\epsilon_{i+1}} f(x) - T_{\epsilon_i} f(x) \right|^\rho \right)^{1/\rho},$$

where the sup is taken over all sequences of real number $\{\epsilon_i\}$ decreasing to zero.

The oscillation and variation for some families of operators have been studied by many authors on probability, ergodic theory, and harmonic analysis; see [2–4]. Recently, some authors [5–8] researched the weighted estimates of the oscillation and variation operators for the commutators of singular integrals.

Let m be a positive integer, let b be a function on $\mathbb{R}$, and let $R_{m+1}(b;x,y)$ be the $m+1$th Taylor series remainder of b at x expander about y, *i.e.*

$$R_{m+1}(b;x,y) = b(x) - \sum_{\alpha \le m} \frac{1}{\alpha!} b^{(\alpha)}(y)(x-y)^\alpha.$$

We consider the family of operators $T^b = \{T_\epsilon^b\}_{\epsilon>0}$, where T_ϵ^b are the multilinear singular integral operators of T_ϵ,

$$T_\epsilon^b f(x) = \int_{|x-y|>\epsilon} \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x,y) f(y)\, dy. \tag{1.5}$$

Note that when $m = 0$, T_ϵ^b is just the commutator of T_ϵ and b, which is denoted by $T_{\epsilon,b}$, that is to say

$$T_{\epsilon,b} f(x) = \int_{|x-y|>\epsilon} \big(b(x) - b(y)\big) K(x,y) f(y)\, dy. \tag{1.6}$$

However, when $m > 0$, T_ϵ^b is a non-trivial generation of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [9–13]).

A locally integrable function b is said to be in Lipschitz space $\mathrm{Lip}_\beta(\mathbb{R})$ if

$$\|b\|_{\dot\Lambda_\beta} = \sup_I \frac{1}{|I|^{1+\beta}} \int_I \left| b(x) - b_I \right| dx < \infty,$$

where

$$b_I = \frac{1}{|I|} \int_I b(x)\, dx.$$

In this paper, we will study the boundedness of oscillation and variation operators for the family of the multilinear singular integral related to a Lipschitz function defined by (1.5) in weighted Lebesgue space. Our main results are as follows.

Theorem 1.1 *Suppose that $K(x,y)$ satisfies* (1.1)-(1.3), $b^{(m)} \in \dot{\wedge}_\beta$, $0 < \beta \leq \delta < 1$, *where δ is the same as in* (1.2). *Let $\rho > 2$, $T = \{T_\epsilon\}_{\epsilon>0}$ and $T^b = \{T^b_\epsilon\}_{\epsilon>0}$ be given by* (1.4) *and* (1.5), *respectively. If $\mathcal{O}(T)$ and $\mathcal{V}_\rho(T)$ are bounded on $L^{p_0}(\mathbb{R}, dx)$ for some $1 < p_0 < \infty$, then, for any $1 < p < 1/\beta$ with $1/q = 1/p - \beta$, $\omega \in A_{p,q}(\mathbb{R})$, $\mathcal{O}(T^b)$ and $\mathcal{V}_\rho(T^b)$ are bounded from $L^p(\mathbb{R}, \omega^p\, dx)$ into $L^q(\mathbb{R}, \omega^q\, dx)$.*

Corollary 1.1 *Suppose that $K(x,y)$ satisfies* (1.1)-(1.3), $b \in \dot{\wedge}_\beta$, $0 < \beta \leq \delta < 1$, *where δ is the same as in* (1.2). *Let $\rho > 2$, $T = \{T_\epsilon\}_{\epsilon>0}$ and $T_b = \{T_{b,\epsilon}\}_{\epsilon>0}$ be given by* (1.4) *and* (1.6), *respectively. If $\mathcal{O}(T)$ and $\mathcal{V}_\rho(T)$ are bounded on $L^{p_0}(\mathbb{R}, dx)$ for some $1 < p_0 < \infty$, then, for any $1 < p < 1/\beta$ with $1/q = 1/p - \beta$, $\omega \in A_{p,q}(\mathbb{R})$, $\mathcal{O}(T_b)$ and $\mathcal{V}_\rho(T_b)$ are bounded from $L^p(\mathbb{R}, \omega^p\, dx)$ into $L^q(\mathbb{R}, \omega^q\, dx)$.*

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2 Some preliminaries

2.1 Weight

A weight ω is a nonnegative, locally integrable function on $\mathbb{R}$. The classical weight theories were introduced by Muckenhoupt and Wheeden in [14] and [15].

A weight ω is said to belong to the Muckenhoup class $A_p(\mathbb{R})$ for $1 < p < \infty$, if there exists a constant C such that

$$\left(\frac{1}{|I|}\int_I \omega(x)\, dx\right)\left(\frac{1}{|I|}\int_I \omega(x)^{-\frac{1}{p-1}}\, dx\right)^{p-1} \leq C$$

for every interval I. The class $A_1(\mathbb{R})$ is defined by replacing the above inequality with

$$\frac{1}{|I|}\int_I \omega(x)\, dx \lesssim \operatorname*{ess\,inf}_{x\in I} w(x) \quad \text{for every ball } I \subset \mathbb{R}.$$

When $p = \infty$, we define $A_\infty(\mathbb{R}) = \bigcup_{1\leq p<\infty} A_p(\mathbb{R})$.

