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Optimal model for 4-band biorthogonal wavelets bases for fast calculation

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Abstract

A class of 4-band symmetric biorthogonal wavelet bases has been constructed, in which any wavelet system the high-pass filters can be determined by exchanging position and changing the sign of the two low-pass filters. Thus, the least restrictive conditions are needed for forming a wavelet so that the free degrees can be reversed for application requirement. Some concrete examples with high vanishing moments are also given, the properties of the transformation matrix are studied and the optimal model is constructed. These wavelets can process the boundary conveniently, and they lead to highly efficient computations in applications.

MSC: 42C40; 42A16

Keywords: multi-band wavelets; circular matrix; biorthogonal wavelets; vanishing movement; symmetry

1 Introduction

As a generalization of orthogonal wavelets, the biorthogonal wavelets have become a fundamental tool in many areas of applied mathematics, from signal processing to numerical analysis [1–8]. It is well known that 2-band orthogonal wavelets, suffering from severe constraint conditions, such as nontrivial symmetric 2-band orthogonal wavelets, do not exist [9]. Biorthogonal wavelets, multi-band wavelets are designed as alternatives for more freedom and flexibility [10–15]. Multi-band wavelets have attracted considerable attention due to their richer parameter space to have a more flexible time-frequency tiling, to zoom in onto narrow band high frequency components in frequency responses, to give better energy compaction than 2-band wavelets. (Bi)orthogonal real-valued multi-band wavelets with symmetry have been reported in [10–12, 16, 17]. In this paper, we can construct innumerable wavelet filters with some structure for fast calculation, among which we can select the best ones for practical applications.

As the case of a dyadic wavelet, a pair $(\phi(x), \widetilde{\phi}(x))$ of dual scaling functions can be expressed as the following dilation equations:

$$\phi(x) = \sum_{k \in Z} h_k \phi(dx - k) \tag{1.1}$$

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and

$$\widetilde{\phi}(x) = \sum_{k \in Z} \widetilde{h}_k \widetilde{\phi}(dx - k), \tag{1.2}$$

respectively, with

$$\langle \phi(\cdot - k), \widetilde{\phi}(\cdot - j) \rangle = \delta_{k-j} \tag{1.3}$$

or equivalently,

$$\sum_k h_k \widetilde{h}_{k+dj} = \delta_j, \tag{1.4}$$

where δ_j denotes the *Dirac* sequence such that $\delta_j = 1$ for $j = 0$ otherwise $\delta_j = 0$.

Recall the sub-band coding scheme or Mallat algorithm associated to a 4-band biorthogonal wavelets. There are eight sequences $h = (h_n)_{n \in Z}$, $g^i = (g_n^i)_{n \in Z}$ ($i = 1, 2, 3$), $\widetilde{h} = (\widetilde{h}_n)_{n \in Z}$, $\widetilde{g}^i = (\widetilde{g}_n^i)_{n \in Z}$ ($i = 1, 2, 3$), four of which are used for decomposition $\{h, g^1, g^2, g^3\}$ and the others for reconstruction $\{\widetilde{h}, \widetilde{g}^1, \widetilde{g}^2, \widetilde{g}^3\}$. Starting from a data sequence $x = (x_n)_{n \in Z}$, we convolve with h, g^1, g^2, g^3 and retain only one sample out of every four for decomposition

$$\begin{aligned} c_n &= \sum_k h_{4n-k} x_k, \\ d_n^i &= \sum_k g_{4n-k}^i x_k, \quad i = 1, 2, 3. \end{aligned} \tag{1.5}$$

The reconstruction operation is

$$x_k = \sum_n \left(\widetilde{h}_{4n-k} c_n + \sum_{i=1}^3 \widetilde{g}_{4n-k}^i d_n^i \right). \tag{1.6}$$

Equations (1.5) and (1.6) can be rewritten as the form of 4-circular matrix [16], which can be defined by the 4-circular operator.

If c^0 is a periodic signal, we rewrite $c^0 = (c_1^0, c_2^0, \ldots, c_{4n}^0)^T$, which is a whole period.

Let $c^1 = (c_1^1, c_2^1, \ldots, c_n^1, d_1^1, d_2^1, \ldots, d_n^1, d_1^2, d_2^2, \ldots, d_n^2, d_1^3, d_2^3, \ldots, d_n^3)^T$. Then there exists a $4n \times 4n$ 4-circular matrix M_{4n} generated by $\{h, g^1, g^2, g^3\}$ such that

$$c^1 = M_{4n} c^0. \tag{1.7}$$

It is easy to see that (1.7) is equivalent to (1.5). Let

$$c^0 = \widetilde{M}_{4n}^T c^1, \tag{1.8}$$

where $\widetilde{M}_{4n}$ is a 4-circular $4n \times 4n$ matrix generated by $\{\widetilde{h}, \widetilde{g}^1, \widetilde{g}^2, \widetilde{g}^3\}$.

