Prederivatives of gamma paraconvex set-valued maps and Pareto optimality conditions for set optimization problems

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Abstract

Prederivatives play an important role in the research of set optimization problems. First, we establish several existence theorems of prederivatives for γ-paraconvex set-valued mappings in Banach spaces with $\gamma > 0$. Then, in terms of prederivatives, we establish both necessary and sufficient conditions for the existence of Pareto minimal solution of set optimization problems.

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1 Introduction

Let X and Y be Banach spaces. We say that $G : X \rightrightarrows Y$ is a set-valued mapping if $G(x)$ is a subset of Y for all $x \in X$. Set-valued problems occur in many situations, such as control problems, feasibility problems, optimality problems, equilibrium problems and variational inequality problems. A powerful tool dealing with set-valued problems is set-valued analysis. We refer the reader to the references [1–4] for more knowledge about set-valued analysis and its applications.

In a pioneering work [5], Ioffe introduced a notion of prederivative which can be viewed as an extension of Clarke generalized gradient. It is well known that the prederivative is an effective tool in dealing with nondifferentiable mapping of nonsmooth analysis. In contrast with the derivative, the prederivative may not be unique. However, in terms of prederivatives, one can establish an inverse function theorem and implicit theorem and solve nondifferential inclusion problems [6]. In the later publication of Pang [7, 8], and Gaydu, Geoffroy and Jean-Alexis [9], some notions of prederivatives were posed and further studied. In 2016, Gaydu, Geoffroy and Marcelin [10] studied the existence of some kinds of prederivatives of convex set-valued mappings and established necessary and sufficient optimality conditions for the weak minimizers and the strong minimizers of set optimization problems. γ-paraconvex set-valued mappings are an extension of convex set-valued mappings, and were studied by some researchers [11, 12]. Moreover, in set optimization problems, Pareto minimizers are more suitable than weak minimizers and strong minimizers in practice [13, 14]. Now, two natural questions are posed. Can we establish some

existence results of some kinds of prederivatives for γ-paraconvex set-valued mappings? Can we give optimality conditions for the Pareto minimizers of set optimization problems by prederivatives?

In this paper, we firstly establish several existence theorems of prederivatives for γ-paraconvex set-valued mappings and cone-γ-paraconvex set-valued mappings. Then we establish necessary and sufficient optimality conditions for the Pareto minimizers of set optimization problems in terms of prederivatives.

2 Preliminaries

Throughout this paper, unless stated otherwise, we always assume that X and Y are real Banach spaces and $G : X \rightrightarrows Y$ is a set-valued mapping. The domain of G is defined by

$$\text{Dom}(G) = \left\{ x \in X \mid G(x) \neq \emptyset \right\}.$$

The graph of G is defined by

$$\text{Gr}(G) = \left\{ (x,y) \in X \times Y \mid y \in G(x) \right\}.$$

We say that G is a closed set-valued mapping if $\text{Gr}(G)$ is a closed subset of $X \times Y$. We say that G has convex values if $G(x)$ is a convex subset of Y for any $x \in X$. Let Ω be a subset of X; we use $\text{cl}(\Omega)$ to denote the closure of Ω, $\text{int}(\Omega)$ to denote the interior of Ω. We use B_X and B_Y to denote the closed unit ball of X and Y, respectively. Let $\bar{x} \in X$. We use $N(\bar{x})$ to denote all open neighborhoods of $\bar{x}$. Let $C \subseteq Y$ be a nonempty set. We say that C is a cone, if $\lambda c \in C$ for any $c \in C$ and $\lambda \geq 0$. We say that C is pointed if $C \cap (-C) = \{0\}$. Define $G + C : X \rightrightarrows Y$ as

$$(G + C)(x) := G(x) + C, \quad \forall x \in X.$$

The following definition is needed in the sequel.

Definition 2.1 ([1])　Let $\Phi : X \rightrightarrows Y$ be a set-valued mapping. We say that Φ is positively homogeneous if $0 \in \Phi(0)$ and $\Phi(\lambda x) = \lambda \Phi(x)$, $\forall x \in X$, $\forall \lambda > 0$.

Definition 2.2 ([12])　Let $C \subseteq Y$ be a convex cone, $\gamma > 0$ and $\eta > 0$. We say that G is a C-γ-paraconvex set-valued mapping with modulus η, if

$$\theta G(x) + (1 - \theta)G(u) \subseteq G\big(\theta x + (1 - \theta)u\big) + \eta \min\{\theta, 1 - \theta\}\|x - u\|^{\gamma} B_Y + C$$

for all $x, u \in X$, $\theta \in [0,1]$. We say that G is a γ-paraconvex set-valued mapping if $C = \{0\}$.

