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Monotonicity, convexity, and inequalities for the generalized elliptic integrals

Tiren Huang^{1*}, Shenyang Tan² and Xiaohui Zhang¹

*Correspondence: huangtiren@163.com
¹Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou, 310018, China
Full list of author information is available at the end of the article

Abstract

We provide the monotonicity and convexity properties and sharp bounds for the generalized elliptic integrals $\mathcal{K}_a(r)$ and $\mathcal{E}_a(r)$ depending on a parameter $a \in (0, 1)$, which contains an earlier result in the particular case $a = 1/2$.

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1 Introduction

For real numbers a , b , and c with $c \neq 0, -1, -2, \dots$, the Gaussian hypergeometric function is defined by

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!} \quad (1.1)$$

for $x \in (-1, 1)$, where (a, n) denotes the shifted factorial function $(a, n) \equiv a(a+1) \cdots (a+n-1)$, $n = 1, 2, \dots$, and $(a, 0) = 1$ for $a \neq 0$. It is well known that the function $F(a, b; c; x)$ has many important applications in geometric function theory, theory of mean values, and several other contexts, and many classes of elementary functions and special functions in mathematical physics are particular or limiting cases of this function [1–10].

In what follows, we suppose $r \in (0, 1)$, $a \in (0, 1)$, and $r' = \sqrt{1-r^2}$. The generalized elliptic integrals of the first and second kinds are defined as

$$\mathcal{K}_a(r) = \frac{\pi}{2} F(a, 1-a; 1; r^2), \quad \mathcal{K}'_a(r) = \mathcal{K}_a(r'), \quad (1.2)$$

$$\mathcal{E}_a(r) = \frac{\pi}{2} F(a-1, 1-a; 1; r^2), \quad \mathcal{E}'_a(r) = \mathcal{E}_a(r'). \quad (1.3)$$

In the particular case $a = 1/2$, the generalized elliptic integrals $\mathcal{K}_a(r)$ and $\mathcal{E}_a(r)$ reduce to the complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$, respectively. Recently, the Gaussian hypergeometric function and generalized elliptic integrals have been the subject of intensive research [2, 3, 5, 8, 11–30].

Anderson, Qiu, and Vamanamurthy [31] considered the monotonicity and convexity of the function

$$f(r) = \frac{\mathcal{E}(r) - r'^2 \mathcal{K}(r)}{r^2} \cdot \frac{r'^2}{\mathcal{E}'(r) - r'^2 \mathcal{K}'(r)}.$$

One of the main results of [31] is the following theorem.

Theorem 1.1 *The function $f(r)$ is increasing and convex from $(0, 1)$ onto $(\pi/4, 4/\pi)$. In particular,*

$$\frac{\pi}{4} < f(r) < \frac{\pi}{4} + \left(\frac{4}{\pi} - \frac{\pi}{4} \right) r \tag{1.4}$$

for $r \in (0, 1)$. Both inequalities given in (1.4) are sharp as $r \rightarrow 0$, whereas the second inequality is also sharp as $r \rightarrow 1$.

Alzer and Richards [32] studied the corresponding properties of the additive counterpart

$$\Delta(r) = \frac{\mathcal{E}(r) - r'^2 \mathcal{K}(r)}{r^2} - \frac{\mathcal{E}'(r) - r'^2 \mathcal{K}'(r)}{r'^2}$$

and obtained the following theorem.

Theorem 1.2 *The function $\Delta(r)$ is strictly increasing and strictly convex from $(0, 1)$ onto $(\pi/4 - 1, 1 - \pi/4)$. Moreover, for all $r \in (0, 1)$, we have*

$$\frac{\pi}{4} - 1 + \alpha r < \Delta(r) < \frac{\pi}{4} - 1 + \beta r \tag{1.5}$$

with the best constants $\alpha = 0$ and $\beta = 2 - \frac{\pi}{2}$.

