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Some refinements of operator reverse AM-GM mean inequalities

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Abstract

In this paper, we prove the operator inequalities as follows: Let A, B be positive operators on a Hilbert space with $0 < m \leq A, B \leq M$ and $\sqrt{\frac{M}{m}} \leq 2.314$. Then for every positive unital linear map Φ,

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \frac{(M+m)^2}{4Mm}\Phi^2(A \sharp B)$$

and

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \frac{(M+m)^2}{4Mm}\left(\Phi(A) \sharp \Phi(B)\right)^2.$$

Moreover, we prove Lin's conjecture when $\sqrt{\frac{M}{m}} \leq 2.314$.

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1 Introduction

Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. Throughout this paper, a capital letter denotes an operator in $\mathcal{B}(\mathcal{H})$, we identify a scalar with the identity operator I multiplied by this scalar. We write $A \geq 0$ to mean that the operator A is positive. A is said to be strictly positive (denoted by $A > 0$) if it is a positive invertible operator. A linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is called positive if $A \geq 0$ implies $\Phi(A) \geq 0$. It is said to be unital if $\Phi(I) = I$. For $A, B > 0$, the geometric mean $A \sharp B$ is defined by

$$A \sharp B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}.$$

Let $0 < m \leq A, B \leq M$. Tominaga [1] showed that the following operator reverse AM-GM inequality holds:

$$\frac{A+B}{2} \leq S(h)A \sharp B, \tag{1.1}$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$ is called Specht's ratio and $h = \frac{M}{m}$. Indeed,

$$S(h) \le \frac{(M+m)^2}{4Mm} \le S^2(h) \quad (h \ge 1) \tag{1.2}$$

was observed by Lin [2, (3.3)].

Let Φ be a positive linear map and $A, B > 0$. Ando [3] gave the following inequality:

$$\Phi(A \sharp B) \le \Phi(A) \sharp \Phi(B). \tag{1.3}$$

By (1.1), (1.2) and (1.3), it is easy to obtain the following inequalities:

$$\Phi\left(\frac{A+B}{2}\right) \le \frac{(M+m)^2}{4Mm} \Phi(A \sharp B) \tag{1.4}$$

and

$$\Phi\left(\frac{A+B}{2}\right) \le \frac{(M+m)^2}{4Mm} \bigl(\Phi(A) \sharp \Phi(B)\bigr). \tag{1.5}$$

Lin [2] proved that (1.4) and (1.5) can be squared:

$$\Phi^2\left(\frac{A+B}{2}\right) \le \left(\frac{(M+m)^2}{4Mm}\right)^2 \Phi^2(A \sharp B) \tag{1.6}$$

and

$$\Phi^2\left(\frac{A+B}{2}\right) \le \left(\frac{(M+m)^2}{4Mm}\right)^2 \bigl(\Phi(A) \sharp \Phi(B)\bigr)^2. \tag{1.7}$$

Meanwhile, Lin [2] conjectured that the following inequalities hold:

$$\Phi^2\left(\frac{A+B}{2}\right) \le S^2(h)\Phi^2(A \sharp B) \tag{1.8}$$

and

$$\Phi^2\left(\frac{A+B}{2}\right) \le S^2(h)\bigl(\Phi(A) \sharp \Phi(B)\bigr)^2. \tag{1.9}$$

For more information on operator inequalities, the reader is referred to [4–7].

In this paper, we will present some operator reverse AM-GM inequalities which are refinements of (1.1), (1.6) and (1.7). Furthermore, we will prove (1.8) and (1.9) if the condition number $\sqrt{\frac{M}{m}}$ is not too big.

2 Main results

We begin this section with the following lemmas.