Remark 2.1　In the special case of $\eta = 0$ and $C = \{0\}$, C-γ-paraconvex set-valued mappings reduce to convex set-valued mappings.

Definition 2.3 ([15])　We say that G is Lipschitz continuous at $\bar{x}$ if there exist $l > 0$ and $U \in N(\bar{x})$ such that

$$G(x) \subseteq G\big(x'\big) + l\|x - x'\|B_Y, \quad \forall x, x' \in U.$$

If the above equation holds on $U = \Omega$, then we say that G is Lipschitz continuous on Ω.

Definition 2.4 ([10]) Let $\Phi : X \rightrightarrows Y$ be a positively homogeneous set-valued mapping, $\bar{x} \in X$ and $\bar{y} \in G(\bar{x})$.

(i) Φ is called an outer prederivative of G at $\bar{x}$, if for any $\delta > 0$ there exists $U \in N(\bar{x})$ such that

$$G(u) \subseteq G(\bar{x}) + \Phi(u - \bar{x}) + \delta \|u - \bar{x}\| B_Y, \quad \forall u \in U.$$

(ii) Φ is called a strict prederivative of G at $\bar{x}$, if for any $\delta > 0$ there exists $U \in N(\bar{x})$ such that

$$G(u) \subseteq G(u') + \Phi(u - u') + \delta \|u - u'\| B_Y, \quad \forall u, u' \in U.$$

(iii) Φ is called a pseudo strict prederivative of G at $(\bar{x}, \bar{y})$, if for any $\delta > 0$ there exist $U \in N(\bar{x})$ and $V \in N(\bar{y})$ such that

$$G(u) \cap V \subseteq G(u') + \Phi(u - u') + \delta \|u - u'\| B_Y, \quad \forall u, u' \in U.$$

3 Prederivatives of gamma paraconvex set-valued mappings

In this section, we establish the existence results of pseudo strict prederivatives for γ-paraconvex set-valued mappings and strict prederivatives for C-γ-paraconvex set-valued mappings, respectively.

Lemma 3.1 ([11]) *Let $G : X \rightrightarrows Y$ be a closed set-valued mapping, $y_0 \in G(X)$, $x_0 \in X$, $\eta > 0$, $\delta > 0$, $\gamma > 0$, G^{-1} be a γ-paraconvex set-valued mapping with modulus r, and $y_0 + \eta B_Y \subseteq G(x_0 + \delta B_X)$. Let $\eta_1 > 0$, $\eta_2 > 0$ with $\eta_1 + \eta_2 = \eta$. Then, for each $y \in y_0 + \eta_1 B_Y$,*

$$d\big(x, G^{-1}(y)\big) \le \frac{d(y, G(x))}{\eta_2} \big(\delta + r\eta_2^{\gamma} + \|x - x_0\|\big), \quad \forall x \in X.$$

Theorem 3.1 *Let $\eta > 0$, $\delta > 0$, $r > 0$, $\gamma > 0$, $G : X \rightrightarrows Y$ be a closed γ-paraconvex set-valued mapping with modulus r, $(\bar{x}, \bar{y}) \in \mathrm{Gr}(G)$ and $\bar{x} + \eta B_X \subseteq G^{-1}(\bar{y} + \delta B_Y)$. Then G has a pseudo strict prederivative Φ at $(\bar{x}, \bar{y})$ with $\Phi(\cdot) = L\| \cdot \| B_Y$, where $L = (\delta + r(\frac{\eta}{2})^{\gamma} + \frac{\eta}{2})\frac{2}{\eta} > 0$.*

Proof Clearly, G^{-1} is a closed set-valued mapping since G is a closed set-valued mapping. Since G is a γ-paraconvex set-mapping and $\bar{x} + \eta B_X \subseteq G^{-1}(\bar{y} + \delta B_Y)$, it follows from Lemma 3.1 that, for each $x \in \bar{x} + \frac{\eta}{2} B_X$,

$$d\big(y, G(x)\big) \le \frac{d(x, G^{-1}(y))}{\frac{\eta}{2}} \left(\delta + r\left(\frac{\eta}{2}\right)^{\gamma} + \|y - \bar{y}\| \right), \quad \forall y \in Y.$$

Then, for any $y \in \bar{y} + \frac{\eta}{2} B_Y$,

$$d\big(y, G(x)\big) \le \left(\delta + r\left(\frac{\eta}{2}\right)^{\gamma} + \frac{\eta}{2} \right) \frac{2}{\eta} d\big(x, G^{-1}(y)\big).$$