It is natural to extend Theorems 1.1 and 1.2 to the generalized elliptic integrals $\mathcal{K}_a(r)$ and $\mathcal{E}_a(r)$. In this paper, we show the monotonicity and convexity of the functions

$$f_a(r) = \frac{\mathcal{E}_a(r) - r'^2 \mathcal{K}_a(r)}{r^2} \cdot \frac{r'^2}{\mathcal{E}'_a(r) - r'^2 \mathcal{K}'_a(r)} \tag{1.6}$$

and

$$g_a(r) = \frac{\mathcal{E}_a(r) - r'^2 \mathcal{K}_a(r)}{r^2} - \frac{\mathcal{E}'_a(r) - r'^2 \mathcal{K}'_a(r)}{r'^2}. \tag{1.7}$$

Moreover, we obtain sharp inequalities for them. If $a = 1/2$, then our results return to Theorems 1.1 and 1.2, which are contained in [31] and [32].

2 Preliminaries and lemmas

In this section, we give several formulas and lemmas to establish our main results stated in Section 1. First, let us recall some known results for $F(a, b; c; x)$.

The following formulas for the hypergeometric function can be found in the literature [33–35]:

$$F(a, b; a + b + 1; x) = (1 - x)F(a + 1, b + 1; a + b + 1; x), \tag{2.1}$$

the differential formula

$$\frac{dF(a, b; c; x)}{dx} = \frac{ab}{c}F(a + 1, b + 1; c + 1; x), \tag{2.2}$$

the asymptotic limit

$$\lim_{x \rightarrow 1^-} F(a, b; c; x) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad c > a + b, \tag{2.3}$$

and the contiguous relation

$$(\sigma - \rho)F(\alpha, \rho; \sigma + 1; z) = \sigma F(\alpha, \rho; \sigma; z) - \rho F(\alpha, \rho + 1; \sigma + 1; z), \tag{2.4}$$

where $\Gamma(x)$ is the Euler gamma function.

Lemma 2.1 ([2], Lemma 5.2) *Let $a \in (0, 1]$. Then the function $[\mathcal{E}_a(r) - r^2\mathcal{K}_a(r)]/r^2$ is increasing and convex from $(0, 1)$ onto $(\pi a/2, [\sin(\pi a)]/[2(1 - a)])$.*

The following formulas were presented in [2]:

$$\frac{d\mathcal{K}_a(r)}{dr} = \frac{2(1 - a)(\mathcal{E}_a(r) - r^2\mathcal{K}_a(r))}{r^2}, \quad \frac{d\mathcal{E}_a(r)}{dr} = \frac{2(1 - a)(\mathcal{K}_a(r) - \mathcal{E}_a(r))}{r}, \tag{2.5}$$

$$\frac{d(\mathcal{E}_a(r) - r^2\mathcal{K}_a(r))}{dr} = 2ar\mathcal{K}_a(r), \quad \frac{d(\mathcal{K}_a(r) - \mathcal{E}_a(r))}{dr} = \frac{2(1 - a)r\mathcal{E}_a(r)}{r^2}. \tag{2.6}$$

Lemma 2.2 ([2], Lemma 2.3) *Let $I \subset \mathbb{R}$ be an interval, and let $f, g : I \rightarrow (0, \infty)$. If both f, g are convex and increasing (decreasing), then the product $f \cdot g$ is convex.*

The following lemma follows from Theorem 1.7 in [1].

Lemma 2.3 *For all $a, b \in (0, \infty)$, the function*

$$I(x) = (1 - x)^4F(a, b; a + b; x) \tag{2.7}$$

is a strictly decreasing automorphism of $(0, 1)$ if and only if $4ab \leq a + b$.

Lemma 2.4 *The function*

$$J(r) = \frac{(1 - r^2)(\mathcal{E}_a(r) - ((4a - 1)r^2 + 1)\mathcal{K}_a(r))}{4a(1 - a)r^3} \tag{2.8}$$

is increasing from $(0, 1)$ onto $(-\infty, 0)$.