A weight $\omega(x)$ is said to belong to the class $A_{p,q}(\mathbb{R})$, $1 < p \leq q < \infty$, if

$$\left(\frac{1}{|I|}\int_I \omega(x)^q\, dx\right)^{1/q}\left(\frac{1}{|I|}\int_I \omega(x)^{-p'}\, dx\right)^{1/p'} \leq C.$$

It is well known that if $\omega \in A_{p.q}(\mathbb{R})$, then $\omega^q \in A_\infty(\mathbb{R})$.

2.2 Function of $\mathrm{Lip}_\beta(\mathbb{R})$

The function of $\mathrm{Lip}_\beta(\mathbb{R})$ has the following important properties.

Lemma 2.1 *Let $b \in \mathrm{Lip}_\beta(\mathbb{R})$. Then*

(1) $1 \leq p < \infty$

$$\sup_I \frac{1}{|I|^\beta}\left(\frac{1}{|I|}\int_I \left|b(x) - b_I\right|^p\, dx\right)^{1/p} \leq C\|b\|_{\dot{\wedge}_\beta};$$

(2) *for any* $I_1 \subset I_2$,

$$\frac{1}{|I_2|}\int_{I_2}\left|b(y)-b_{I_1}\right|dy \lesssim \frac{|I_2|}{|I_1|}|I_2|^{\beta}\|b\|_{\dot{\Lambda}_\beta}.$$

2.3 Maximal function

We recall the definition of Hardy-Littlewood maximal operator and fractional maximal operator. The Hardy-Littlewood maximal operator is defined by

$$M(f)(x)=\sup_{I\ni x}\frac{1}{|I|}\int_I\left|f(y)\right|dy.$$

The fractional maximal function is defined as

$$M_{\beta,r}(f)(x)=\sup_{I\ni x}\left(\frac{1}{|I|^{1-r\beta}}\int_I\left|f(y)\right|^r dy\right)^{1/r}$$

for $1\le r<\infty$. In order to simplify the notation, we set $M_\beta(f)(x)=M_{\beta,1}(f)(x)$.

Lemma 2.2 *Let* $1<p<\infty$ *and* $\omega\in A_\infty(\mathbb{R})$. *Then*

$$\|Mf\|_{L^p(\omega)}\lesssim\left\|M^\sharp f\right\|_{L^p(\omega)}$$

for all f *such that the left hand side is finite.*

Lemma 2.3 *Suppose* $0<\beta<1$, $1\le r<p<1/\beta$, $1/q=1/p-\beta$. *If* $\omega\in A_{p,q}(\mathbb{R})$, *then*

$$\|M_{\beta,r}f\|_{L^q(\omega^q)}\lesssim\|f\|_{L^p(\omega^p)}.$$

2.4 Taylor series remainder

The following lemma gives an estimate on Taylor series remainder.

Lemma 2.4 [10] *Let* b *be a function on* $\mathbb{R}$ *and* $b^{(m)}\in L^s(\mathbb{R})$ *for any* $s>1$. *Then*

$$\left|R_m(b;x,y)\right|\lesssim|x-y|^m\left(\frac{1}{|I_x^y|}\int_{I_x^y}\left|b^{(m)}(z)\right|^s dz\right)^{1/s},$$

where I_x^y *is the interval* $(x-5|x-y|,x+5|x-y|)$.

2.5 Oscillation and variation operators

We consider the operator

$$\mathcal{O}'(Tf)(x)=\left(\sum_{i=1}^{\infty}\sup_{t_{i+1}<\delta_i<t_i}\left|T_{t_{i+1}}f(x)-T_{\delta_i}f(x)\right|^2\right)^{1/2}.$$

It is easy to check that

$$\mathcal{O}'(Tf)\approx\mathcal{O}(Tf).$$

Following [4], we denote by E the mixed norm Banach space of two variable function h defined on $\mathbb{R} \times \mathbb{N}$ such that

$$\|h\|_E \equiv \left(\sum_i \left(\sup_s \left| h(s,i) \right| \right)^2 \right)^{1/2} < \infty.$$

Given $T = \{T_\epsilon\}_{\epsilon>0}$, where T_ϵ defined as (1.4), for a fixed decreasing sequence $\{t_i\}$ with $t_i \searrow 0$, let $J_i = (t_{i+1}, t_i]$ and define the E-valued operator $\mathcal{U}(T) : f \to \mathcal{U}(T)f$ by

$$\mathcal{U}(T)f(x) = \left\{ T_{t_{i+1}}f(x) - T_s f(x) \right\}_{s\in J_i, i\in\mathbb{N}} = \left\{ \int_{\{t_{i+1}<|x-y|<s\}} K(x,y)f(y)\,dy \right\}_{s\in J_i, i\in\mathbb{N}}.$$

Then

$$\begin{aligned} \mathcal{O}'(Tf)(x) &= \left\| \mathcal{U}(T)f(x) \right\|_E = \left\| \left\{ T_{t_{i+1}}f(x) - T_s f(x) \right\}_{s\in J_i, i\in\mathbb{N}} \right\|_E \\ &= \left\| \left\{ \int_{\{t_{i+1}<|x-y|<s\}} K(x,y)f(y)\,dy \right\}_{s\in J_i, i\in\mathbb{N}} \right\|_E. \end{aligned}$$

On the other hand, let $\Theta = \{\beta : \beta = \{\epsilon_i\}, \epsilon_i \in \mathbb{R}, \epsilon_i \searrow 0\}$. We denote by F_ρ the mixed norm space of two variable functions $g(i,\beta)$ such that

$$\|g\|_{F_\rho} \equiv \sup_\beta \left(\sum_i \left| g(i,\beta) \right|^\rho \right)^{1/\rho}.$$