Let $L := (\delta + r(\frac{\eta}{2})^{\gamma} + \frac{\eta}{2})\frac{2}{\eta}$. We have

$$d\big(y, G(x)\big) \le L d\big(x, G^{-1}(y)\big), \quad \forall x \in \bar{x} + \frac{\eta}{2} B_X, \forall y \in \bar{y} + \frac{\eta}{2} B_Y.$$

This implies that

$$y \in G(x) + Ld\big(x, G^{-1}(y)\big)B_Y, \quad \forall x \in \bar{x} + \frac{\eta}{2}B_X, \forall y \in \bar{y} + \frac{\eta}{2}B_Y.$$

Then, for any $\tilde{\delta} > 0$,

$$G\big(x'\big) \cap \left(\bar{y} + \frac{\eta}{2}B_Y\right) \subseteq G(x) + L\|x - x'\|B_Y + \tilde{\delta}\|x - x'\|B_Y, \quad \forall x, x' \in \bar{x} + \frac{\eta}{2}B_X.$$

Therefore $\Phi(\cdot) := L\| \cdot \|B_Y$ is a pseudo strict prederivative of G at $(\bar{x}, \bar{y})$. $\qquad\square$

Remark 3.1 Theorem 3.1 extends [10, Theorem 3.3] from convex set-valued mappings to γ-paraconvex set-valued mappings. Recall [16] that G is said to be open at $(\bar{x}, \bar{y}) \in \mathrm{Gr}(G)$ if $G(U)$ is a neighborhood of $\bar{y}$ for every neighborhood U of $\bar{x}$. The assumption $\bar{x} + \eta B_X \subseteq G^{-1}(\bar{y} + \delta B_Y)$ means that $G^{-1}(\bar{y} + \delta B_Y)$ is a neighborhood of $\bar{x}$, which is very close to the openness property of G^{-1} at $(\bar{x}, \bar{y})$. However, the coefficients δ and η are fixed in our assumption.

Lemma 3.2 ([17, Lemma 1]) *Let A, B and D are subsets of X. If B is a closed convex set, D is a bounded set and $A + D \subseteq B + D$, then $A \subseteq B$.*

Theorem 3.2 *Let $C \subseteq Y$ be a nonempty closed convex cone, $\eta > 0$, $\gamma \geq 1$, $r > 0$, $\alpha > 0$, $G : X \rightrightarrows Y$ be a C-γ-paraconvex set-valued mapping with modulus r, $\bar{x} + \alpha B_X \subseteq \mathrm{Dom}(G)$. Assume that*

$$G(x) = A(x) + E(x), \quad \forall x \in \bar{x} + \alpha B_X, \tag{3.1}$$

where $A(x)$ is a subset of Y and $E(x)$ is a convex set with $A(x) \subseteq \eta B_Y$ and $0 \in E(x) \subseteq C$. Let $\Phi : X \rightrightarrows Y$ be defined by

$$\Phi(x) = \left(\frac{16\eta}{\alpha} + r\left(\frac{3\alpha}{4}\right)^{\gamma-1}\right)\|x\|B_Y, \quad \forall x \in X.$$

Then the following conclusions hold:

(i) *$G + C$ is Lipschitz on $\bar{x} + \frac{\alpha}{4}B_X$ with modulus $\frac{16\eta}{\alpha} + r(\frac{3\alpha}{4})^{\gamma-1}$;*

(ii) *Φ is a strict prederivative of $G + C$ at each $x \in \bar{x} + \frac{\alpha}{8}B_X$;*

(iii) *$\Phi + C$ is a strict prederivative of G at each $x \in \bar{x} + \frac{\alpha}{8}B_X$.*

Proof (i) Take $x_1, x_2 \in \bar{x} + \frac{\alpha}{4}B_X$ with $x_1 \neq x_2$. Let $\lambda := \frac{\alpha}{4}$ and define

$$z := x_1 - \lambda \frac{x_2 - x_1}{\|x_2 - x_1\|}.$$

Then

$$x_1 = \frac{\lambda}{\|x_2 - x_1\| + \lambda}x_2 + \frac{\|x_2 - x_1\|}{\|x_2 - x_1\| + \lambda}z.$$

