He et al. *Journal of Inequalities and Applications* (2017) 2017:260
DOI 10.1186/s13660-017-1536-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some new sharp bounds for the spectral radius of a nonnegative matrix and its application

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Abstract

In this paper, we give some new sharp upper and lower bounds for the spectral radius of a nonnegative irreducible matrix. Using these bounds, we obtain some new and improved bounds for the signless Laplacian spectral radius of a graph or a digraph.

MSC: 05C50; 05C35; 05C20; 15A18

Keywords: nonnegative matrix; graph; digraph; spectral radius

1 Introduction

Let $G=(V,E)$ be a graph with vertex set $V(G)=\{v_1,\ldots,v_n\}$ and edge set $E(G)$. Let $N=\{1,\ldots,n\}$, for $i\in N$. We assume that d_i is the degree of vertex v_i. Let $D(G)=\operatorname{diag}(d_1,d_2,\ldots,d_n)$ be the degree diagonal matrix of the graph G and $A(G)=(a_{ij})$ be the adjacency matrix of the graph G. Then the matrix $Q(G)=D(G)+A(G)$ is called the signless Laplacian matrix of the graph G. The largest modulus of eigenvalues of $Q(G)$ is denoted by $\rho(G)$, which is also called the signless Laplacian spectral radius of G.

Let $\overrightarrow{G}=(V,E)$ be a digraph with vertex set $V(\overrightarrow{G})=\{v_1,\ldots,v_n\}$ and arc set $E(\overrightarrow{G})$. Let d_i^+ be the out-degree of vertex v_i, $D(\overrightarrow{G})=\operatorname{diag}(d_1^+,d_2^+,\ldots,d_n^+)$ be the out-degree diagonal matrix of the digraph $\overrightarrow{G}$, and $A(\overrightarrow{G})=(a_{ij})$ be the adjacency matrix of the digraph $\overrightarrow{G}$. Then the matrix $Q(\overrightarrow{G})=D(\overrightarrow{G})+A(\overrightarrow{G})$ is called the signless Laplacian matrix of the digraph $\overrightarrow{G}$. The largest modulus of eigenvalues of $Q(\overrightarrow{G})$ is denoted by $\rho(\overrightarrow{G})$, which is also called the signless Laplacian spectral radius of $\overrightarrow{G}$.

In recent decades, there are many bounds on the signless Laplacian spectral radius of a graph (digraph) [1–3]. Let $m_i=\frac{\sum_{i\sim j}d_j}{d_i}$ be the average degree of the neighbours of v_i in G and $m_i^+=\frac{\sum_{i\sim j}d_j^+}{d_i^+}$ be the average out-degree of the out-neighbours of v_i in $\overrightarrow{G}$. In this paper, we assume that the graph (digraph) is simple and connected (strong connected).

In 2013, Maden, Das, and Cevik [4] obtained the following bounds for the signless Laplacian spectral radius of a graph:

$$\rho(G)\le\max_{i\sim j}\left\{\frac{d_i+2d_j-1+\sqrt{(d_i-2d_j+1)^2+4d_i}}{2}\right\}. \tag{1}$$

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In 2016, Xi and Wang [5] obtained the following bounds for the signless Laplacian spectral radius of a digraph:

$$\rho(\overrightarrow{G}) \leq \max_{i\sim j}\left\{\frac{d_i^+ + 2d_j^+ - 1 + \sqrt{(d_i^+ - 2d_j^+ + 1)^2 + 4d_i^+}}{2}\right\}. \tag{2}$$

In this paper, we improve the bounds for the signless Laplacian spectral radius of a graph (digraph) that are given in (1) and (2).

2 Main result

In this section, some upper and lower bounds for the spectral radius of a nonnegative irreducible matrix are given. We need the following lemma.