Clearly, (1.8) is equivalent to (1.6). Then

$$\|c^1\|^2 = (c^1)^T c^1 = (c^0)^T (M_{4n}^T M_{4n}) c^0. \tag{1.9}$$

Since $M_{4n}^T M_{4n}$ is a positive definite matrix, its eigenvalues λ_i ($i = 1, 2 \ldots, 4n$) are positive and there exists an orthonormal matrix Q such that

$$M_{4n}^T M_{4n} = Q^T \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{4n}) Q. \tag{1.10}$$

Let $s = Qc^0 = (s_1, s_2, \ldots, s_{4n})^T$. Then $\|s\|^2 = \|c^0\|^2$. It follows from (1.9) and (1.10) that

$$\left\| c^1 \right\|^2 = \left(Qc^0 \right)^T \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{4n}) \left(Qc^0 \right) = \sum_{i=1}^{4n} \lambda_i s_i^2.$$

Thus,

$$\min\{\lambda_i\} \left\| c^0 \right\|^2 \leq \left\| c^1 \right\|^2 \leq \max\{\lambda_i\} \left\| c^0 \right\|^2. \tag{1.11}$$

Similarly,

$$\min\{\widetilde{\lambda}_i\} \|c^0\|^2 \leq \|c^1\|^2 \leq \max\{\widetilde{\lambda}_i\} \|c^0\|^2, \tag{1.12}$$

where $\widetilde{\lambda}_i$ ($i = 1, 2 \ldots, 4n$) are the eigenvalues of $\widetilde{M}_{4n}^T \widetilde{M}_{4n}$.

We try to calculate the eigenvalues of $M_{4n}^T M_{4n}$ or $\widetilde{M}_{4n}^T \widetilde{M}_{4n}$. It has been shown that the eigenvalues of $M_{4n}^T M_{4n}$ appear in pairs of reciprocals, $M_{4n}^T M_{4n}$ and $\widetilde{M}_{4n}^T \widetilde{M}_{4n}$ have the same eigenvalues. It is obvious that $\max\{\lambda_i\} = \max\{\widetilde{\lambda}_i\} = \frac{1}{\min\{\lambda_i\}} = \frac{1}{\min\{\widetilde{\lambda}_i\}}$.

This paper is organized as follows. In Section 2, we design a type of 4-band biorthogonal wavelets filters for fast calculation, give the construction method. In Section 3, some results of the wavelet transform matrix are developed. So a model to minimize the maximum eigenvalue is built in Section 4. An example is provided to illustrate our results in this paper. Conclusions and future work are summarized in Section 5.

2 4-band biorthogonal wavelets filters for fast calculation

In this section, a class of 4-band symmetric biorthogonal wavelet filters for fast calculation is designed, and the corresponding wavelet filters are constructed.

Assume that

$$h = \{t_0, t_1, \ldots, t_{2L-1}, t_{2L-1}, \ldots, t_1, t_0\}, \tag{2.1}$$

$$\widetilde{h} = \{\widetilde{t}_0, \widetilde{t}_1, \ldots, \widetilde{t}_{2L-1}, \widetilde{t}_{2L-1}, \ldots, \widetilde{t}_1, \widetilde{t}_0\}. \tag{2.2}$$

The high-pass filters for decomposition are

$$
\begin{aligned}
&g_{2i}^1 = (-1)^{L-i} h_{2i+1}, \qquad g_{2i+1}^1 = (-1)^{L-i+1} h_{2i} \quad (0 \leq i \leq 2L - 1), \\
&g_i^2 = (-1)^i \widetilde{h}_{4L-1-i} \quad (0 \leq i \leq 4L - 1), \\
&g_i^3 = (-1)^i \widetilde{g}_{4L-1-i}^1 \quad (0 \leq i \leq 4L - 1).
\end{aligned}
\tag{2.3}
$$

Note that g^1 is symmetric and g^2 and g^3 are antisymmetric.