We also consider the F_ρ-valued operator $\mathcal{V}(T) : f \to \mathcal{V}(T)f$ given by

$$\mathcal{V}(T)f(x) = \left\{ T_{t_{i+1}}f(x) - T_{t_i}f(x) \right\}_{\beta=\{\epsilon_i\}\in\Theta}.$$

Then

$$\mathcal{V}_\rho(T)f(x) = \left\| \mathcal{V}(T)f(x) \right\|_{F_\rho}.$$

Next, let B be a Banach space and φ be a B-valued function, we define the sharp maximal operator as follows:

$$\varphi^\sharp(x) = \sup_{x\in I} \frac{1}{|I|} \int_I \left\| \varphi(y) - \frac{1}{|I|} \int_I \varphi(z)\,dz \right\|_B dy \approx \sup_{x\in I} \inf_c \frac{1}{|I|} \int_I \left\| \varphi(y) - c \right\|_B dy.$$

Then

$$M^\sharp\left(\mathcal{O}'(Tf)\right) \le 2\left(\mathcal{U}(T)f\right)^\sharp(x)$$

and

$$M^\sharp\left(\mathcal{V}_\rho(Tf)\right) \le 2\left(\mathcal{V}(T)f\right)^\sharp(x).$$

Finally, let us recall some results about oscillation and variation operators.

Lemma 2.5 ([5]) *Suppose that $K(x,y)$ satisfies* (1.1)-(1.3), $\rho > 2$. *Let $T = \{T_\epsilon\}_{\epsilon>0}$ be given by* (1.4). *If $O(T)$ and $V_\rho(T)$ are bounded on $L^{p_0}(R)$ for some $1 < p_0 < \infty$, then, for any $1 < p < \infty$, $\omega \in A_p(\mathbb{R})$,*

$$\|\mathcal{O}'(Tf)\|_{L^p(\omega)} \le \|\mathcal{O}(Tf)\|_{L^p(\omega)} \lesssim \|f\|_{L^p(\omega)}$$

and

$$\|\mathcal{V}_\rho(Tf)\|_{L^p(\omega)} \lesssim \|f\|_{L^p(\omega)}.$$

3 The proof of main results

Note that if $\omega \in A_{p,q}(\mathbb{R})$, then $\omega^q \in A_\infty(\mathbb{R})$. By Lemma 2.2 and Lemma 2.3, we only need to prove

$$M^\sharp\big(\mathcal{O}'\big(T^b\big)f\big)(x) \lesssim \|b^{(m)}\|_{\dot{\wedge}_\beta}\big(M_{\beta,r}(f)(x) + M_\beta(f)(x)\big) \tag{3.1}$$

and

$$M^\sharp\big(\mathcal{V}_\rho\big(T^b\big)f\big)(x) \lesssim \|b^{(m)}\|_{\dot{\wedge}_\beta}\big(M_{\beta,r}(f)(x) + M_\beta(f)(x)\big) \tag{3.2}$$

hold for any $1 < r < \infty$.

We will prove only inequality (3.1), since (3.2) can be obtained by a similar argument. Fix f and x_0 with an interval $I = (x_0 - l, x_0 + l)$. Write $f = f_1 + f_2 = f\chi_{5I} + f\chi_{\mathbb{R}\setminus 5I}$, and let

$$C_I = \left\{\int_{\{t_{i+1}<|x_0-y|<s\}} \frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m} K(x_0,y) f_2(y)\,dy\right\}_{s\in J_i, i\in\mathbb{N}} = \mathcal{U}\big(T^b\big)f_2(x_0).$$

Then

$$\begin{aligned}\mathcal{U}\big(T^b\big)f(x) &= \left\{\int_{\{t_{i+1}<|x-y|<s\}} \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x,y) f(y)\,dy\right\}_{s\in J_i, i\in\mathbb{N}}\\ &= \mathcal{U}(T)\left(\frac{R_{m+1}(b;x,\cdot)}{|x-\cdot|^m} f_1\right) + \mathcal{U}\big(T^b\big)f_2(x).\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{|I|}\int_I \big\|\mathcal{U}\big(T^b\big)f(x) - C_I\big\|_E\,dx\\ &\quad\le \frac{1}{|I|}\int_I \left\|\mathcal{U}(T)\left(\frac{R_{m+1}(b;x,\cdot)}{|x-\cdot|^m} f_1\right)\right\|_E dx + \frac{1}{|I|}\int_I \big\|\mathcal{U}\big(T^b\big)f_2(x) - \mathcal{U}\big(T^b\big)f_2(x_0)\big\|_E\,dx\\ &\quad= M_1 + M_2.\end{aligned}$$

For $x \in I$, $k = 0, -1, -2, \ldots$, let $E_k = \{y : 2^{k-1}\cdot 6l \le |y-x| < 2^k\cdot 6l\}$, let $I_k = \{y : |y-x| < 2^k\cdot 6l\}$, and let $b_k(z) = b(z) - \frac{1}{m!}(b^{(m)})_{I_k} z^m$. By [10] we have $R_{m+1}(b;x,y) = R_{m+1}(b_k;x,y)$ for any $y \in E_k$.