Lemma 2.1 ([6]) *Let A be a nonnegative matrix with the spectral radius $\rho(A)$ and the row sum $r_1, r_2, \ldots, r_n$. Then $\min_{1\leq i\leq n} r_i \leq \rho(A) \leq \max_{1\leq i\leq n} r_i$. Moreover, if the matrix A is irreducible, then the equalities hold if and only if*

$$r_1 = r_2 = \cdots = r_n.$$

Theorem 2.1 *Let $A = (a_{ij})$ be an irreducible and nonnegative matrix with $a_{ii} = 0$ for all $i \in N$ and the row sum $r_1, r_2, \ldots, r_n$. Let $B = A + M$, where $M = \operatorname{diag}(t_1, t_2, \ldots, t_n)$ with $t_i \geq 0$ for any $i \in N$, $s_i = \sum_{j=1}^n a_{ij} r_j$, $s_{ij} = s_i - a_{ij} r_j$. Let $\rho(B)$ be the spectral radius of B and let*

$$f(i,j) = \frac{t_i + t_j + \frac{s_{ij}}{r_i} + \sqrt{(t_i - t_j + \frac{s_{ij}}{r_i})^2 + \frac{4 s_j a_{ij}}{r_i}}}{2},$$

for any $i, j \in N$. Then

$$\min_{1\leq i\leq n}\max_{\substack{1\leq j\leq n\\ j\neq i}}\bigl\{f(i,j), a_{ij}\neq 0\bigr\} \leq \rho(B) \leq \max_{1\leq i\leq n}\min_{\substack{1\leq j\leq n\\ j\neq i}}\bigl\{f(i,j), a_{ij}\neq 0\bigr\}. \tag{3}$$

Moreover, either of the equalities in (3) *holds if and only if $t_i + \frac{s_i}{r_i} = t_j + \frac{s_j}{r_j}$ for any distinct $i, j \in N$.*

Proof Let $R = \operatorname{diag}(r_1, r_2, \ldots, r_n)$. Since the matrix A is nonnegative irreducible, the matrix $R^{-1}BR$ is also nonnegative and irreducible. By the famous Perron-Frobenius theorem [6], there is a positive eigenvector $x = (x_1, x_2, \ldots, x_n)^T$ corresponding to the spectral radius of $R^{-1}BR$.

Upper bounds: Let $x_p > 0$ be an arbitrary component of x, $x_q = \max\{x_k, 1 \leq k \leq n\}$. Obviously, $p \neq q$, $a_{pq} \neq 0$. By $R^{-1}BRx = \rho(B)x$, we have

$$\rho(B)x_p = t_p x_p + \sum_{k=1,k\neq p}^n \frac{a_{pk} r_k x_k}{r_p} \leq t_p x_p + \frac{x_q}{r_p}\sum_{k=1}^n a_{pk} r_k \leq t_p x_p + \frac{x_q s_p}{r_p}. \tag{4}$$

Similarly, we have

$$\rho(B)x_q = t_q x_q + \sum_{k=1,k\neq q}^n \frac{a_{qk} r_k x_k}{r_q} \leq \left(t_q + \frac{s_q - a_{qp} r_p}{r_q}\right)x_q + \frac{a_{qp} r_p}{r_q} x_p. \tag{5}$$

By (4), (5), and $\rho(B) - t_p > 0$, $\rho(B) - t_q > 0$, we have

$$\bigl(\rho(B) - t_p\bigr)\left(\rho(B) - t_q - \frac{s_q - a_{qp}r_p}{r_q}\right) \le \frac{s_p a_{qp}}{r_q}.$$

Therefore,

$$\rho(B) \le \frac{t_p + t_q + \frac{s_{qp}}{r_q} + \sqrt{(t_p - t_q - \frac{s_{qp}}{r_q})^2 + \frac{4s_p a_{qp}}{r_q}}}{2}. \tag{6}$$

This must be true for every $p \ne q$. Then

$$\rho(B) \le \min_{j\ne q} \frac{t_j + t_q + \frac{s_{qj}}{r_q} + \sqrt{(t_j - t_q - \frac{s_{qj}}{r_q})^2 + \frac{4s_j a_{qj}}{r_q}}}{2}. \tag{7}$$