The high-pass filters for reconstruction are

$$\widetilde{g}^1_{2i} = (-1)^{L-i}\widetilde{h}_{2i+1}, \qquad \widetilde{g}^1_{2i+1} = (-1)^{L-i+1}\widetilde{h}_{2i} \quad (0 \le i \le 2L-1),$$

$$\widetilde{g}^2_i = (-1)^i h_{4L-1-i} \quad (0 \le i \le 4L-1),$$

$$\widetilde{g}^3_i = (-1)^i g^1_{4L-1-i} \quad (0 \le i \le 4L-1).$$

(2.4)

We can see that $\widetilde{g}^1$ is symmetric and $\widetilde{g}^2$ and $\widetilde{g}^3$ are antisymmetric.

For $L = 3$, an example of a filter bank $\{h, g^1, g^2, g^3\}$ is as follows:

$$\begin{bmatrix} t_0 & t_1 & t_2 & t_3 & t_4 & t_5 & t_5 & t_4 & t_3 & t_2 & t_1 & t_0 \\ t_1 & -t_0 & -t_3 & t_2 & t_5 & -t_4 & -t_4 & t_5 & t_2 & -t_3 & -t_0 & t_1 \\ \widetilde{t}_0 & -\widetilde{t}_1 & \widetilde{t}_2 & -\widetilde{t}_3 & \widetilde{t}_4 & -\widetilde{t}_5 & \widetilde{t}_5 & -\widetilde{t}_4 & \widetilde{t}_3 & -\widetilde{t}_2 & \widetilde{t}_1 & -\widetilde{t}_0 \\ \widetilde{t}_1 & \widetilde{t}_0 & -\widetilde{t}_3 & -\widetilde{t}_2 & \widetilde{t}_5 & \widetilde{t}_4 & -\widetilde{t}_4 & -\widetilde{t}_5 & \widetilde{t}_2 & \widetilde{t}_3 & -\widetilde{t}_0 & -\widetilde{t}_1 \end{bmatrix}.$$

Clearly, this type of wavelet filter banks are only determined the two low-pass filters h, $\widetilde{h}$, *i.e.*, the high-pass filters can be determined by exchanging position and changing the sign of the two low-pass filters. Therefore, it can reduce the computational complexity and facilitate fast computation.

In addition, we assume $\sum_k h_k = \sum_k \widetilde{h}_k = 2$. At this time, the basic conditions of biorthogonal wavelets are

$$\sum_k h_k \widetilde{h}_{k+4j} = \delta_j,$$

$$\sum_k h_k \widetilde{g}^n_{k+4j} = 0, \quad n = 1,2,3,$$

$$\sum_k g^n_k \widetilde{g}^m_{k+4j} = \delta_{n-m}\delta_j, \quad 1 \le n, m \le 3.$$

We have the following theorem for the biorthogonality condition.

Theorem 2.1 *If*

$$\sum_k h_k \widetilde{h}_{k+4j} = \delta_j,$$

$$\sum_k g^1_k \widetilde{h}_{k+4j} = 0,$$

(2.5)

then the wavelet filter banks defined by (2.1)-(2.4) *satisfy the biorthogonality condition.*

Proof By direct calculation, we can see that $\sum_k h_k \widetilde{g}^n_{k+4j} = 0$ ($n = 2,3$), $\sum_k g^1_k \widetilde{g}^n_{k+4j} = 0$ ($n = 2,3$), $\sum_k g^n_k \widetilde{h}_{k+4j} = 0$ ($n = 2,3$), $\sum_k g^n_k \widetilde{g}^1_{k+4j} = 0$ ($n = 2,3$).

From $\sum_k h_k \widetilde{h}_{k+4j} = \delta_j$, we can obtain $\sum_k g^n_k \widetilde{g}^n_{k+4j} = \delta_j$ ($n = 1,2,3$). From $\sum_k g^1_k \widetilde{h}_{k+4j} = 0$, we can obtain $\sum_k h_k \widetilde{g}^1_{k+4j} = 0$, $\sum_k g^2_k \widetilde{g}^3_{k+4j} = 0$, $\sum_k g^3_k \widetilde{g}^2_{k+4j} = 0$. The proof is complete. □

Remark 2.1 According to Theorem 2.1, for $4L$ parameters in the wavelet system, (2.5) contains $2L$ nonlinear equations. We can add constraints such as high vanishing movements for the surplus $2L$ parameters.