By Lemma 2.5, we know $\mathcal{O}'(T)$ is bounded on $L^u(\mathbb{R})$ for $u > 1$. Then, using Hölder's inequality, we deduce

$$\begin{aligned}
M_1 &\lesssim \left(\frac{1}{|I|}\int_I \left\|\mathcal{U}(T)\left(\frac{R_{m+1}(b;x,\cdot)}{|x-\cdot|^m}f_1\right)\right\|_E^u dx\right)^{1/u} \\
&\lesssim \left(\frac{1}{|I|}\int_{\{y:|y-x|<6l\}}\left|\frac{R_{m+1}(b;\cdot,y)}{|y-\cdot|^m}f(y)\right|^u dy\right)^{1/u} \\
&= \left(\frac{1}{|I|}\sum_{k=-\infty}^{0}\int_{E_k}\left|\left(\frac{R_{m+1}(b_k;\cdot,y)}{|y-\cdot|^m}f(y)\right)\right|^r dy\right)^{1/r} \\
&\lesssim \left(\frac{1}{|I|}\sum_{k=-\infty}^{0}\int_{E_k}\left|\left(\left(\frac{R_m(b_k;\cdot,y)}{|y-\cdot|^m}-\frac{1}{m!}\frac{(y-\cdot)^m b_k^{(m)}(y)}{|y-\cdot|^m}\right)f(y)\right)\right|^u dy\right)^{1/u} \\
&\lesssim \left(\frac{1}{|I|}\sum_{k=-\infty}^{0}\int_{E_k}\left|\frac{R_m(b_k;\cdot,y)}{|y-\cdot|^m}f(y)\right|^u dy\right)^{1/u} \\
&\quad + \left(\frac{1}{|I|}\sum_{k=-\infty}^{0}\int_{E_k}\left|\frac{1}{m!}\frac{(y-\cdot)^m b_k^{(m)}(y)}{|y-\cdot|^m}f(y)\right|^u dy\right)^{1/u} \\
&= M_{11}+M_{12}.
\end{aligned}$$

By Lemma 2.4 and Lemma 2.1,

$$\begin{aligned}
\left|R_m(b_k;x,y)\right| &\lesssim |x-y|^m\left(\frac{1}{|I_x^y|}\int_{I_x^y}\left|b_k^{(m)}(z)\right|^s dz\right)^{1/s} \\
&\lesssim |x-y|^m\left(\frac{1}{2^k\cdot 30l}\int_{|y-x|<2^k\cdot 30l}\left|b^{(m)}(y)-\left(b^{(m)}\right)_{I_k}\right|^s dz\right)^{1/s} \\
&\lesssim |x-y|^m\left(2^k l\right)^\beta \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}.
\end{aligned}$$

Then

$$\begin{aligned}
M_{11} &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta} l^\beta \left(\frac{1}{|I|}\sum_{k=-\infty}^{0}2^{k\beta u}\int_{E_k}\left|f(y)\right|^u dy\right)^{1/u} \\
&\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta} l^\beta \left(\frac{1}{|I|}\sum_{k=-\infty}^{0}\int_{E_k}\left|f(y)\right|^u dy\right)^{1/u} \\
&\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta} l^\beta \left(\frac{1}{|I|}\int_{7I}\left|f(y)\right|^u dy\right)^{1/u} \\
&\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta} l^\beta \left(\frac{1}{|I|}\int_{7I}\left|f(y)\right|^r dy\right)^{1/r} \\
&\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta} M_{\beta,r}(f)(x_0).
\end{aligned}$$

Since $b_k^{(m)}(y) = b^{(m)}(y) - (b^{(m)})_{I_k}$, then, applying Hölder's inequality and Lemma 2.1, we get

$$\begin{aligned}
M_{12} &\lesssim \left(\frac{1}{|I|}\sum_{k=-\infty}^{0}\int_{E_k}\left|\left(b^{(m)}(y)-\left(b^{(m)}\right)_{I_k}\right)f(y)\right|^u dy\right)^{1/u}\\
&\lesssim \left(\frac{1}{|I|}\sum_{k=-\infty}^{0}\left(\int_{I_k}\left|f(y)\right|^r dy\right)^{u/r}\left(\int_{I_k}\left|b^{(m)}(y)-\left(b^{(m)}\right)_{I_k}\right|^{\frac{ur}{r-u}}\right)^{1-u/r}\right)^{1/u}\\
&\lesssim \left\|b^{(m)}\right\|_{\dot{\wedge}_\beta}\left(\frac{1}{|I|}\sum_{k=-\infty}^{0}\left(\int_{I_k}\left|f(y)\right|^r dy\right)^{u/r}|I_k|^{\beta u+1-u/r}\right)^{1/u}\\
&\lesssim \left\|b^{(m)}\right\|_{\dot{\wedge}_\beta}M_{\beta,r}(f)(x_0)\left(\frac{1}{|I|}\sum_{k=-\infty}^{0}|I_k|\right)^{1/u}\\
&\lesssim \left\|b^{(m)}\right\|_{\dot{\wedge}_\beta}M_{\beta,r}(f)(x_0).
\end{aligned}$$

We now estimate M_2. For $x \in I$, we have

$$\begin{aligned}
&\left\|\mathcal{U}\left(T^b\right)f_2(x)-\mathcal{U}\left(T^b\right)f_2(x_0)\right\|_E\\
&\quad=\left\|\left\{\int_{\{t_{i+1}<|x-y|<s\}}\frac{R_{m+1}(b;x,y)}{|x-y|^m}K(x,y)f_2(y)\,dy\right.\right.\\
&\qquad\left.\left.-\int_{\{t_{i+1}<|x_0-y|<s\}}\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}K(x_0,y)f_2(y)\,dy\right\}_{s\in J_i,i\in\mathbb{N}}\right\|_E\\
&\quad\le\left\|\left\{\int_{\{t_{i+1}<|x-y|<s\}}\left(\frac{R_{m+1}(b;x,y)}{|x-y|^m}K(x,y)-\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}K(x_0,y)\right)f_2(y)\,dy\right\}_{s\in J_i,i\in\mathbb{N}}\right\|_E\\
&\qquad+\left\|\left\{\int_R\left(\chi_{\{t_{i+1}<|x-y|<s\}}(y)-\chi_{\{t_{i+1}<|x_0-y|<s\}}(y)\right)\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}K(x_0,y)f_2(y)\,dy\right\}_{s\in J_i,i\in\mathbb{N}}\right\|_E\\
&\quad=N_1+N_2.
\end{aligned}$$