Let $\theta := \frac{\|x_2 - x_1\|}{\|x_2 - x_1\| + \lambda}$. Then $\theta \in (0, 1)$ and $x_1 = (1 - \theta)x_2 + \theta z$. Since $z - x_2 = \frac{x_1 - x_2}{\theta}$ and $\gamma \geq 1$, we have

$$\min\{\theta, 1 - \theta\}\|x_2 - z\|^\gamma$$

$$= \min\{\theta, 1 - \theta\}\frac{\|x_1 - x_2\|^\gamma}{\theta^\gamma} \leq \theta \frac{\|x_1 - x_2\|^\gamma}{\theta^\gamma}$$

$$= \frac{\|x_1 - x_2\|^\gamma}{\theta^{\gamma - 1}} = \frac{\|x_1 - x_2\|^\gamma}{\left(\frac{\|x_1 - x_2\|}{\|x_1 - x_2\| + \lambda}\right)^{\gamma - 1}}$$

$$\leq \left(\|x_1 - x_2\| + \lambda\right)^{\gamma - 1}\|x_1 - x_2\|$$

$$\leq \left(\frac{\alpha}{2} + \frac{\alpha}{4}\right)^{\gamma - 1}\|x_1 - x_2\| = \left(\frac{3\alpha}{4}\right)^{\gamma - 1}\|x_1 - x_2\|. \tag{3.2}$$

Since G is a C-γ-paraconvex set-valued mapping with modulus r, it is easy to verify that $G + C$ is a γ-paraconvex set-valued mapping with modulus r. Taking into account inequality (3.2), we have

$$(1 - \theta)(G + C)(x_2) + \theta(G + C)(z)$$

$$\subseteq (G + C)(x_1) + r\min\{\theta, 1 - \theta\}\|x_2 - z\|^\gamma B_Y$$

$$\subseteq (G + C)(x_1) + r\left(\frac{3\alpha}{4}\right)^{\gamma - 1}\|x_1 - x_2\|B_Y. \tag{3.3}$$

Due to the convexity of $(G + C)(x)$ for each $x \in X$, we have

$$(1 - \theta)(G + C)(x_2) + \theta(G + C)(x_2) = (G + C)(x_2). \tag{3.4}$$

Adding $\theta(G + C)(x_2)$ on both sides of equation (3.3), and using (3.4), we get

$$(G + C)(x_2) + \theta(G + C)(z) \subseteq (G + C)(x_1) + \theta(G + C)(x_2) + r\left(\frac{3\alpha}{4}\right)^{\gamma - 1}\|x_1 - x_2\|B_Y.$$

Clearly, $C + \theta C = C + C = 2C = C$ since C is a convex cone. Therefore, the above equation can be rewritten as

$$G(x_2) + \theta G(z) + C \subseteq G(x_1) + \theta G(x_2) + C + r\left(\frac{3\alpha}{4}\right)^{\gamma - 1}\|x_1 - x_2\|B_Y. \tag{3.5}$$

Since $A(x) \subseteq \eta B_Y$ for any $x \in \bar{x} + \alpha B_X$, we obtain

$$A(x) \subseteq A(x') + 2\eta B_Y, \quad \forall x, x' \in \bar{x} + \alpha B_X. \tag{3.6}$$

Next, we show that $z \in \bar{x} + \alpha B_X$. Since $x_1 = (1 - \theta)x_2 + \theta z$, we have

$$\|z - \bar{x}\| \leq \|z - x_2\| + \|x_2 - \bar{x}\|$$

$$\leq \frac{\|x_1 - x_2\|}{\theta} + \frac{\alpha}{4}$$

$$= \|x_1 - x_2\| + \lambda + \frac{\alpha}{4}$$

$$\leq \|x_1 - \bar{x}\| + \|\bar{x} - x_2\| + \frac{\alpha}{4} + \frac{\alpha}{4}$$

$$\leq \frac{\alpha}{4} + \frac{\alpha}{4} + \frac{\alpha}{2} = \alpha.$$

Therefore $z \in \bar{x} + \alpha B_X$. As $x_2, z \in \bar{x} + \alpha B_X$, combined with (3.6), we have

$$A(x_2) \subseteq A(z) + 2\eta B_Y.$$

By the assumption (3.1), $G(z) = A(z) + E(z)$, $G(x_2) = A(x_2) + E(x_2)$, it follows from (3.5) that

$$G(x_2) + \theta\big(A(z) + E(z)\big) + C$$

$$\subseteq G(x_1) + \theta\big(A(x_2) + E(x_2)\big) + C + r\left(\frac{3\alpha}{4}\right)^{\gamma-1}\|x_1 - x_2\|B_Y$$

$$\subseteq G(x_1) + \theta\big(A(z) + 2\eta B_Y\big) + \theta E(x_2) + C + r\left(\frac{3\alpha}{4}\right)^{\gamma-1}\|x_1 - x_2\|B_Y. \tag{3.7}$$