This must be true for any $q \in N$. Then

$$\rho(B) \le \max_{1\le i\le n} \min_{j\ne i} \left\{ \frac{t_i + t_j + \frac{s_{ij}}{r_i} + \sqrt{(t_i - t_j + \frac{s_{ij}}{r_i})^2 + \frac{4s_j a_{ij}}{r_i}}}{2}, a_{ij} \ne 0 \right\}. \tag{8}$$

Lower bounds: Let $x_p > 0$ be an arbitrary component of x, $x_q = \min\{x_k, 1 \le k \le n\}$. Obviously, $p \ne q$, $a_{pq} \ne 0$. By $R^{-1}BRx = \rho(B)x$, we have

$$\rho(B)x_p = t_p x_p + \sum_{k=1,k\ne p}^{n} \frac{a_{pk} r_k x_k}{r_p} \ge t_p x_p + \frac{x_q}{r_p} \sum_{k=1}^{n} a_{pk} r_k \ge t_p x_p + \frac{x_q s_p}{r_p}. \tag{9}$$

Similarly, we have

$$\rho(B)x_q = t_q x_q + \sum_{k=1,k\ne q}^{n} \frac{a_{qk} r_k x_k}{r_q} \ge \left(t_q + \frac{s_q - a_{qp} r_p}{r_q}\right) x_q + \frac{a_{qp} r_p}{r_q} x_p. \tag{10}$$

By (9), (10), and $\rho(B) - t_p > 0$, $\rho(B) - t_q > 0$, we have

$$\bigl(\rho(B) - t_p\bigr)\left(\rho(B) - t_q - \frac{s_q - a_{qp}r_p}{r_q}\right) \ge \frac{s_p a_{qp}}{r_q}. \tag{11}$$

Therefore,

$$\rho(B) \ge \frac{t_p + t_q + \frac{s_{qp}}{r_q} + \sqrt{(t_p - t_q - \frac{s_{qp}}{r_q})^2 + \frac{4s_p a_{qp}}{r_q}}}{2}. \tag{12}$$

This must be true for every $p \ne q$. Then

$$\rho(B) \ge \max_{j\ne q} \frac{t_j + t_q + \frac{s_{qj}}{r_q} + \sqrt{(t_j - t_q - \frac{s_{qj}}{r_q})^2 + \frac{4s_j a_{qj}}{r_q}}}{2}. \tag{13}$$

This must be true for all $q \in N$. Then

$$\rho(B) \ge \min_{1\le i\le n} \max_{j\ne i} \left\{ \frac{t_i + t_j + \frac{s_{ij}}{r_i} + \sqrt{(t_i - t_j + \frac{s_{ij}}{r_i})^2 + \frac{4s_j a_{ij}}{r_i}}}{2}, a_{ij} \ne 0 \right\}. \tag{14}$$

From (4), (5), and $x_p > 0$ as an arbitrary component of x, we get $x_k = x_q = x_p$ for all k. Then we see easily that the right equality holds in (8) for $t_i + \frac{s_i}{r_i} = t_j + \frac{s_j}{r_j}$ for any distinct $i, j \in N$. The proof of the left equality in (3) is similar to the proof of the right equality, and we omit it here.

Thus, we complete the proof. □

3 Signless Laplacian spectral radius of a graph

In this section, we will apply Theorem 2.1 to obtain some new results on the signless Laplacian spectral radius $\rho(G)$ of a graph.

Theorem 3.1 *Let $G = (V, E)$ be a simple connected graph on n vertices. Then*

$$\begin{aligned}&\min_{1\le i\le n}\max_{i\sim j}\left\{\frac{d_i + 2d_j - 1 + \sqrt{(d_i - 2d_j + 1)^2 + 4d_i}}{2}\right\}\\&\quad\le \rho(G) \le \max_{1\le i\le n}\min_{i\sim j}\left\{\frac{d_i + 2d_j - 1 + \sqrt{(d_i - 2d_j + 1)^2 + 4d_i}}{2}\right\}.\end{aligned} \tag{15}$$

Moreover, one of the equalities in (15) *holds if and only if G is a regular graph.*

Proof We apply Theorem 2.1 to $Q(G)$. Let $t_i = 0$ for any $i \in N$. Then $f(i, j) = \frac{d_i+2d_j-1+\sqrt{(d_i-2d_j+1)^2+4d_i}}{2}$. Thus (15) holds.