For $k = 0, 1, 2, \dots$, let $F_k = \{y : 2^k \cdot 4l \le |y - x_0| < 2^{k+1} \cdot 4l\}$, let $\widetilde{I}_k = \{y : |y - x_0| < 2^k \cdot 4l\}$, and let $\widetilde{b}_k(z) = b(z) - \frac{1}{m!}(b^{(m)})_{\widetilde{I}_k} z^m$. Note that

$$\begin{aligned}
&\frac{R_{m+1}(b;x,y)}{|x-y|^m}K(x,y)-\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}K(x_0,y)\\
&\quad=\frac{R_{m+1}(\widetilde{b}_k;x,y)}{|x-y|^m}K(x,y)-\frac{R_{m+1}(\widetilde{b}_k;x_0,y)}{|x_0-y|^m}K(x_0,y)\\
&\quad=\frac{1}{|x-y|^m}\left(R_m(\widetilde{b}_k;x,y)-R_m(\widetilde{b}_k;x_0,y)\right)K(x,y)\\
&\qquad+R_m(\widetilde{b}_k;x_0,y)\left(\frac{1}{|x-y|^m}-\frac{1}{|x_0-y|^m}\right)K(x,y)\\
&\qquad-\frac{1}{m!}\widetilde{b}_k^{(m)}(y)\left(\frac{(x-y)^m}{|x-y|^m}-\frac{(x_0-y)^m}{|x_0-y|^m}\right)K(x,y)\\
&\qquad+\frac{R_{m+1}(\widetilde{b}_k;x_0,y)}{|x_0-y|^m}\left(K(x,y)-K(x_0,y)\right).
\end{aligned}$$

By Minkowski's inequalities and $\|\{\chi_{\{t_{i+1}<|x-y|<s\}}\}_{s\in J_i,i\in\mathbb{N}}\|_E \leq 1$, we obtain

$$\begin{aligned}
N_1 &\leq \int_{\mathbb{R}} \big\|\{\chi_{\{t_{i+1}<|x-y|<s\}}\}_{s\in J_i,i\in\mathbb{N}}\big\|_E \\
&\quad \times \left|\frac{R_{m+1}(\widetilde{b}_k;x,y)}{|x-y|^m}K(x,y) - \frac{R_{m+1}(\widetilde{b}_k;x_0,y)}{|x_0-y|^m}K(x_0,y)\right|\big|f_2(y)\big|\,dy \\
&\leq \sum_{k=0}^{\infty}\int_{F_k}\frac{1}{|x-y|^m}\big|R_m(\widetilde{b}_k;x,y) - R_m(\widetilde{b}_k;x_0,y)\big|\big|K(x,y)\big|\big|f_2(y)\big|\,dy \\
&\quad + \sum_{k=0}^{\infty}\int_{F_k}\big|R_m(\widetilde{b}_k;x_0,y)\big|\left|\frac{1}{|x-y|^m} - \frac{1}{|x_0-y|^m}\right|\big|K(x,y)\big|\big|f_2(y)\big|\,dy \\
&\quad + \sum_{k=0}^{\infty}\int_{F_k}\frac{1}{m!}\big|\widetilde{b}_k^{(m)}(y)\big|\left|\frac{(x-y)^m}{|x-y|^m} - \frac{(x_0-y)^m}{|x_0-y|^m}\right|\big|K(x,y)\big|\big|f_2(y)\big|\,dy \\
&\quad + \sum_{k=0}^{\infty}\int_{F_k}\left|\frac{R_{m+1}(\widetilde{b}_k;x_0,y)}{|x_0-y|^m}\right|\big|K(x,y) - K(x_0,y)\big|\big|f_2(y)\big|\,dy \\
&= N_{11} + N_{12} + N_{13} + N_{14}.
\end{aligned}$$

From the mean value theorem, there exists $\eta \in I$ such that

$$R_m(\widetilde{b}_k;x,y) - R_m(\widetilde{b}_k;x_0,y) = (x-x_0)R_{m-1}\big(\widetilde{b}_k';\eta,y\big).$$

For $\eta, x \in I$, $y \in F_k$, we have $|y-x_0| \approx |y-x| \approx |y-\eta|$ and $5|y-\eta| \approx 5|y-x_0| \leq 2^{k+1}\cdot 20l$. By Lemma 2.4 and Lemma 2.1 we get

$$\begin{aligned}
\big|R_{m-1}\big(\widetilde{b}_k';\eta,y\big)\big| &\lesssim |\eta-y|^{m-1}\left(\frac{1}{|I_\eta^y|}\int_{I_\eta^y}\big|\widetilde{b}_k^{(m)}(z)\big|^s\,dz\right)^{1/s} \\
&\lesssim |x-y|^{m-1}\left(\frac{1}{2^{k+1}\cdot 20l}\int_{|z-x_0|<2^{k+1}\cdot 20l}\big|b^{(m)}(z) - \big(b^{(m)}\big)_{\widetilde{I}_k}\big|^s\,dz\right)^{1/s} \\
&\lesssim \big\|b^{(m)}\big\|_{\dot{\Lambda}_\beta}\big(2^k l\big)^\beta|x-y|^{m-1}.
\end{aligned}$$