Since $0 \in E(x_2) \subseteq C$ and $0 \in E(z) \subseteq C$, we have

$$\theta E(x_2) + C = C, \qquad \theta E(z) + C = C.$$

Equation (3.7) yields

$$G(x_2) + C + \theta A(z)$$

$$\subseteq G(x_1) + \theta A(z) + 2\eta\theta B_Y + C + r\left(\frac{3\alpha}{4}\right)^{\gamma-1}\|x_1 - x_2\|B_Y$$

$$\subseteq \mathrm{cl}\left(G(x_1) + 2\eta\theta B_Y + C + r\left(\frac{3\alpha}{4}\right)^{\gamma-1}\|x_1 - x_2\|B_Y\right) + \theta A(z). \tag{3.8}$$

Since $\mathrm{cl}(G(x_1) + 2\eta\theta B_Y + C + r(\frac{3\alpha}{4})^{\gamma-1}\|x_1 - x_2\|B_Y)$ is a closed convex set and $A(z)$ is a bounded set, it follows from Lemma 3.2 and (3.8) that

$$G(x_2) + C \subseteq \mathrm{cl}\left(G(x_1) + 2\eta\theta B_Y + C + r\left(\frac{3\alpha}{4}\right)^{\gamma-1}\|x_1 - x_2\|B_Y\right)$$

$$\subseteq G(x_1) + 4\eta\theta B_Y + C + r\left(\frac{3\alpha}{4}\right)^{\gamma-1}\|x_1 - x_2\|B_Y$$

$$= G(x_1) + C + \left(4\eta\theta + r\left(\frac{3\alpha}{4}\right)^{\gamma-1}\|x_1 - x_2\|\right)B_Y$$

$$= G(x_1) + C + \left(\frac{4\eta\|x_2 - x_1\|}{\|x_2 - x_1\| + \lambda} + r\left(\frac{3\alpha}{4}\right)^{\gamma-1}\|x_1 - x_2\|\right)B_Y$$

$$\subseteq G(x_1) + C + \left(\frac{4\eta}{\lambda} + r\left(\frac{3\alpha}{4}\right)^{\gamma-1}\right)\|x_2 - x_1\|B_Y$$

$$\subseteq G(x_1) + C + \left(\frac{16\eta}{\alpha} + r\left(\frac{3\alpha}{4}\right)^{\gamma-1}\right)\|x_2 - x_1\|B_Y, \tag{3.9}$$

where the last inequality holds since $\lambda = \frac{\alpha}{4}$. Therefore, $G + C$ is Lipschitz with modulus $\frac{16\eta}{\alpha} + r(\frac{3\alpha}{4})^{\gamma-1}$ on $\bar{x} + \frac{\alpha}{4}B_X$ since x_1 and x_2 are two arbitrary elements of $\bar{x} + \frac{\alpha}{4}B_X$.

(ii) Let $\Phi : X \rightrightarrows Y$ be defined by

$$\Phi(x) = \left(\frac{16\eta}{\alpha} + r \left(\frac{3\alpha}{4} \right)^{\gamma-1} \right) \|x\| B_Y, \quad \forall x \in X. \tag{3.10}$$

Clearly, Φ is a positively homogeneous mapping with bounded closed values. By (3.9), we get

$$G(x_2) + C \subseteq G(x_1) + C + \Phi(x_2 - x_1) + \delta\|x_2 - x_1\| B_Y, \quad \forall x_1, x_2 \in \bar{x} + \frac{\alpha}{4} B_X \tag{3.11}$$

for any $\delta > 0$. This implies that Φ is a strict prederivative of $G + C$ at each $x \in \bar{x} + \frac{\alpha}{8} B_X$.

(iii) Since C is a cone, it follows from (3.10) that $0 \in (\Phi + C)(0)$, and for any $t > 0$ and $x \in X$,

$$(\Phi + C)(tx) = t\Phi(x) + C = t(\Phi + C)(x),$$

and hence $\Phi + C$ is positively homogeneous. Let $\tilde{x} \in \bar{x} + \frac{\alpha}{8} B_X$. Then there exists $\tilde{r} > 0$ such that $\tilde{x} + \tilde{r} B_X \subseteq \bar{x} + \frac{\alpha}{4} B_X$. Since $0 \in C$, it follows from (3.11) that for any $\delta > 0$,

$$G(x_2) \subseteq G(x_1) + (\Phi + C)(x_2 - x_1) + \delta\|x_2 - x_1\| B_Y, \quad \forall x_1, x_2 \in \tilde{x} + \tilde{r} B_X.$$