Example 2.1 We relax the condition of the sum of highest vanishing moments, and let $L = 3$ and the high-pass wavelet filters defined by (2.3) have 3, 4, 3 order vanishing moments, respectively. The parameterized filters are as follows:

$$t_0 = \frac{-1}{10} \frac{49{,}152x^3 - 25{,}088x^2 + 4{,}120x - 219}{2{,}048x^2 - 960x + 113},$$

$$t_1 = \frac{-1}{40} \frac{65{,}536x^3 - 36{,}864x^2 + 6{,}240x - 307}{2{,}048x^2 - 960x + 113},$$

$$t_2 = \frac{1}{40} \frac{65{,}536x^3 - 16{,}384x^2 - 800x + 183}{2{,}048x^2 - 960x + 113},$$

$$t_3 = \frac{1}{20} \frac{98{,}304x^3 - 39{,}936x^2 + 4{,}720x - 153}{2{,}048x^2 - 960x + 113},$$

$$t_4 = \frac{1}{4} \frac{2{,}048x^2 - 1{,}216x + 169}{2{,}048x^2 - 960x + 113},$$

$$t_5 = \frac{1}{4} \frac{2{,}048x^2 - 1{,}216x + 177}{2{,}048x^2 - 960x + 113},$$

$$\widetilde{t}_0 = x - \frac{3}{16},$$

$$\widetilde{t}_1 = x - \frac{5}{32},$$

$$\widetilde{t}_2 = x - \frac{3}{32},$$

$$\widetilde{t}_3 = x,$$

$$\widetilde{t}_4 = -2x + \frac{11}{16},$$

$$\widetilde{t}_5 = -2x + \frac{3}{4},$$

where x is a free parameter.

3 The properties of the wavelet transformation matrix

A so-called 4-circular matrix [8], which is generated by the filters h, g^1, g^2, g^3, is denoted as M_{4n}. For $n = 3$, M_{12} is as follows:

$$M_{12} = \begin{bmatrix} h_0 & h_1 & h_2 & h_3 & 0 & 0 & 0 & 0 & h_{-4} & h_{-3} & h_{-2} & h_{-1} \\ h_{-4} & h_{-3} & h_{-2} & h_{-1} & h_0 & h_1 & h_2 & h_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_{-4} & h_{-3} & h_{-2} & h_{-1} & h_0 & h_1 & h_2 & h_3 \\ g_0^1 & g_1^1 & g_2^1 & g_3^1 & 0 & 0 & 0 & 0 & g_{-4}^1 & g_{-3}^1 & g_{-2}^1 & g_{-1}^1 \\ g_{-4}^1 & g_{-3}^1 & g_{-2}^1 & g_{-1}^1 & g_0^1 & g_1^1 & g_2^1 & g_3^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{-4}^1 & g_{-3}^1 & g_{-2}^1 & g_{-1}^1 & g_0^1 & g_1^1 & g_2^1 & g_3^1 \\ g_0^2 & g_1^2 & g_2^2 & g_3^2 & 0 & 0 & 0 & 0 & g_{-4}^2 & g_{-3}^2 & g_{-2}^2 & g_{-1}^2 \\ g_{-4}^2 & g_{-3}^2 & g_{-2}^2 & g_{-1}^2 & g_0^2 & g_1^2 & g_2^2 & g_3^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{-4}^2 & g_{-3}^2 & g_{-2}^2 & g_{-1}^2 & g_0^2 & g_1^2 & g_2^2 & g_3^2 \\ g_0^3 & g_1^3 & g_2^3 & g_3^3 & 0 & 0 & 0 & 0 & g_{-4}^3 & g_{-3}^3 & g_{-2}^3 & g_{-1}^3 \\ g_{-4}^3 & g_{-3}^3 & g_{-2}^3 & g_{-1}^3 & g_0^3 & g_1^3 & g_2^3 & g_3^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{-4}^3 & g_{-3}^3 & g_{-2}^3 & g_{-1}^3 & g_0^3 & g_1^3 & g_2^3 & g_3^3 \end{bmatrix}.$$

Define

$$
MM^T =
\begin{bmatrix}
HH^T & HG_1^T & HG_2^T & HG_3^T \\
G_1H^T & G_1G_1^T & G_1G_2^T & G_1G_3^T \\
G_2H^T & G_2G_1^T & G_2G_2^T & G_2G_3^T \\
G_3H^T & G_3G_1^T & G_3G_2^T & G_3G_3^T
\end{bmatrix},
$$