Then

$$\big|R_m(\widetilde{b}_k;x,y) - R_m(\widetilde{b}_k;x_0,y)\big| \lesssim \big\|b^{(m)}\big\|_{\dot{\Lambda}_\beta}\big(2^k l\big)^\beta|x-x_0||x-y|^{m-1}.$$

Since $|K(x,y)| \leq C|x_0-y|^{-1}$,

$$\begin{aligned}
N_{11} &\lesssim \big\|b^{(m)}\big\|_{\dot{\Lambda}_\beta}\sum_{k=0}^{\infty}\big(2^k l\big)^\beta\int_{2^k\cdot 4l\leq|x_0-y|<2^{k+1}\cdot 4l}\frac{l}{(2^k\cdot 4l)^2}\big|f(y)\big|\,dy \\
&\lesssim \big\|b^{(m)}\big\|_{\dot{\Lambda}_\beta}\sum_{k=0}^{\infty}\frac{1}{2^k}\frac{(2^k l)^\beta}{2^k l}\int_{|x_0-y|<2^{k+1}\cdot 4l}\big|f(y)\big|\,dy \\
&\lesssim \big\|b^{(m)}\big\|_{\dot{\Lambda}_\beta}M_\beta(f)(x_0).
\end{aligned}$$

For N_{12}, since $x \in I$, $y \in F_k$,

$$\left|R_m(\widetilde{b}_k; x, y)\right| \lesssim |x-y|^m \left(\frac{1}{|I_x^y|}\int_{I_x^y}\left|\widetilde{b}_k^{(m)}(z)\right|^s dz\right)^{1/s} \lesssim \|b^{(m)}\|_{\dot{\wedge}_\beta}\left(2^k l\right)^\beta |x-y|^m$$

and

$$\left|\frac{1}{|x-y|^m}-\frac{1}{|x_0-y|^m}\right| \lesssim \frac{|x-x_0|}{|x-y|^{m+1}}.$$

Thus

$$N_{12} \lesssim \|b^{(m)}\|_{\dot{\wedge}_\beta}\sum_{k=0}^{\infty}\left(2^k l\right)^\beta \int_{2^k\cdot 4l\le|x_0-y|<2^{k+1}\cdot 4l}\frac{l}{(2^k\cdot 4l)^2}\left|f(y)\right| dy \lesssim \|b^{(m)}\|_{\dot{\wedge}_\beta} M_\beta(f)(x_0).$$

As for N_{13}, due to

$$\left|\frac{(x-y)^m}{|x-y|^m}-\frac{(x_0-y)^m}{|x_0-y|^m}\right| \lesssim \frac{|x-x_0|}{|x-y|},$$

and noting $\widetilde{b}_k^{(m)}(y) = b^{(m)}(y) - (b^{(m)})_{\widetilde{I}_k}$, we have

$$\begin{aligned}
N_{13} &\lesssim \sum_{k=0}^{\infty}\int_{F_k}\left|b^{(m)}(y)-\left(b^{(m)}\right)_{\widetilde{I}_k}\right|\frac{|x-x_0|}{|x_0-y|^2}\left|f(y)\right| dy \\
&\lesssim \sum_{k=0}^{\infty}\frac{1}{2^k}\frac{1}{2^k\cdot 4l}\int_{|x_0-y|<2^k\cdot 4l}\left|b^{(m)}(y)-\left(b^{(m)}\right)_{\widetilde{I}_k}\right|\left|f(y)\right| dy \\
&\lesssim \sum_{k=0}^{\infty}\frac{1}{2^k}\left(\frac{1}{2^k\cdot 4l}\int_{|x_0-y|<2^k\cdot 4l}\left|f(y)\right|^r dy\right)^{1/r} \\
&\quad\times\left(\frac{1}{2^k\cdot 4l}\int_{|x_0-y|<2^k\cdot 4l}\left|b^{(m)}(y)-\left(b^{(m)}\right)_{\widetilde{I}_k}\right|^{r'} dy\right)^{1/r'} \\
&\lesssim \|b^{(m)}\|_{\dot{\wedge}_\beta} M_{r,\beta}(f)(x_0)\sum_{k=0}^{\infty}\frac{1}{2^k} \lesssim \|b^{(m)}\|_{\dot{\wedge}_\beta} M_{\beta,r}(f)(x_0).
\end{aligned}$$

Notice

$$\begin{aligned}
\left|R_{m+1}(\widetilde{b}_k; x_0, y)\right| &\le \left|R_m(\widetilde{b}_k; x_0, y)\right| + \frac{1}{m!}\left|\widetilde{b}_k^{(m)}(y)(x_0-y)^m\right| \\
&\lesssim \|b^{(m)}\|_{\dot{\wedge}_\beta}\left(2^k l\right)^\beta |x_0-y|^m + \left|b^{(m)}(y)-\left(b^{(m)}\right)_{\widetilde{I}_k}\right||x_0-y|^m
\end{aligned}$$

and by (1.2),

$$\left|K(x,y)-K(x_0,y)\right| \lesssim \frac{|x-x_0|^\delta}{|x_0-y|^{1+\delta}}.$$