Therefore, $\Phi + C$ is a strict prederivative of G at $\tilde{x}$. □

Remark 3.2 In [10, Theorem 3.8], Gaydu, Geoffroy and Marcelin proved the following result. Let Y be a finite dimensional Banach space, $C \subseteq Y$ be a nonempty closed convex cone, $G : X \rightrightarrows Y$ be a C-convex set-valued mapping, $\bar{x} \in \mathrm{int}(\mathrm{dom}(G))$. Assume that there exist $\alpha > 0$ and $\eta > 0$ such that $G(x) + \mathrm{cl}(C)$ is a closed set and $G(x) \subseteq \eta B_Y$ for all $x \in \bar{x} + \alpha B_X$. Then there exists $U \in N(\bar{x})$ such that

(i) $G + C$ is Lipschitz on U;

(ii) there exists a positively homogeneous mapping $\Phi : X \rightrightarrows Y$ with bounded closed values such that Φ is a strict prederivative of $G + C$ at each $x \in U$;

(iii) $\Phi + C$ is a strict prederivative of G at each $x \in U$.

In contrast with [10, Theorem 3.8], Theorem 3.2 has some improvements. Firstly, we extend Y from finite dimensional spaces to general Banach spaces. Secondly, we extend G from C-convex set-valued mappings to C-γ-paraconvex set-valued mappings. Thirdly, we do not need the boundedness of $G(x)$.

In the following, we give an example to illustrate Theorem 3.2.

Example 3.1 Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$, $G : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined by

$$G(x) = \left[\left| |x| - 1 \right|, +\infty \right), \quad \forall x \in \mathbb{R}.$$

It follows from [12, Example 3.1] that G is a C-1-paraconvex set-valued mapping with modulus 1, but not a C-convex set-valued mapping. Take $\bar{x} = 0$, $A(x) = ||x| - 1|$ and $E(x) = \mathbb{R}_+$. Then

$$G(x) = A(x) + E(x), \quad \forall x \in \bar{x} + B_X$$

with $A(x) \subseteq [-1, 1]$ for all $x \in \bar{x} + B_X$. All conditions of Theorem 3.2 are justified. By Theorem 3.2, $G + C$ is Lipschitz on $\bar{x} + \frac{1}{4}B_X$ with modulus 17, and $\Phi(\cdot) = 17| \cdot |B_Y$ satisfies (ii) and (iii) of Theorem 3.2 on $\bar{x} + \frac{1}{8}B_X$.

Corollary 3.1 *Let $C \subseteq Y$ be a nonempty closed convex cone, $\eta > 0$, $\gamma \geq 1$, $r > 0$, $\alpha > 0$, $G : X \rightrightarrows Y$ be a C-γ-paraconvex set-valued mapping with modulus r and $\bar{x} + \alpha B_X \subseteq \mathrm{Dom}(G)$. Assume that*

$$G(x) \subseteq \eta B_Y, \quad \forall x \in \bar{x} + \alpha B_X.$$

Let $\Phi : X \rightrightarrows Y$ be defined by

$$\Phi(x) = \left(\frac{16\eta}{\alpha} + r\left(\frac{3\alpha}{4} \right)^{\gamma-1} \right) \|x\| B_Y, \quad \forall x \in X.$$

Then the following conclusions hold:

(i) *$G + C$ is Lipschitz on $\bar{x} + \frac{\alpha}{4}B_X$ with modulus $\frac{16\eta}{\alpha} + r(\frac{3\alpha}{4})^{\gamma-1}$;*

(ii) *Φ is a strict prederivative of $G + C$ at each $x \in \bar{x} + \frac{\alpha}{8}B_X$;*

(iii) *$\Phi + C$ is a strict prederivative of G at each $x \in \bar{x} + \frac{\alpha}{8}B_X$.*

Proof Take $A(x) = G(x)$, $E(x) = \{0\}$ for all $x \in X$ in Theorem 3.2. Then the conclusions follow from Theorem 3.2 directly. □

4 Pareto minimizer and prederivative

In this section, we always assume that C is a pointed closed convex cone of Y. Consider the following set optimization problem:

$$(\mathrm{SP}): \quad \begin{cases} \min_C & G(x), \\ \mathrm{s.t.} & x \in \Omega, \end{cases}$$

where Ω is a nonempty closed subset of X with $\mathrm{Dom}(G) \cap \Omega \neq \emptyset$.

Definition 4.1 ([13]) We say that $(\bar{u}, \bar{v}) \in \mathrm{Gr}(G)$ is a Pareto minimizer of the optimization problem (SP), if $\bar{u} \in \Omega$ and $(\bar{v} - C) \cap G(\Omega) = \{\bar{v}\}$.

First, we establish a necessary condition for Pareto minimizers of the optimization problem (SP).