$$
\widetilde{M}\widetilde{M}^T =
\begin{bmatrix}
\widetilde{H}\widetilde{H}^T & \widetilde{H}\widetilde{G}_1^T & \widetilde{H}\widetilde{G}_2^T & \widetilde{H}\widetilde{G}_3^T \\
\widetilde{G}_1\widetilde{H}^T & \widetilde{G}_1\widetilde{G}_1^T & \widetilde{G}_1\widetilde{G}_2^T & \widetilde{G}_1\widetilde{G}_3^T \\
\widetilde{G}_2\widetilde{H}^T & \widetilde{G}_2\widetilde{G}_1^T & \widetilde{G}_2\widetilde{G}_2^T & \widetilde{G}_2\widetilde{G}_3^T \\
\widetilde{G}_3\widetilde{H}^T & \widetilde{G}_3\widetilde{G}_1^T & \widetilde{G}_3\widetilde{G}_2^T & \widetilde{G}_3\widetilde{G}_3^T
\end{bmatrix},
$$

(3.1)

where H, G_1, G_2, G_3 and $\widetilde{H}$, $\widetilde{G}_1$, $\widetilde{G}_2$, $\widetilde{G}_3$ are all $n \times 4n$ 4-circular matrices generated by $\{h, g^1, g^2, g^3\}$, $\{\widetilde{h}, \widetilde{g}^1, \widetilde{g}^2, \widetilde{g}^3\}$, respectively.

Theorem 3.1 *Assume that the type of wavelet filter banks defined by* (2.1)-(2.4) *satisfy* (2.5). *Then, for any large enough integer n, $M_n M_n^T$ and $\widetilde{M}_n \widetilde{M}_n^T$ defined by* (3.1) *are similar matrices.*

Proof The element at the jth row and the kth column in $G_1 G_1^T$ can be written as

$$
\sum_i g_{i+4j}^1 g_{i+4k}^1 = \sum_{i \in 2Z} (-1)^{(i+4j)/2} h_{(i+4j)+1} (-1)^{(i+4k)/2} h_{(i+4k)+1}
$$

$$
+ \sum_{i \in 2Z+1} (-1)^{(4L+1-(i+4j-1)/2)} h_{(i+4j)-1} (-1)^{(4L+1-(i+4k-1)/2)} h_{(i+4k)-1}
$$

$$
= \sum_{i \in 2Z} h_{i+4j+1} h_{i+4k+1} + \sum_{i \in 2Z+1} h_{i+4j-1} h_{i+4k-1}
$$

$$
= \sum_i h_{i+4j} h_{i+4k}.
$$

It is just the element in HH^T at the same position. Therefore,

$$
HG_1^T = G_1 H^T = G_2 G_3^T = G_3 G_2^T = 0,
$$

$$
HG_2^T = G_3 G_1^T = -G_2 H^T = -G_1 G_3^T,
$$

$$
HG_3^T = G_1 G_2^T = -G_2 G_1^T = -G_3 H^T,
$$

$$
G_2 G_2^T = G_3 G_3^T.
$$

Similarly, we can show that

$$
\widetilde{G}_1 \widetilde{G}_1^T = \widetilde{H}\widetilde{H}^T,
$$

$$
\widetilde{G}_2 \widetilde{G}_2^T = \widetilde{G}_3 \widetilde{G}_3^T,
$$

$$
\widetilde{H}\widetilde{G}_2^T = \widetilde{G}_3 \widetilde{G}_1^T = -\widetilde{G}_2 \widetilde{H}^T = -\widetilde{G}_1 \widetilde{G}_3^T,
$$

$$
\widetilde{H}\widetilde{G}_3^T = \widetilde{G}_1 \widetilde{G}_2^T = -\widetilde{G}_2 \widetilde{G}_1^T = -\widetilde{G}_3 \widetilde{H}^T,
$$

$$
\widetilde{H}\widetilde{G}_1^T = \widetilde{G}_1 \widetilde{H}^T = \widetilde{G}_2 \widetilde{G}_3^T = \widetilde{G}_3 \widetilde{G}_2^T = 0.
$$

The element at the jth row and the kth column in $G_2 G_2^T$ can be written as

$$\sum_i g_{i+4j}^2 g_{i+4k}^2 = \sum_i (-1)^{(i+4j)} \widetilde{h}_{4L-(i+4j)-1} (-1)^{(i+4k)} \widetilde{h}_{4L-(i+4k)-1}$$

$$= \sum_i \widetilde{h}_{4L-(i+4j)-1} \widetilde{h}_{4L-(i+4k)-1}$$

$$= \sum_i \widetilde{h}_{-i+4j} \widetilde{h}_{-i+4k}$$

$$= \sum_i \widetilde{h}_{i+4j} \widetilde{h}_{i+4k}.$$

It is just the element in $\widetilde{H}\widetilde{H}^T$ at the same position.