Proof Let

$$f_1(x) = (1-x)(F(a-1, 1-a; 1; x) - ((4a-1)x+1)F(a, 1-a; 1; x)).$$

By the series expansion for $F(a, b; c; x)$ we have

$$\begin{aligned} f_1(x) &= (1-x) \left(\sum_{n=0}^{\infty} \frac{(a-1, n)(1-a, n)}{n!} \cdot \frac{x^n}{n!} - ((4a-1)x+1) \sum_{n=0}^{\infty} \frac{(a, n)(1-a, n)}{n!} \cdot \frac{x^n}{n!} \right) \\ &= (1-x) \left(\sum_{n=0}^{\infty} \left(\frac{(a-1, n)(1-a, n)}{n!} - \frac{(a, n)(1-a, n)}{n!} \right) \cdot \frac{x^n}{n!} \right. \\ &\quad \left. - \sum_{n=0}^{\infty} \frac{(4a-1)(a, n)(1-a, n)}{n!} \cdot \frac{x^{n+1}}{n!} \right) \\ &= (1-x) \sum_{n=1}^{\infty} \frac{(a, n-1)(1-a, n-1)}{n!n!} (-4an^2 + an)x^n \\ &= \sum_{n=1}^{\infty} \frac{(a, n-1)(1-a, n-1)}{n!n!} (-4an^2 + an)x^n \\ &\quad - \sum_{n=1}^{\infty} \frac{(a, n-1)(1-a, n-1)}{n!n!} (-4an^2 + an)x^{n+1} \\ &= -3ax \\ &\quad + \sum_{n=2}^{\infty} \frac{(a, n-2)(1-a, n-2)}{n!n!} an((4n+4a^2-4a-6)n+a+2-a^2)x^n. \end{aligned} \tag{2.9}$$

By the definition of the generalized elliptic integrals of the first and second kinds (1.2) we have

$$\begin{aligned} J(r) &= \frac{\frac{\pi}{2}f_1(r^2)}{4a(1-a)r^3} \\ &= \frac{\pi}{8a(1-a)} \left(-\frac{3a}{r} \right. \\ &\quad \left. + \sum_{n=2}^{\infty} \frac{(a, n-2)(1-a, n-2)}{n!n!} an((4n+4a^2-4a-6)n+a+2-a^2)r^{2n-3} \right). \end{aligned}$$

Since $0 < a < 1, n \geq 2$, we have $(4n+4a^2-4a-6)n+a+2-a^2 > 0$, and hence $J(r)$ is an increasing function on $(0, 1)$. From this formula it is easy to see that $\lim_{r \rightarrow 0^+} J(r) = -\infty$. By Lemma 2.3 we have that $\lim_{r \rightarrow 1^-} J(r) = 0$. □

Lemma 2.5 ([6], Lemma 2.1) *For $-\infty < a < b < \infty$, let $f, g : [a, b] \rightarrow R$ be continuous on $[a, b]$ and differentiable on (a, b) . Let $g'(x) \neq 0$ on (a, b) . If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b) , then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

3 Main results and proofs

In this section, we present and prove two main theorems.

Theorem 3.1 *The function $f_a(r)$ in (1.6) is increasing and convex from $(0, 1)$ onto $(\frac{\pi a(1-a)}{\sin(\pi a)}, \frac{\sin(\pi a)}{\pi a(1-a)})$. In particular,*

$$\frac{\pi a(1-a)}{\sin(\pi a)} + \alpha r < f_a(r) < \frac{\pi a(1-a)}{\sin(\pi a)} + \left(\frac{\sin(\pi a)}{\pi a(1-a)} - \frac{\pi a(1-a)}{\sin(\pi a)} \right) r \tag{3.1}$$

for $r \in (0, 1)$ with the best constant $\alpha = 0$, $\beta = \frac{\sin(\pi a)}{\pi a(1-a)} - \frac{\pi a(1-a)}{\sin(\pi a)}$. These two inequalities are sharp as $r \rightarrow 0$, whereas the second inequality is sharp as $r \rightarrow 1$.

Proof Let

$$f_a^1(r) = \frac{\mathcal{E}_a(r) - r'^2 \mathcal{K}_a(r)}{r^2}.$$

Then

$$f_a(r) = \frac{\mathcal{E}_a(r) - r'^2 \mathcal{K}_a(r)}{r^2} \cdot \frac{r'^2}{\mathcal{E}'_a(r) - r'^2 \mathcal{K}'_a(r)} = f_a^1(r) \cdot \frac{1}{f_a^1(r')}.$$

By Lemma 2.1, $f_a^1(r)$, $1/f_a^1(r')$ are positive increasing functions on $(0, 1)$, and hence $f_a(r)$ is also an increasing function on $(0, 1)$. Since $f_a^1(r)$ is a convex function by Lemma 2.1, the desired convexity of $f_a(r)$ will follow from Lemma 2.2 if we prove that $1/f_a^1(r')$ is a convex function on $(0, 1)$.