Similar to the estimates for N_{11}, we have

$$\sum_{k=0}^{\infty}\int_{F_k}\frac{|R_m(\widetilde{b}_k; x_0, y)|}{|x-y|^m}\frac{|x-x_0|^\delta}{|x_0-y|^{1+\delta}}\left|f(y)\right| dy \lesssim \|b^{(m)}\|_{\dot{\wedge}_\beta} M_\beta(f)(x_0).$$

Similar to the estimates for N_{13}, we have

$$\sum_{k=0}^{\infty}\int_{F_k}\frac{|\widetilde{b}_k^{(m)}(y)(x_0-y)^m|}{|x-y|^m}\frac{|x-x_0|^{\delta}}{|x_0-y|^{1+\delta}}\big|f(y)\big|\,dy\lesssim\big\|b^{(m)}\big\|_{\dot\Lambda_\beta}M_{\beta,r}(f)(x_0).$$

Then

$$N_{14}\lesssim\big\|b^{(m)}\big\|_{\dot\Lambda_\beta}\big(M_\beta(f)(x_0)+M_{\beta,r}(f)(x_0)\big).$$

Finally, let us estimate N_2. Notice that the integral

$$\int_R\big(\chi_{\{t_{i+1}<|x-y|<s\}}(y)-\chi_{\{t_{i+1}<|x_0-y|<s\}}(y)\big)\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}K(x_0,y)f_2(y)\,dy$$

will be non-zero in the following cases:

(i) $t_{i+1}<|x-y|<s$ and $|x_0-y|\le t_{i+1}$;

(ii) $t_{i+1}<|x-y|<s$ and $|x_0-y|\ge s$;

(iii) $t_{i+1}<|x_0-y|<s$ and $|x-y|\le t_{i+1}$;

(iv) $t_{i+1}<|x_0-y|<s$ and $|x-y|\ge s$.

In case (i) we have $t_{i+1}<|x-y|\le|x_0-x|+|x_0-y|<l+t_{i+1}$ as $|x-x_0|<l$. Similarly, in case (iii) we have $t_{i+1}<|x_0-y|<l+t_{i+1}$ as $|x-x_0|<l$. In case (ii) we have $s<|x_0-y|<l+s$ and in case (iv) we have $s<|x-y|<l+s$. By (1.1) and taking $1<t<r$, we have

$$\begin{aligned}&\int_{\mathbb{R}}\big(\chi_{\{t_{i+1}<|x-y|<s\}}(y)-\chi_{\{t_{i+1}<|x_0-y|<s\}}(y)\big)\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}K(x_0,y)f_2(y)\,dy\\
&\quad\lesssim\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x-y|<s\}}(y)\chi_{\{t_{i+1}<|x-y|<l+t_{i+1}\}}(y)\left|\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}\right|\frac{|f_2(y)|}{|x_0-y|}\,dy\\
&\qquad+\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x-y|<s\}}(y)\chi_{\{s<|x_0-y|<l+s\}}(y)\left|\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}\right|\frac{|f_2(y)|}{|x_0-y|}\,dy\\
&\qquad+\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x_0-y|<s\}}(y)\chi_{\{t_{i+1}<|x_0-y|<l+t_{i+1}\}}(y)\left|\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}\right|\frac{|f_2(y)|}{|x_0-y|}\,dy\\
&\qquad+\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x_0-y|<s\}}(y)\chi_{\{s<|x-y|<l+s\}}(y)\left|\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}\right|\frac{|f_2(y)|}{|x_0-y|}\,dy\\
&\quad\lesssim l^{1/t'}\left(\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x-y|<s\}}(y)\left|\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}\right|^t\frac{|f_2(y)|^t}{|x_0-y|^t}\,dy\right)^{1/t}\\
&\qquad+l^{1/t'}\left(\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x_0-y|<s\}}(y)\left|\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}\right|^t\frac{|f_2(y)|^t}{|x_0-y|^t}\,dy\right)^{1/t}.\end{aligned}$$

Then

$$\begin{aligned}N_2&\lesssim l^{1/t'}\left\|\left\{\left(\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x-y|<s\}}(y)\left|\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}\right|^t\frac{|f_2(y)|^t}{|x_0-y|^t}\,dy\right)^{1/t}\right\}_{s\in J_i,i\in\mathbb{N}}\right\|_E\\
&\quad+l^{1/t'}\left\|\left\{\left(\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x_0-y|<s\}}(y)\left|\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}\right|^t\frac{|f_2(y)|^t}{|x_0-y|^t}\,dy\right)^{1/t}\right\}_{s\in J_i,i\in\mathbb{N}}\right\|_E\\
&=N_{21}+N_{22}.\end{aligned}$$

Notice

$$\left|R_{m+1}(\widetilde{b}_k;x_0,y)\right| \lesssim \left\|b^{(m)}\right\|_{\dot{\wedge}_\beta}\left(2^k l\right)^\beta |x_0-y|^m + \left|b^{(m)}(y)-\left(b^{(m)}\right)_{\widetilde{I}_k}\right||x_0-y|^m.$$