Lemma 1 ([8]) *Let $A, B > 0$. Then the following norm inequality holds*:

$$\|AB\| \le \frac{1}{4}\|A+B\|^2. \tag{2.1}$$

Lemma 2 ([9]) *Let $A > 0$. Then for every positive unital linear map Φ,*

$$\Phi\big(A^{-1}\big) \geq \Phi^{-1}(A). \tag{2.2}$$

Theorem 1 *If $0 < m \leq A, B \leq M$ for some scalars $m \leq M$, then*

$$\frac{A+B}{2} \leq \frac{M+m}{2\sqrt{Mm}} A \sharp B. \tag{2.3}$$

Proof Put $C = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$. Since $\frac{m}{M} \leq C \leq \frac{M}{m}$, it follows that

$$\left[C^{\frac{1}{2}} - \frac{1}{2}\left(\sqrt{\frac{m}{M}} + \sqrt{\frac{M}{m}}\right)\right]^2 \leq \frac{1}{4}\left(\sqrt{\frac{M}{m}} - \sqrt{\frac{m}{M}}\right)^2,$$

and hence

$$C + 1 \leq \left(\sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}}\right) C^{\frac{1}{2}}.$$

This implies

$$B + A \leq \left(\sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}}\right) A \sharp B.$$

Thus

$$\frac{A+B}{2} \leq \frac{M+m}{2\sqrt{Mm}} A \sharp B.$$

This completes the proof. □

Remark 1 By (1.2), it is easy to know that (2.3) is tighter than (1.1).

Theorem 2 *If $0 < m \leq A, B \leq M$ and $\sqrt{\frac{M}{m}} \leq 2.314$ for some scalars $m \leq M$, then*

$$\left(\frac{A+B}{2}\right)^2 \leq \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 (A \sharp B)^2. \tag{2.4}$$

Proof Inequality (2.4) is equivalent to

$$\left\| \frac{A+B}{2} (A \sharp B)^{-1} \right\| \leq \frac{M+m}{2\sqrt{Mm}}. \tag{2.5}$$

If $0 < m \leq A, B \leq \frac{M+m}{2}$, we have

$$A + \frac{M+m}{2} m A^{-1} \leq \frac{M+m}{2} + m \tag{2.6}$$

and

$$B + \frac{M+m}{2} m B^{-1} \leq \frac{M+m}{2} + m. \tag{2.7}$$

Compute

$$\begin{aligned}\left\| \frac{A+B}{2}\frac{M+m}{2}m(A\sharp B)^{-1}\right\| &\le \frac{1}{4}\left\| \frac{A+B}{2}+\frac{M+m}{2}m(A\sharp B)^{-1}\right\|^2 \quad \bigl(\text{by (2.1)}\bigr)\\ &\le \frac{1}{4}\left\| \frac{A+B}{2}+\frac{M+m}{2}m\frac{A^{-1}+B^{-1}}{2}\right\|^2\\ &\le \frac{1}{4}\left(\frac{M+m}{2}+m\right)^2 \quad \bigl(\text{by (2.6), (2.7)}\bigr).\end{aligned}$$

That is,

$$\left\| \frac{A+B}{2}(A\sharp B)^{-1}\right\| \le \frac{(\frac{M+m}{2}+m)^2}{4\frac{M+m}{2}m}.$$

Since $1\le \sqrt{\frac{M}{m}}\le 2.314$, it follows that

$$\left(\sqrt{\frac{M}{m}}-1\right)^2\left[\left(\sqrt{\frac{M}{m}}\right)^3-\frac{2M}{m}+\sqrt{\frac{M}{m}}-4\right]\le 0. \tag{2.8}$$

It is easy to know that $\frac{(\frac{M+m}{2}+m)^2}{4\frac{M+m}{2}m}\le \frac{M+m}{2\sqrt{Mm}}$ is equivalent to (2.8).
Thus,

$$\left\| \frac{A+B}{2}(A\sharp B)^{-1}\right\| \le \frac{M+m}{2\sqrt{Mm}}.$$

If $\frac{M+m}{2}\le A,B\le M$, we have

$$A+\frac{M+m}{2}MA^{-1}\le \frac{M+m}{2}+M \tag{2.9}$$

and

$$B+\frac{M+m}{2}MB^{-1}\le \frac{M+m}{2}+M. \tag{2.10}$$

Similarly, we get

$$\left\| \frac{A+B}{2}(A\sharp B)^{-1}\right\| \le \frac{(\frac{M+m}{2}+M)^2}{4\frac{M+m}{2}M}\le \frac{(\frac{M+m}{2}+m)^2}{4\frac{M+m}{2}m}\le \frac{M+m}{2\sqrt{Mm}}.$$