Therefore,

$$\widetilde{G}_2 \widetilde{G}_2^T = HH^T, \qquad HG_2^T = -\widetilde{H}\widetilde{G}_2^T, \qquad HG_3^T = -\widetilde{H}\widetilde{G}_3^T.$$

Define

$$B = \begin{bmatrix} O & O & O & I \\ O & O & -I & O \\ O & -I & O & O \\ I & O & O & O \end{bmatrix}, \qquad B^{-1} = \begin{bmatrix} O & O & O & I \\ O & O & -I & O \\ O & -I & O & O \\ I & O & O & O \end{bmatrix},$$

where I_n, O_n are $n \times n$ identity matrix and zero matrix, respectively, for convenience, we omit the subscript $n \times n$ when it does not cause any confusion.

We can see that $B_{4n}^{-1}(M_n M_n^T) B_{4n} = \widetilde{M}_n \widetilde{M}_n^T$. It implies that $M_n M_n^T$ and $\widetilde{M}_n \widetilde{M}_n^T$ are similar matrices. □

Corollary 3.1 *If λ is an eigenvalue of matrix $M_n M_n^T$, then $1/\lambda$ is also an eigenvalue of matrix $M_n M_n^T$.*

Proof Note that $M_n M_n^T (\widetilde{M}_n \widetilde{M}_n^T) = I_{4n}$. It implies that both $M_n M_n^T$ and $\widetilde{M}_n \widetilde{M}_n^T$ are positive definite matrices, and all of eigenvalues of $M_n M_n^T$ and $\widetilde{M}_n \widetilde{M}_n^T$ are positive. It follows from Theorem 3.1 that $M_n M_n^T$ and $\widetilde{M}_n \widetilde{M}_n^T$ have the same eigenvalues. The proof is completed. □

Now we can state the lower bound theorem as follows.

Theorem 3.2

$$\max_{1 \le i \le 4n} \lambda_i \ge \max \left\{ \frac{\sum_i h_i^2 + \sum_i \widetilde{h}_i^2}{2}, \frac{2}{\sum_i h_i^2 + \sum_i \widetilde{h}_i^2} \right\}. \tag{3.2}$$

Proof Assume that the type of wavelet filter banks defined by (2.1)-(2.4) satisfy (2.5). The elements at the diagonal of matrix $M_{4n} M_{4n}^T$ are $\sum_i h_i^2$ or $\sum_i g_i^2$.

Thus,

$$\sum_{i=1}^{4n} \lambda_i = 2n \left(\sum_i h_i^2 + \sum_i \widetilde{h}_i^2 \right).$$

According to Corollary 3.1, we have

$$\max_{i} \lambda_i \geq \frac{\sum_{i=1}^{4n} \lambda_i}{4n} \geq \min_{i} \lambda_i = \frac{1}{\max_i \lambda_i},$$

this leads to

$$\max_{1 \leq i \leq 4n} \lambda_i \geq \max\left\{ \frac{\sum_i h_i^2 + \sum_i \widetilde{h}_i^2}{2}, \frac{2}{\sum_i h_i^2 + \sum_i \widetilde{h}_i^2} \right\}.$$

The proof is complete.   □

Remark 3.1 Theorem 3.2 shows that there exists a lower bound on the maximum eigenvalue of the wavelet transform matrices which is only related to the filters and independent of the dimension of the transformation matrix.

4 Optimal model for 4-band biorthogonal wavelets bases

In the former section, it has been shown that the eigenvalues of $M_{4n}^T M_{4n}$ appear in pairs of reciprocal, $M_{4n}^T M_{4n}$ and $\widetilde{M}_{4n}^T \widetilde{M}_{4n}$ have the same eigenvalues. It is obvious that $\max\{\lambda_i\} = \max\{\widetilde{\lambda}_i\} = \frac{1}{\min\{\lambda_i\}} = \frac{1}{\min\{\lambda_i\}}$. In this section, we shall design wavelets based on minimizing the maximum eigenvalue.