Theorem 4.1 *Let $\bar{u} \in \mathrm{int}(\Omega)$, $(\bar{u}, \bar{v}) \in \mathrm{Gr}(G)$. Suppose that Φ is a pseudo strict prederivative of G at $(\bar{u}, \bar{v})$ and $(\bar{u}, \bar{v})$ is a Pareto minimizer of the optimization problem (SP). Then, for any $\delta > 0$ and $u \in \Omega$,*

$$\Phi\left(-(u - \bar{u})\right) + \delta \|u - \bar{u}\| B_Y \nsubseteq C \setminus \{0\}. \tag{4.1}$$

Proof Let $(\bar{u}, \bar{v})$ be a Pareto minimizer of the optimization problem (SP). Suppose that the conclusion is not true. Then there exist $\delta_0 > 0$ and $u_0 \in \Omega$ such that

$$\Phi\left(-(u_0 - \bar{u})\right) + \delta_0 \|u_0 - \bar{u}\| B_Y \subseteq C \setminus \{0\}. \tag{4.2}$$

Since Φ is a pseudo strict prederivative of G at $(\bar{u}, \bar{v})$, there exist $\eta_1 > 0$ and $\eta_2 > 0$ such that

$$G(u) \cap (\bar{v} + \eta_2 B_Y) \subseteq G(\tilde{u}) + \Phi(u - \tilde{u}) + \delta_0 \|u - \tilde{u}\| B_Y, \quad \forall u, \tilde{u} \in \bar{u} + \eta_1 B_X.$$

Choose $\theta > 0$ such that $\bar{u} + \theta(u_0 - \bar{u}) \in (\bar{u} + \eta_1 B_X) \cap \Omega$. By the above equation, we have

$$G(\bar{u}) \cap (\bar{v} + \eta_2 B_Y) \subseteq G(\bar{u} + \theta(u_0 - \bar{u})) + \Phi(-\theta(u_0 - \bar{u})) + \delta_0 \|\theta(u_0 - \bar{u})\| B_Y.$$

Since $\bar{v} \in G(\bar{u}) \cap (\bar{v} + \eta_2 B_Y)$, the above equation implies that there exists $\hat{v} \in G(\bar{u} + \theta(u_0 - \bar{u}))$ such that

$$\bar{v} - \hat{v} \in \Phi(-\theta(u_0 - \bar{u})) + \delta_0 \|\theta(u_0 - \bar{u})\| B_Y.$$

Since Φ is positively homogeneous, we get

$$\theta^{-1}(\bar{v} - \hat{v}) \in \Phi(-(u_0 - \bar{u})) + \delta_0 \|u_0 - \bar{u}\| B_Y.$$

Combined with (4.2), we have $\bar{v} - \hat{v} \in C \setminus \{0\}$. Noting that $\hat{v} \in G(\bar{u} + \theta(u_0 - \bar{u})) \subseteq G(\Omega)$, we have

$$\hat{v} \in (\bar{v} - C \setminus \{0\}) \cap G(\Omega).$$

This contradicts the assumption that $(\bar{u}, \bar{v})$ is a Pareto minimizer of the optimization problem (SP). Therefore (4.1) holds. $\qquad\square$

Definition 4.2 ([18]) Let $G : X \rightrightarrows Y$ be a set-valued mapping, $\bar{u} \in \text{Dom}(G)$. We say that G is C-starshaped at $\bar{u}$, if for any $u \in X$, $\theta \in [0, 1]$,

$$(1 - \theta)G(\bar{u}) + \theta G(u) \subseteq G((1 - \theta)\bar{u} + \theta u) + C.$$

The following theorem provides a sufficient optimality condition for a Pareto minimizer of the optimization problem (SP).

Theorem 4.2 *Let $\bar{u} \in \Omega$, Φ be an outer prederivative of G at $\bar{u}$, $\bar{v} \in G(\bar{u})$, $G(\bar{u}) \subseteq \bar{v} + C$ and G be C-starshaped at $\bar{u}$. If there exist $\delta > 0$ and $\eta > 0$ such that*

$$(\Phi(u - \bar{u}) + \delta \|u - \bar{u}\| B_Y) \cap (-C) \subseteq \{0\} \tag{4.3}$$

for all $u \in \bar{u} + \eta B_X$, then $(\bar{u}, \bar{v})$ is a Pareto minimizer of the optimization problem (SP).