If $m\le A\le \frac{M+m}{2}\le B\le M$, we have

$$\begin{aligned}\left\| \frac{A+B}{2}\frac{M+m}{2}\sqrt{Mm}(A\sharp B)^{-1}\right\| &\le \frac{1}{4}\left\| \frac{A+B}{2}+\frac{M+m}{2}\sqrt{Mm}(A\sharp B)^{-1}\right\|^2 \quad \bigl(\text{by (2.1)}\bigr)\\ &= \frac{1}{4}\left\| \frac{A+B}{2}+\frac{M+m}{2}\bigl[\bigl(mA^{-1}\bigr)\sharp\bigl(MB^{-1}\bigr)\bigr]\right\|^2\\ &\le \frac{1}{4}\left\| \frac{A+B}{2}+\frac{M+m}{2}\frac{mA^{-1}+MB^{-1}}{2}\right\|^2\\ &\le \frac{1}{4}(M+m)^2 \quad \bigl(\text{by (2.6), (2.10)}\bigr).\end{aligned}$$

That is,

$$\left\| \frac{A+B}{2}(A \sharp B)^{-1} \right\| \leq \frac{(M+m)^2}{4\frac{M+m}{2}\sqrt{Mm}} = \frac{M+m}{2\sqrt{Mm}}.$$

If $m \leq B \leq \frac{M+m}{2} \leq A \leq M$, similarly, by (2.1), (2.7) and (2.9), we have

$$\left\| \frac{A+B}{2}(A \sharp B)^{-1} \right\| \leq \frac{M+m}{2\sqrt{Mm}}.$$

This completes the proof. □

Theorem 3 *Let* Φ *be a positive unital linear map. If* $0 < m \leq A, B \leq M$ *and* $\sqrt{\frac{M}{m}} \leq 2.314$ *for some scalars* $m \leq M$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \frac{(M+m)^2}{4Mm}\Phi^2(A \sharp B) \tag{2.11}$$

and

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \frac{(M+m)^2}{4Mm}\big(\Phi(A) \sharp \Phi(B)\big)^2. \tag{2.12}$$

Proof Inequality (2.11) is equivalent to

$$\left\| \Phi\left(\frac{A+B}{2}\right)\Phi^{-1}(A \sharp B) \right\| \leq \frac{M+m}{2\sqrt{Mm}}.$$

If $0 < m \leq A, B \leq \frac{M+m}{2}$, compute

$$\begin{aligned}
&\left\| \Phi\left(\frac{A+B}{2}\right)\frac{M+m}{2}m\Phi^{-1}(A \sharp B) \right\| \\
&\quad \leq \frac{1}{4}\left\| \Phi\left(\frac{A+B}{2}\right) + \frac{M+m}{2}m\Phi^{-1}(A \sharp B) \right\|^2 \quad \big(\text{by (2.1)}\big) \\
&\quad \leq \frac{1}{4}\left\| \Phi\left(\frac{A+B}{2}\right) + \frac{M+m}{2}m\Phi\big((A \sharp B)^{-1}\big) \right\|^2 \quad \big(\text{by (2.2)}\big) \\
&\quad = \frac{1}{4}\left\| \Phi\left(\frac{A+B}{2} + \frac{M+m}{2}m(A \sharp B)^{-1}\right) \right\|^2 \\
&\quad \leq \frac{1}{4}\left\| \Phi\left(\frac{A+B}{2} + \frac{M+m}{2}m\frac{A^{-1}+B^{-1}}{2}\right) \right\|^2 \\
&\quad \leq \frac{1}{4}\left(\frac{M+m}{2} + m\right)^2 \quad \big(\text{by (2.6), (2.7)}\big).
\end{aligned}$$