And the equality holds in (15) for regular graphs if and only if G is a regular graph. □

Remark 3.1 Obviously, we have

$$\begin{aligned}&\max_{1\le i\le n}\min_{i\sim j}\left\{\frac{d_i + 2d_j - 1 + \sqrt{(d_i - 2d_j + 1)^2 + 4d_i}}{2}\right\}\\&\quad\le \max_{i\sim j}\left\{\frac{d_i + 2d_j - 1 + \sqrt{(d_i - 2d_j + 1)^2 + 4d_i}}{2}\right\}.\end{aligned}$$

That is to say, our upper bound in Theorem 3.1 is always better than the upper bound (1) in [4].

Theorem 3.2 *Let $G = (V, E)$ be a simple connected graph on n vertices. Then*

$$\rho(G) \ge \min_{1\le i\le n}\max_{i\sim j}\left\{\frac{d_i + d_j + m_j - d_i/d_j + \sqrt{(d_i - d_j - m_j + d_i/d_j) + 4d_i}}{2}\right\} \tag{16}$$

and

$$\rho(G) \le \max_{1\le i\le n}\min_{i\sim j}\left\{\frac{d_i + d_j + m_j - d_i/d_j + \sqrt{(d_i - d_j - m_j + d_i/d_j) + 4d_i}}{2}\right\}. \tag{17}$$

Moreover, one of the equalities in (16), (17) *holds if and only if G is a regular graph or a bipartite semi-regular graph.*

Proof We apply Theorem 2.1 to $Q(G)$. Let $t_i = d_i$, $s_i = \sum_{j=1}^{n} a_{ij}r_j = d_i m_i$ for any $1 \le i \le n$. Then $f(i, j) = \frac{d_i+d_j+m_j-d_i/d_j+\sqrt{(d_i-d_j-m_j+d_i/d_j)+4d_i}}{2}$. Thus (16), (17) hold.

And the equality holds if and only if G is a regular graph or a bipartite semi-regular graph. □

4 Signless Laplacian spectral radius of a digraph

In this section, we will apply Theorem 2.1 to obtain some new results on the signless Laplacian spectral radius $\rho(\overrightarrow{G})$ of a digraph.

Theorem 4.1 *Let $\overrightarrow{G} = (V,E)$ be a strong connected digraph on n vertices. Then*

$$\begin{aligned}&\min_{1\le i\le n}\max_{i\sim j}\left\{\frac{d_i^+ + 2d_j^+ - 1 + \sqrt{(d_i^+ - 2d_j^+ + 1)^2 + 4d_i^+}}{2}\right\}\\ &\quad\le \rho(\overrightarrow{G}) \le \max_{1\le i\le n}\min_{i\sim j}\left\{\frac{d_i^+ + 2d_j^+ - 1 + \sqrt{(d_i^+ - 2d_j^+ + 1)^2 + 4d_i^+}}{2}\right\}.\end{aligned} \tag{18}$$

Moreover, one of the equalities in (18) *holds if and only if $\overrightarrow{G}$ is a regular digraph.*

Proof We apply Theorem 2.1 to $Q(\overrightarrow{G})$. Let $t_i = 0$ for any $1 \le i \le n$. Then $f(i,j) = \frac{d_i^+ + 2d_j^+ - 1 + \sqrt{(d_i^+ - 2d_j^+ + 1)^2 + 4d_i^+}}{2}$. Then the inequality (18) holds.