Proof Let $u \in \Omega$. Since Φ is an outer prederivative of G at $\bar{u}$, for the given δ in the assumption, there exists $\bar{\eta} \in (0, \eta)$ such that

$$G(\tilde{u}) \subseteq G(\bar{u}) + \Phi(\tilde{u} - \bar{u}) + \delta \|\tilde{u} - \bar{u}\| B_Y, \quad \forall \tilde{u} \in \bar{u} + \bar{\eta} B_X. \tag{4.4}$$

Choose $\theta \in (0, 1)$ such that $(1 - \theta)\bar{u} + \theta u \in \bar{u} + \bar{\eta} B_X$. Since G is C-starshaped at $\bar{u}$, we have

$$(1 - \theta)G(\bar{u}) + \theta G(u) \subseteq G((1 - \theta)\bar{u} + \theta u) + C. \tag{4.5}$$

Combined (4.4) with (4.5), we get

$$(1 - \theta)G(\bar{u}) + \theta G(u)$$
$$\subseteq G(\bar{u}) + \Phi\big((1 - \theta)\bar{u} + \theta u - \bar{u}\big) + \delta\big\|(1 - \theta)\bar{u} + \theta u - \bar{u}\big\|B_Y + C$$
$$= G(\bar{u}) + \Phi\big(\theta(u - \bar{u})\big) + \delta\big\|\theta(u - \bar{u})\big\|B_Y + C.$$

As $G(\bar{u}) \subseteq \bar{v} + C$, we get

$$(1 - \theta)\bar{v} + \theta G(u) \subseteq (1 - \theta)G(\bar{u}) + \theta G(u)$$
$$\subseteq G(\bar{u}) + \Phi\big(\theta(u - \bar{u})\big) + \delta\big\|\theta(u - \bar{u})\big\|B_Y + C$$
$$\subseteq \bar{v} + \Phi\big(\theta(u - \bar{u})\big) + \delta\big\|\theta(u - \bar{u})\big\|B_Y + C,$$

that is,

$$\theta\big(G(u) - \bar{v}\big) \subseteq \Phi\big(\theta(u - \bar{u})\big) + \delta\big\|\theta(u - \bar{u})\big\|B_Y + C. \tag{4.6}$$

Since $(1 - \theta)\bar{u} + \theta u \in \bar{u} + \bar{\eta}B_X$, with (4.3), we have

$$\big(\Phi\big(\theta(u - \bar{u})\big) + \delta\big\|\theta(u - \bar{u})\big\|B_Y\big) \cap (-C) \subseteq \{0\}.$$

It follows from (4.6) that

$$\big(G(u) - \bar{v}\big) \cap (-C) \subseteq \{0\}.$$

Therefore, $(\bar{u}, \bar{v})$ is a Pareto minimizer of the optimization problem (SP) since u is an arbitrary element of Ω. □

Remark 4.1 In [10], Gaydu, Geoffroy and Marcelin established necessary condition and sufficient conditions for the weak minimizers and the strong minimizers of the optimization problem (SP). It is well known that each strong minimizer is a Pareto minimizer and each Pareto minimizer is a weak minimizer, but the converses are not true.

In the following, we give an example to illustrate Theorem 4.2.

Example 4.1 In (SP), let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}_+^2$, $\Omega = (-\infty, 0]$, $G : X \rightrightarrows Y$ be defined by

$$G(u) = \begin{cases} \{(t_1, t_2) \mid t_1 \geq u, t_2 \geq -u\}, & u \leq 0, \\ \emptyset, & u > 0. \end{cases}$$

Take $\bar{u} = 0$, $\bar{v} = (0, 0) \in G(\bar{u})$ and $\delta = \frac{1}{2}$. Then $G(\bar{u}) \subseteq \bar{v} + \mathbb{R}_+^2$. It is easy to verify that G is C-starshaped at $\bar{u}$, $\Phi = (1, -1)$ is an outer prederivative of G at $\bar{u}$ and

$$\big(\Phi(u - \bar{u}) + \delta\|u - \bar{u}\|B_{\mathbb{R}^2}\big) \cap \big(-\mathbb{R}_+^2\big)$$
$$= \Big((u, -u) + \frac{1}{2}|u|B_{\mathbb{R}^2}\Big) \cap \big(-\mathbb{R}_+^2\big) \subseteq \{(0, 0)\}, \quad \forall u \in \bar{u} + \frac{1}{2}B_X = \Big[-\frac{1}{2}, \frac{1}{2}\Big].$$

All conditions of Theorem 4.2 are verified. Therefore, $(\bar{u}, \bar{v})$ is a Pareto minimizer of the optimization problem (SP).

5 Conclusion

In this paper, we establish two existence theorems of prederivatives for γ-paraconvex set-valued mappings, and give optimality conditions for the Pareto minimizers of set optimization problems. These results improve the corresponding one obtained in [10]. Moreover, the coefficients in Theorems 3.1 and 3.2 can be calculated. Theorems 3.1 and 3.2 give sufficient conditions for the existence of Φ of Theorem 4.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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