According to (2.6), we have

$$\left(\frac{1}{f_a^1(r)} \right)' = \left(\frac{r^2}{\mathcal{E}_a(r) - r'^2 \mathcal{K}_a(r)} \right)' = \frac{g_1(r)}{g_2(r)},$$

where

$$g_1(r) = 2(\mathcal{E}_a(r) - \mathcal{K}_a(r) + (1-a)r^2 \mathcal{K}_a(r)), \quad g_2(r) = \frac{(\mathcal{E}_a(r) - r'^2 \mathcal{K}_a(r))^2}{r}.$$

Obviously, $g_1(0^+) = 0$. By Lemma 2.1 we get $g_2(0^+) = 0$. Moreover,

$$\frac{g'_1(r)}{g'_2(r)} = \frac{4a(1-a)r^3}{(1-r^2)(\mathcal{E}_a(r) - ((4a-1)r^2 + 1)\mathcal{K}_a(r))} = \frac{1}{J(r)},$$

where $J(r)$ is defined by (2.8). Hence, by Lemma 2.4 and Lemma 2.5, $(1/f_a^1(r))'$ is decreasing, so that $(1/f_a^1(r'))'$ is increasing, and $1/f_a^1(r')$ is convex on $(0, 1)$. □

Theorem 3.2 *The function $g_a(r)$ in (1.7) is strictly increasing and strictly convex from $(0, 1)$ onto $(\frac{\pi a}{2} - \frac{\sin(\pi a)}{2(1-a)}, \frac{\sin(\pi a)}{2(1-a)} - \frac{\pi a}{2})$. Moreover, for all $r \in (0, 1)$, we have*

$$\frac{\pi a}{2} - \frac{\sin(\pi a)}{2(1-a)} + \alpha r < g_a(r) < \frac{\pi a}{2} - \frac{\sin(\pi a)}{2(1-a)} + \beta r \tag{3.2}$$

with the best constants $\alpha = 0$ and $\beta = \frac{\sin(\pi a)}{1-a} - \pi a$. These two inequalities are sharp as $r \rightarrow 0$, whereas the second inequality is sharp as $r \rightarrow 1$.

Proof Let

$$M_a(r) = \frac{\pi}{2r^2} (F(a - 1, 1 - a; 1; r^2) - r^2 F(a, 1 - a; 1; r^2)).$$

By the series expansion for $F(a, b; c; x)$ we obtain

$$M_a(r) = \frac{a\pi}{2} F(a, 1 - a; 2; r^2). \tag{3.3}$$

Then

$$g_a(r) = M_a(r) - M_a(r') = \frac{a\pi}{2} (F(a, 1 - a; 2; r^2) - F(a, 1 - a; 2; 1 - r^2)). \tag{3.4}$$

Using the differentiation formula (2.2), we have

$$g'_a(r) = \frac{a^2(1 - a)\pi}{2} r (F(a + 1, 2 - a; 3; r^2) + F(a + 1, 2 - a; 3; 1 - r^2)), \tag{3.5}$$

$$g''_a(r) = \frac{a^2(1 - a)\pi}{2} \left(F(a + 1, 2 - a; 3; r^2) + F(a + 1, 2 - a; 3; 1 - r^2) + \frac{2(a + 1)(2 - a)}{3} r^2 (F(a + 2, 3 - a; 4; r^2) - F(a + 2, 3 - a; 4; 1 - r^2)) \right). \tag{3.6}$$

By formula (2.1), we get

$$F(a + 2, 3 - a; 4; 1 - r^2) = \frac{1}{r^2} F(a + 1, 2 - a; 4; 1 - r^2). \tag{3.7}$$

Using the contiguous relation (2.4), we take $\alpha = a + 1$, $\rho = 2 - a$, $\sigma = 3$, and $z = 1 - r^2$ and obtain

$$(a + 1)F(a + 1, 2 - a; 4; 1 - r^2) = 3F(a + 1, 2 - a; 3; 1 - r^2) - (2 - a)F(a + 1, 3 - a; 4; 1 - r^2).$$