By $1 \leq \sqrt{\frac{M}{m}} \leq 2.314$ and (2.8), we have

$$\left\| \Phi\left(\frac{A+B}{2}\right)\Phi^{-1}(A \sharp B) \right\| \leq \frac{M+m}{2\sqrt{Mm}}.$$

If $0 < \frac{M+m}{2} \le A, B \le M$, similarly, by (2.1), (2.2), (2.8), (2.9), (2.10) and $\frac{(\frac{M+m}{2}+M)^2}{M} \le \frac{(\frac{M+m}{2}+m)^2}{m}$, we have

$$\left\|\Phi\left(\frac{A+B}{2}\right)\Phi^{-1}(A \sharp B)\right\| \le \frac{M+m}{2\sqrt{Mm}}.$$

If $m \le A \le \frac{M+m}{2} \le B \le M$, we have

$$\begin{aligned}
&\left\|\Phi\left(\frac{A+B}{2}\right)\frac{M+m}{2}\sqrt{Mm}\Phi^{-1}(A \sharp B)\right\| \\
&\quad\le \frac{1}{4}\left\|\Phi\left(\frac{A+B}{2}\right)+\frac{M+m}{2}\sqrt{Mm}\Phi^{-1}(A \sharp B)\right\|^2 \quad \big(\text{by (2.1)}\big) \\
&\quad\le \frac{1}{4}\left\|\Phi\left(\frac{A+B}{2}\right)+\frac{M+m}{2}\sqrt{Mm}\Phi\big((A \sharp B)^{-1}\big)\right\|^2 \quad \big(\text{by (2.2)}\big) \\
&\quad= \frac{1}{4}\left\|\Phi\left(\frac{A+B}{2}+\frac{M+m}{2}\sqrt{Mm}(A \sharp B)^{-1}\right)\right\|^2 \\
&\quad\le \frac{1}{4}\left\|\Phi\left(\frac{A+B}{2}+\frac{M+m}{2}\big(mA^{-1} \sharp MB^{-1}\big)\right)\right\|^2 \\
&\quad\le \frac{1}{4}\left\|\Phi\left(\frac{A+B}{2}+\frac{M+m}{2}\frac{mA^{-1}+MB^{-1}}{2}\right)\right\|^2 \\
&\quad\le \frac{1}{4}(M+m)^2 \quad \big(\text{by (2.6), (2.10)}\big).
\end{aligned}$$

That is,

$$\left\|\Phi\left(\frac{A+B}{2}\right)\Phi^{-1}(A \sharp B)\right\| \le \frac{M+m}{2\sqrt{Mm}}.$$

If $m \le B \le \frac{M+m}{2} \le A \le M$, similarly, by (2.1), (2.2), (2.7), (2.9), we have

$$\left\|\Phi\left(\frac{A+B}{2}\right)\Phi^{-1}(A \sharp B)\right\| \le \frac{M+m}{2\sqrt{Mm}}.$$

Thus (2.11) holds.

A and B are replaced by $\Phi(A)$ and $\Phi(B)$ in (2.4), respectively, we get (2.12).
This completes the proof. □

Remark 2 Since $0 < m \le M$, then $\frac{(M+m)^2}{4Mm} \le [\frac{(M+m)^2}{4Mm}]^2$. Thus (2.11) and (2.12) are refinements of (1.6) and (1.7), respectively, when $\sqrt{\frac{M}{m}} \le 2.314$.

By (1.2) and Theorem 3, we know that Lin's conjecture (1.8) and (1.9) hold when $\sqrt{\frac{M}{m}} \le 2.314$.

Corollary 1 *Let Φ be a positive unital linear map. If $0 < m \le A, B \le M$ and $\sqrt{\frac{M}{m}} \le 2.314$ for some scalars $m \le M$, then*

$$\Phi^2\left(\frac{A+B}{2}\right) \le S^2(h)\Phi^2(A \sharp B)$$

and

$$\Phi^2\left(\frac{A+B}{2}\right) \le S^2(h)\big(\Phi(A)\sharp\Phi(B)\big)^2,$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$, $h = \frac{M}{m}$.

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Competing interests
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Authors' contributions
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