And the equality holds in (18) if and only if $\overrightarrow{G}$ is a regular digraph. □

Remark 4.1 Obviously, we have

$$\begin{aligned}&\max_{1\le i\le n}\min_{i\sim j}\left\{\frac{d_i^+ + 2d_j^+ - 1 + \sqrt{(d_i^+ - 2d_j^+ + 1)^2 + 4d_i^+}}{2}\right\}\\ &\quad\le \max_{i\sim j}\left\{\frac{d_i^+ + 2d_j^+ - 1 + \sqrt{(d_i^+ - 2d_j^+ + 1)^2 + 4d_i^+}}{2}\right\}.\end{aligned}$$

That is to say, our upper bound in Theorem 4.1 is always better than the upper bound (2) in [5].

Theorem 4.2 *Let $\overrightarrow{G} = (V,E)$ be a strong connected digraph on n vertices. Then*

$$\rho(\overrightarrow{G}) \ge \min_{1\le i\le n}\max_{i\sim j}\left\{\frac{d_i^+ + d_j^+ + m_j^+ - d_i^+/d_j^+ + \sqrt{(d_i^+ - d_j^+ - m_j^+ + d_i^+/d_j^+) + 4d_i^+}}{2}\right\} \tag{19}$$

and

$$\rho(\overrightarrow{G}) \le \max_{1\le i\le n}\min_{i\sim j}\left\{\frac{d_i^+ + d_j^+ + m_j^+ - d_i^+/d_j^+ + \sqrt{(d_i^+ - d_j^+ - m_j^+ + d_i^+/d_j^+) + 4d_i^+}}{2}\right\}. \tag{20}$$

Moreover, one of the equalities in (19), (20) *holds if and only if $\overrightarrow{G}$ is a regular digraph or a bipartite semi-regular digraph.*

Proof We apply Theorem 2.1 to $Q(\overrightarrow{G})$. Let $t_i = d_i^+$, $s_i = \sum_{j=1}^n a_{ij} r_j = d_i^+ m_i^+$ for any $1 \le i \le n$. Then $f(i,j) = \frac{d_i^+ + d_j^+ + m_j^+ - d_i^+/d_j^+ + \sqrt{(d_i^+ - d_j^+ - m_j^+ + d_i^+/d_j^+) + 4d_i^+}}{2}$. Thus (19), (20) hold.

One sees easily that the equality holds if and only if $\overrightarrow{G}$ is a regular digraph or a bipartite semi-regular digraph. □

5 Conclusion

In this paper, we give some new sharp upper and lower bounds for the spectral radius of a nonnegative irreducible matrix. Using these bounds, we obtain some new and improved bounds for the signless Laplacian spectral radius of a graph or a digraph which are better than the bounds in [4, 5].

Acknowledgements
Jun He is supported by the Science and Technology Foundation of Guizhou Province (Qian ke he Ji Chu [2016]1161); Guizhou Province Natural Science Foundation in China (Qian Jiao He KY [2016]255); the Doctoral Scientific Research Foundation of Zunyi Normal College (BS[2015]09); High-level Innovative Talents of Guizhou Province (Zun Ke He Ren Cai[2017]8). Yan-Min Liu is supported by National Science Foundations of China (71461027); Science and Technology Talent Training Object of Guizhou Province outstanding youth (Qian ke he ren zi [2015]06); Guizhou Province Natural Science Foundation in China (Qian Jiao He KY [2014]295); 2013, 2014 and 2015 Zunyi 15851 Talents Elite Project funding; Zhunyi Innovative Talent Team (Zunyi KH(2015)38). Tian is supported by Guizhou Province Natural Science Foundation in China (Qian Jiao He KY [2015]451); Science and Technology Foundation of Guizhou Province (Qian ke he J zi [2015]2147). Xiang-Hu Liu is supported by the Guizhou Province Department of Education fund (KY[2015]391, [2016]046); Guizhou Province Department of Education Teaching Reform Project [2015]337; Guizhou Province Science and Technology fund (Qian Ke He Ji Chu[2016]1160).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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Received: 3 August 2017 Accepted: 10 October 2017 Published online: 19 October 2017

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