Hence, it follows from (3.6), (3.7), and the last formula that

$$\begin{aligned} & \frac{2}{a^2(1 - a)\pi} g''_a(r) \\ &= F(a + 1, 2 - a; 3; r^2) + F(a + 1, 2 - a; 3; 1 - r^2) \\ & \quad + \frac{2(a + 1)(2 - a)}{3} r^2 \left(F(a + 2, 3 - a; 4; r^2) - \frac{F(a + 2, 3 - a; 4; 1 - r^2)}{r^2} \right) \\ &= F(a + 1, 2 - a; 3; r^2) + \frac{2(a + 1)(2 - a)r^2}{3} F(a + 2, 3 - a; 4; r^2) \\ & \quad + (2a - 3)F(a + 1, 2 - a; 3; 1 - r^2) + \frac{2(2 - a)^2}{3} F(a + 1, 3 - a; 4; 1 - r^2) \\ &> 1 + (2a - 3)F(a + 1, 2 - a; 3; 1 - r^2) + \frac{2(2 - a)^2}{3} F(a + 1, 3 - a; 4; 1 - r^2). \end{aligned}$$

By the series expansion for $F(a, b; c; x)$ we have

$$\begin{aligned}
 & (2a - 3)F(a + 1, 2 - a; 3; 1 - r^2) + \frac{2(2 - a)^2}{3}F(a + 1, 3 - a; 4; 1 - r^2) \\
 &= \sum_{n=0}^{\infty} \left(\frac{2(a + 1, n)(3 - a, n)(2 - a)^2}{(3, n + 1)} + \frac{(a + 1, n)(2 - a, n)(2a - 3)}{(3, n)} \right) \frac{(1 - r^2)^n}{n!} \\
 &= \sum_{n=0}^{\infty} \frac{(2 - a, n)(n + 2a^2 - 2a - 1)(a + 1, n)}{(3, n + 1)} \frac{(1 - r^2)^n}{n!} \\
 &> \frac{2a^2 - 2a - 1}{3} + \frac{a(2 - a)(a^2 - 1)}{6}(1 - r^2). \tag{3.8}
 \end{aligned}$$

Hence

$$\frac{2}{a^2(1 - a)\pi}g''_a(r) > 1 + \frac{2a^2 - 2a - 1}{3} + \frac{a(2 - a)(a^2 - 1)}{6}(1 - r^2). \tag{3.9}$$

Through direct calculation we have

$$\begin{aligned}
 & 1 + \frac{2a^2 - 2a - 1}{3} + \frac{a(2 - a)(a^2 - 1)}{6} \\
 &= \frac{-a^4 + 2a^3 + 5a^2 - 6a + 4}{6} > 0, \quad \forall a \in (0, 1). \tag{3.10}
 \end{aligned}$$

Then we get $g''_a(r) > 0$. Thus $g_a(r)$ is strictly convex on $(0, 1)$. According to (3.3) and (2.3), we have

$$M_a(0) = \frac{a\pi}{2}, \quad M'_a(0) = 0, \quad \lim_{r \rightarrow 1^-} M_a(r) = \frac{\sin(\pi a)}{2(1 - a)}. \tag{3.11}$$

Applying Lemma 2.3 and (2.6), we have

$$g'_a(0) = \lim_{r \rightarrow 0} \frac{M_a(r) - M_a(0)}{r} - \frac{M_a(r') - M_a(1)}{r} = \lim_{x \rightarrow 1} \frac{x'}{x^4} ((2 - x^2)\mathcal{K}_a(x) - 2\mathcal{E}_a(x)) = 0.$$

Because of $g''_a(r) > 0$, $g'_a(r)$ is increasing on $(0, 1)$, and $g'_a(0) = 0$. Then the monotonicity of $g_a(r)$ on $(0, 1)$ is obtained. It follows from the convexity of $g_a(r)$ that, for $x \in (0, 1)$,

$$\frac{\pi a}{2} - \frac{\sin(\pi a)}{2(1 - a)} < g_a(r) < \frac{\pi a}{2} - \frac{\sin(\pi a)}{2(1 - a)} + \left(\frac{\sin(\pi a)}{1 - a} - \pi a \right) r. \tag{3.12}$$

□

Corollary 3.3 *Let*

$$L_a(p, q) = g_a(pq) - g_a(p) - g_a(q). \tag{3.13}$$

Then we have

$$\frac{\pi a}{2} - \frac{\sin(\pi a)}{2(1 - a)} < L_a(p, q) < \frac{\sin(\pi a)}{2(1 - a)} - \frac{a\pi}{2} \tag{3.14}$$

for all $p, q \in (0, 1)$.