Choosing $1<r<p$ with $t=\sqrt{r}$, we have

$$\begin{aligned}
N_{21} &\lesssim l^{1/t'}\left\{\sum_{i\in\mathbb{N}}\sup_{s\in J_i}\left(\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x-y|<s\}}(y)\left|\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}\right|^t\frac{|f_2(y)|^t}{|x_0-y|^t}\,dy\right)^{2/t}\right\}^{1/2}\\
&\lesssim l^{1/t'}\left\{\sum_{i\in\mathbb{N}}\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x-y|<t_i\}}(y)\left|\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}\right|^t\frac{|f_2(y)|^t}{|x_0-y|^t}\,dy\right\}^{1/t}\\
&\lesssim l^{1/t'}\left\{\int_{\mathbb{R}}\left|\frac{R_{m+1}(b;x_0,y)}{|x_0-y|^m}\right|^t\frac{|f_2(y)|^t}{|x_0-y|^t}\,dy\right\}^{1/t}\\
&\lesssim l^{1/t'}\left\{\sum_{k=0}^{\infty}\int_{F_k}\left|\frac{R_{m+1}(\widetilde{b}_k;x_0,y)}{|x_0-y|^m}\right|^t\frac{|f_2(y)|^t}{|x_0-y|^t}\,dy\right\}^{1/t}\\
&\lesssim \left\|b^{(m)}\right\|_{\dot{\wedge}_\beta} l^{1/t'}\left\{\sum_{k=0}^{\infty}\left(2^k l\right)^{\beta t}\int_{F_k}\frac{|f(y)|^t}{|x_0-y|^t}\,dy\right\}^{1/t}\\
&\quad + l^{1/t'}\left\{\sum_{k=0}^{\infty}\int_{F_k}\left(\left|b^{(m)}(y)-\left(b^{(m)}\right)_{\widetilde{I}_k}\right|\right)^t\frac{|f(y)|^t}{|x_0-y|^t}\,dy\right\}^{1/t}.
\end{aligned}$$

But

$$\begin{aligned}
&l^{1/t'}\left\{\sum_{k=0}^{\infty}\left(2^k l\right)^{\beta t}\int_{F_k}\frac{|f(y)|^t}{|x_0-y|^t}\,dy\right\}^{1/t}\\
&\quad\lesssim l^{1/t'}\left(\sum_{k=1}^{\infty}\frac{(2^k l)^{\beta t}}{(2^k\cdot 4l)^t}\int_{|x_0-y|<2^{k+1}\cdot 4l}\left|f(y)\right|^t\,dy\right)^{1/t}\\
&\quad\lesssim \left(\sum_{k=1}^{\infty}\frac{1}{2^{k(t-1)}}\frac{(2^k l)^{\beta t}}{2^k\cdot 5l}\int_{|x_0-y|<2^k\cdot 5l}\left|f(y)\right|^t\,dy\right)^{1/t}\\
&\quad\lesssim \left(\sum_{k=1}^{\infty}\frac{1}{2^{k(t-1)}}\left(\frac{(2^k l)^{\beta t^2}}{2^k\cdot 5l}\int_{|x_0-y|<2^k\cdot 5l}\left|f(y)\right|^{t^2}\,dy\right)^{1/t}\right)^{1/t}\\
&\quad\lesssim \left(\sum_{k=1}^{\infty}\frac{1}{2^{k(t-1)}}\right)^{1/t} M_{\beta,r}(f)(x_0)\lesssim M_{\beta,r}(f)(x_0)
\end{aligned}$$

and

$$\begin{aligned}
&l^{1/t'}\left\{\sum_{k=0}^{\infty}\int_{F_k}\left(\left|b^{(m)}(y)-\left(b^{(m)}\right)_{\widetilde{I}_k}\right|\right)^t\frac{|f(y)|^t}{|x_0-y|^t}\,dy\right\}^{1/t}\\
&\quad\lesssim \left(\sum_{k=0}^{\infty}\frac{1}{2^{k(t-1)}}\frac{1}{2^k\cdot 4l}\int_{|x_0-y|<2^k\cdot 4l}\left|b^{(m)}(y)-\left(b^{(m)}\right)_{\widetilde{I}_k}\right|^t\left|f(y)\right|^t\,dy\right)^{1/t}
\end{aligned}$$

$$\lesssim \left(\sum_{k=0}^{\infty} \frac{1}{2^{k(t-1)}} \left(\frac{1}{2^k \cdot 4l} \int_{|x_0-y|<2^k \cdot 4l} \left| f(y) \right|^{t^2} dy \right)^{1/t} \right.$$

$$\left. \times \left(\frac{1}{2^k \cdot 4l} \int_{|x_0-y|<2^k \cdot 4l} \left| b^{(m)}(y) - \left(b^{(m)}\right)_{\widetilde{I}_k} \right|^{tt'} \right)^{1/t'} \right)^{1/t}$$

$$\lesssim \left\| b^{(m)} \right\|_{\dot{\Lambda}_\beta} \left(\sum_{k=0}^{\infty} \frac{1}{2^{k(t-1)}} \left(\frac{(2^k \cdot 4l)^{r\beta}}{2^k \cdot 4l} \int_{|x_0-y|<2^k \cdot 4l} \left| f(y) \right|^{t^2} dy \right)^{1/t} \right)^{1/t}$$

$$\lesssim \left\| b^{(m)} \right\|_{\dot{\Lambda}_\beta} M_{\beta,r}(f)(x_0) \left(\sum_{k=0}^{\infty} \frac{1}{2^{k(t-1)}} \right)^{1/t}$$

$$\lesssim \left\| b^{(m)} \right\|_{\dot{\Lambda}_\beta} M_{\beta,r}(f)(x_0).$$

Therefore

$$N_{21} \lesssim \left\| b^{(m)} \right\|_{\dot{\Lambda}_\beta} M_{\beta,r}(f)(x_0).$$

Similarly,

$$N_{22} \lesssim \left\| b^{(m)} \right\|_{\dot{\Lambda}_\beta} M_{\beta,r}(f)(x_0).$$

This completes the proof of (3.1). Hence, Theorem 1.1 is proved.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper together. They also read and approved the final manuscript.

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