Optimal model. Assume that the type of wavelet filter banks defined by (2.1)-(2.4) with parameter set Ω and $\lambda(\omega)$ are the eigenvalues of $M_{4n}^T M_{4n}$. The optimal model for 4-band biorthogonal wavelet bases is

$$\min_{\omega \in \Omega} \max_{1 \leq i \leq 4n} \lambda_i(\omega). \tag{4.1}$$

Example 4.1 For $L = 3$, the parameterized filters have been given in Example 2.1. We can use an extra degree of freedom to minimize the maximum eigenvalue. From (4.1), we can obtain the solution is $x \approx 0.11097$. The wavelets are denoted as Op(12-12). At this time,

$$t_0 = 0.01129264, \qquad t_1 = -0.01660958, \qquad t_2 = -0.01418315,$$

$$t_3 = 0.02102888, \qquad t_4 = 0.4676785, \qquad t_5 = 0.5307927,$$

$$\widetilde{t}_0 = -0.07653, \qquad \widetilde{t}_1 = -0.04528, \qquad \widetilde{t}_2 = 0.01722, \qquad \widetilde{t}_3 = 0.11097,$$

$$\widetilde{t}_4 = 0.46556, \qquad \widetilde{t}_5 = 0.52806.$$

See Figure 1 for the graphs of the scaling functions and wavelets in this example. We have

$$\lambda(M_{20}) = \{0.7775, 0.7775, 0.7775, 0.7775, 0.8555, 0.8555, 0.8555, 0.8555, 1, 1,$$

$$1, 1, 1.1689, 1.1689, 1.1689, 1.1689, 1.2863, 1.2863, 1.2863, 1.2863\}.$$

Figure 1 The graphs of Op(12-12) in Example 4.1.
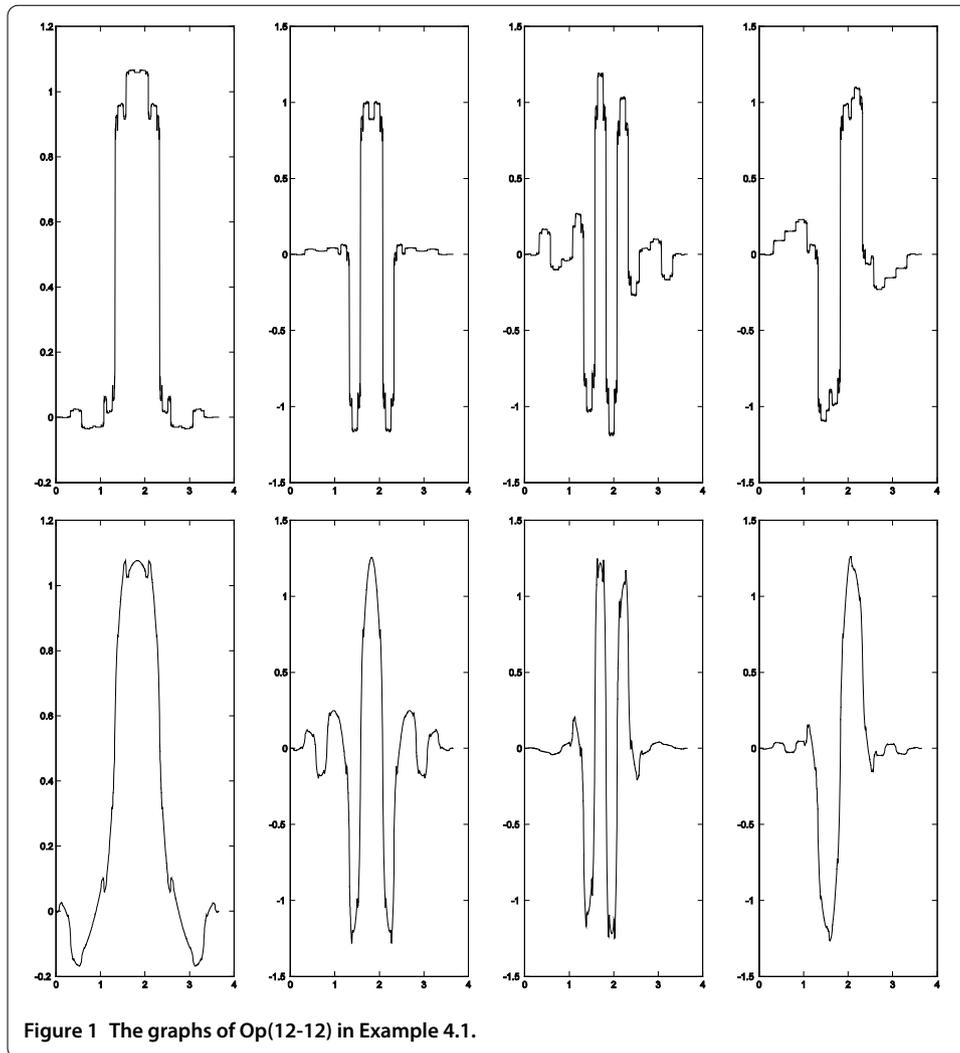
Note that $x = 1/9$; very near 0.11097. We can obtain the wavelets with rational filter banks as follows:

$$t_0 = 857/76{,}830, \qquad t_1 = -4{,}813{,}531{,}065{,}998{,}413/288{,}230{,}376{,}151{,}711{,}744,$$

$$t_2 = -2{,}081{,}007{,}309{,}269{,}241/144{,}115{,}188{,}075{,}855{,}872,$$

$$t_3 = 3{,}020{,}922{,}951{,}921{,}967/144{,}115{,}188{,}075{,}855{,}872,$$

$$t_4 = 4{,}793/10{,}244, \qquad t_5 = 5{,}441/10{,}244,$$

$$\widetilde{t}_0 = -11/144, \qquad \widetilde{t}_1 = -13/288, \qquad \widetilde{t}_2 = 5/288, \qquad \widetilde{t}_3 = 1/9,$$

$$\widetilde{t}_4 = 67/144, \qquad \widetilde{t}_5 = 19/36.$$

5 Conclusions and future work

We have constructed a class of 4-band symmetric biorthogonal wavelet bases, in which any wavelet system the high-pass filters can be determined by exchanging position and changing the sign of the two low-pass filters. The transformation matrix is studied and the optimal model is constructed. A concrete example with high vanishing moments is

also given which leads to highly efficient computations. We will further study the related topic that this wavelet bases are applied in numerical calculation and image compression coding.

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References
1. Daubechies, I: Ten Lectures on Wavelets. SIAM, Philadelphia (1992)
2. Wang, G: Matrix methods of constructing wavelet filters and discrete hyper-wavelet transforms. Opt. Eng. **39**, 1080-1087 (2000)
3. Wang, G, Yuan, W: Optimal model for biorthogonal wavelet filters. Opt. Eng. **42**, 350-356 (2003)
4. Wang, G, Zou, Q, Yang, M: Sub-band operators and saddle point wavelets. Appl. Math. Comput. **227**, 27-42 (2014)
5. Cui, L, Zhai, B, Zhang, T: Existence and design of biorthogonal matrix-valued wavelets. Nonlinear Anal., Real World Appl. **10**, 2679-2687 (2009)
6. Vetterli, M, Le, D: Perfect reconstruction FIR filter banks: some properties and factorizations. IEEE Trans. Acoust. Speech Signal Process. **37**, 1057-1071 (1989)
7. Vaidyanathan, P, Hong, P: Lattice structures for optimal design and robust implementation of two-band perfect reconstruction QMF banks. IEEE Trans. Acoust. Speech Signal Process. **36**, 81-94 (1988)
8. Zou, Q, Wang, G, Yang, M: Spectral radius of biorthogonal wavelets with its application. J. Korean Math. Soc. **51**, 941-953 (2014)
9. Cohen, A, Daubechies, I, Feauveau, J: Biorthogonal bases of compactly supported wavelets. Commun. Pure Appl. Math. **45**, 485-560 (1992)
10. Han, B: Symmetric orthonormal scaling functions and wavelets with dilation factor 4. Adv. Comput. Math. **8**, 221-247 (1998)
11. Bi, N, Dai, X, Sun, Q: Construction of compactly supported M-band wavelets. Appl. Comput. Harmon. Anal. **6**, 113-131 (1999)
12. Chui, C, Lian, J: Construction of compactly supported symmetric and antisymmetric orthonormal wavelets with scale = 3. Appl. Comput. Harmon. Anal. **2**, 68-84 (1995)
13. Jiang, Q: Biorthogonal wavelets with 4-fold axial symmetry for quadrilateral surface multiresolution processing. Adv. Comput. Math. **34**, 127-165 (2011)
14. He, T: Biorthogonal wavelets with certain regularities. Appl. Comput. Harmon. Anal. **11**, 227-242 (2001)
15. Zhuang, X: Matrix extension with symmetry and construction of biorthogonal multiwavelets with any integer dilation. Appl. Comput. Harmon. Anal. **33**, 159-181 (2012)
16. Wang, G: Four-bank compactly supported bi-symmetric orthonormal wavelets bases. Opt. Eng. **43**, 2362-2368 (2004)
17. Zou, Q, Wang, G: Construction of biorthogonal 3-band wavelets with symmetry and high vanishing moments. J. Math. **35**, 12-22 (2015)