Proof By direct calculation we obtain

$$\frac{\partial}{\partial p} L_a(p, q) = sg'_a(pq) - g'_a(p), \quad \frac{\partial^2}{\partial p \partial q} L_a(p, q) = g'_a(pq) + pqg''_a(pq).$$

Considering the positivity of g'_a and g''_a on $(0, 1)$, we have

$$\frac{\partial^2}{\partial p \partial q} L_a(p, q) > 0,$$

This means that $\frac{\partial}{\partial p} L_a(p, q)$ is strictly increasing with respect to q . So we have

$$\frac{\partial}{\partial p} L_a(p, q) < \frac{\partial}{\partial p} L_a(p, q) \Big|_{q=1} = 0. \tag{3.15}$$

Then the monotonicity of $L_a(p, q)$ with respect to p is obtained, which leads to

$$\frac{\pi a}{2} - \frac{\sin(\pi a)}{2(1-a)} < L_a(p, q) < \frac{\sin(\pi a)}{2(1-a)} - \frac{a\pi}{2}. \quad \square$$

Remark 3.4 Taking $a = 1/2$ in Theorems 3.1 and 3.2, we get Theorems 1.1 and 1.2.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

Author details

¹Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou, 310018, China. ²Taizhou Institute of Sci. & Tech., NUST, Taizhou, 225300, China.

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References

1. Anderson, GD, Barnard, RW, Richards, KC, Vamanamurthy, MK, Vuorinen, M: Inequalities for zero-balanced hypergeometric functions. *Trans. Am. Math. Soc.* **347**, 1713-1723 (1995)
2. Anderson, GD, Qiu, SL, Vamanamurthy, MK, Vuorinen, M: Generalized elliptic integrals and modular equations. *Pac. J. Math.* **192**, 1-37 (2000)
3. Baricz, Á: Turán type inequalities for generalized complete elliptic integrals. *Math. Z.* **256**, 895-911 (2007)
4. Barnard, RW, Pearce, K, Richards, KC: An inequality involving the generalized hypergeometric function and the arc length of an ellipse. *SIAM J. Math. Anal.* **31**, 693-699 (2000)
5. Neuman, E: Inequalities and bounds for generalized complete elliptic integrals. *J. Math. Anal. Appl.* **373**, 203-213 (2011)
6. Ponnusamy, S, Vuorinen, M: Asymptotic expansions and inequalities for hypergeometric functions. *Mathematika* **44**, 278-301 (1997)
7. Ponnusamy, S, Vuorinen, M: Univalence and convexity properties for Gaussian hypergeometric functions. *Rocky Mt. J. Math.* **31**, 327-353 (2001)
8. Yang, ZH, Chu, YM, Wang, M-K: Monotonicity criterion for the quotient of power series with applications. *J. Math. Anal. Appl.* **428**, 587-604 (2015)
9. Sun, MB, Yang, XP: Inequalities for the weighted mean of r -convex functions. *Proc. Am. Math. Soc.* **133**(6), 1639-1646 (2005)

10. Sun, MB, Yang, XP: Generalized Hadamard's inequality and r -convex functions in Carnot groups. *J. Math. Anal. Appl.* **294**(2), 387-398 (2004)
11. Qiu, SL, Vuorinen, M: Duplication inequalities for the ratios of hypergeometric functions. *Forum Math.* **12**, 109-133 (2000)
12. Heikkala, V, Linden, H, Vamanamurthy, MK, Vuorinen, M: Generalized elliptic integrals and the Legendre M-function. *J. Math. Anal. Appl.* **338**(1), 223-243 (2008)
13. Wang, MK, Chu, YM, Qiu, SL: Sharp bounds for generalized elliptic integrals of the first kind. *J. Math. Anal. Appl.* **429**, 744-757 (2015)
14. Chu, YM, Wang, MK, Qiu, YF: On Alzer and Qiu's conjecture for complete elliptic integral and inverse hyperbolic tangent function. *Abstr. Appl. Anal.* **2011**, Article ID 697547 (2011)
15. Chu, YM, Wang, MK, Qiu, SL, Jiang, YP: Bounds for complete elliptic integrals of the second kind with applications. *Comput. Math. Appl.* **63**(7), 1177-1184 (2012)
16. Chu, YM, Wang, MK, Qiu, YF: Hölder mean inequalities for the complete elliptic integrals. *Integral Transforms Spec. Funct.* **23**(7), 521-527 (2012)
17. Song, YQ, Zhou, PG, Chu, YM: Inequalities for the Gaussian hypergeometric function. *Sci. China Math.* **57**(11), 2369-2380 (2014)
18. Wang, GD, Zhang, XH, Chu, YM: Inequalities for the generalized elliptic integrals and modular functions. *J. Math. Anal. Appl.* **331**(2), 1275-1283 (2007)
19. Wang, GD, Zhang, XH, Chu, YM: A power mean inequality involving the complete elliptic integrals. *Rocky Mt. J. Math.* **44**(5), 1661-1667 (2014)
20. Wang, MK, Li, YM, Chu, YM: Inequalities and infinite product formula for Ramanujan generalized modular equation function. *Ramanujan J.* (2017). doi:10.1007/s11139-017-9888-3
21. Wang, MK, Chu, YM: Asymptotical bounds for complete elliptic integrals of the second kind. *J. Math. Anal. Appl.* **402**(1), 119-126 (2013)
22. Wang, MK, Chu, YM, Qiu, YF, Qiu, SL: An optimal power mean inequality for the complete elliptic integrals. *Appl. Math. Lett.* **24**(6), 887-890 (2011)
23. Wang, MK, Chu, YM, Jiang, YP: Ramanujan's cubic transformation inequalities for zero-balanced hypergeometric functions. *Rocky Mt. J. Math.* **46**(2), 679-691 (2016)
24. Wang, MK, Chu, YM: Refinements of transformation inequalities for zero-balanced hypergeometric functions. *Acta Math. Sci.* **37B**(3), 607-622 (2017)
25. Wang, MK, Chu, YM, Song, YQ: Asymptotical formulas for Gaussian and generalized hypergeometric functions. *Appl. Math. Comput.* **276**, 44-60 (2016)
26. Wang, MK, Qiu, SL, Chu, YM, Jiang, YP: Generalized Hersch-Pfluger distortion function and complete elliptic integrals. *J. Math. Anal. Appl.* **385**(1), 221-229 (2012)
27. Wang, MK, Qiu, SL, Chu, YM, Jiang, YP: Convexity of the complete elliptic integrals of the first kind with respect to Hölder means. *J. Math. Anal. Appl.* **388**(2), 1141-1146 (2012)
28. Yang, ZH, Chu, YM: A monotonicity property involving the generalized elliptic integral of the first kind. *Math. Inequal. Appl.* **20**(3), 729-735 (2017)
29. Zhang, XH, Wang, DG, Chu, YM: Convexity with respect to Hölder mean involving zero-balanced hypergeometric functions. *J. Math. Anal. Appl.* **353**(1), 256-259 (2009)
30. Zhang, XH, Wang, DG, Chu, YM: Remarks on generalized elliptic integrals. *Proc. R. Soc. Edinb. A* **139**(2), 417-426 (2009)
31. Anderson, GD, Qiu, SL, Vamanamurthy, MK: Elliptic integral inequalities, with applications. *Constr. Approx.* **14**, 195-207 (1998)
32. Alzer, H, Richards, K: A note on a function involving complete elliptic integrals: monotonicity, convexity, inequalities. *Anal. Math.* **41**(3), 133-139 (2015)
33. Abramowitz, M, Stegun, IA: *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*. Dover, New York (1966)
34. Prudnikov, AP, Brychkov, YA, Marichev, OI: *Integrals and Series, vol. 3*. Gordon & Breach, Amsterdam (1990)
35. Qiu, SL, Vuorinen, M: Special functions in geometric function theory. In: Kühnau, R (ed.) *Handbook of Complex Analysis: Geometric Function Theory, vol. 2*, pp. 621-659. Elsevier, Amsterdam (2005)

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