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Nonexistence of stable F-stationary maps of a functional related to pullback metrics

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Abstract

Let M^m be a compact convex hypersurface in R^{m+1}. In this paper, we prove that if the principal curvatures λ_i of M^m satisfy $0 < \lambda_1 \leq \cdots \leq \lambda_m$ and $3\lambda_m < \sum_{j=1}^{m-1} \lambda_j$, then there exists no nonconstant stable F-stationary map between M and a compact Riemannian manifold when (6) or (7) holds.

MSC: 58E20; 53C21

Keywords: F-stationary map; compact convex hypersurfaces

1 Introduction

Let $u : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds (M^m, g) and (N^n, h). Recently, Kawai and Nakauchi [1] introduced a functional related to the pullback metric u^*h as follows:

$$\Phi(u) = \frac{1}{4} \int_M \|u^*h\|^2 \, dv_g, \tag{1}$$

(see [2–4]), where u^*h is the symmetric 2-tensor defined by

$$\bigl(u^*h\bigr)(X, Y) = h\bigl(du(X), du(Y)\bigr)$$

for any vector fields X, Y on M and $\|u^*h\|$ is given by

$$\|u^*h\|^2 = \sum_{i,j=1}^{m} \bigl[h\bigl(du(e_i), du(e_j)\bigr)\bigr]^2,$$

with respect to a local orthonormal frame $(e_1, \ldots, e_m)$ on (M, g). The map u is stationary for Φ if it is a critical point of $\Phi(u)$ with respect to any compact supported variation of u, and u is stable if the second variation for the functional $\Phi(u)$ is nonnegative. They showed the nonexistence of a nonconstant stable stationary map for Φ, either from S^m ($m \geq 5$) to any manifold, or from any compact Riemannian manifold to S^n ($n \geq 5$). In this paper, for a smooth function $F : [0, \infty) \to [0, \infty)$ such that $F(0) = 0$ and $F'(t) > 0$ on $t \in (0, \infty)$, we are concerned with the instability of F-stationary maps which is the generalization of a stationary map for Φ introduced by Asserda in [4]. In [4], they obtained some monotonicity formulas for F-stationary maps via the coarea formula and the comparison theorem. Also, by using monotonicity formulas, they got some Liouville type results for these maps.

The authors in [5] obtained the first and second variation formula for F-stationary maps. By using the second variation formula, they proved that every stable F-stationary map from $S^m(1)$ to any Riemannian manifold is constant if

$$\int_{S^m} \|u^*h\|^2 \left\{ F''\left(\frac{\|u^*h\|^2}{4}\right)\|u^*h\|^2 + (4-m)F'\left(\frac{\|u^*h\|^2}{4}\right)\right\} dv_g < 0, \tag{2}$$

or every F-stationary map from any compact Riemannian manifold N^n to S^m is constant if

$$\int_{N^n} \|u^*h\|^2 \left\{ F''\left(\frac{\|u^*h\|^2}{4}\right)\|u^*h\|^2 + (4-m)F'\left(\frac{\|u^*h\|^2}{4}\right)\right\} dv_g < 0. \tag{3}$$

In this paper, we obtain the results on the instability of F-stationary maps which are from or into the compact convex hypersurfaces in the Euclidean space.

2 Preliminaries

Let $F : [0,\infty) \to [0,\infty)$ be a C^2-function such that $F(0) = 0$ and $F'(t) > 0$ on $t \in (0,\infty)$. For a smooth map $u : (M,g) \to (N,h)$ between compact Riemannian manifolds (M,g) and (N,h) with Riemannian metrics g and h, respectively, following Ara [6] for an F-harmonic map (also see [7–10]), Asserda in [4] gave the following definition.

Definition 2.1 We call u an F-stationary map for Φ_F if

$$\frac{d}{dt}\Phi_F(u_t)|_{t=0} = 0$$

for any compactly supported variation $u_t : M \to N$ with $u_0 = u$, where

$$\Phi_F(u) = \int_{M^m} F\left(\frac{\|u^*h\|^2}{4}\right) dv_g.$$

Let ∇ and ${}^N\nabla$ always denote the Levi-Civita connections of M and N, respectively. Let $\widetilde{\nabla}$ be the induced connection on $u^{-1}TN$ defined by $\widetilde{\nabla}_X W =^N \nabla_{du(X)} W$, where X is a tangent vector of M and W is a section of $u^{-1}TN$. We choose a local orthonormal frame field $\{e_i\}$ on M. We define the F-tension field $\tau_{\Phi_F}(u)$ of u by

$$\begin{aligned}\tau_{\Phi_F}(u) &= -\delta\left(F'\left(\frac{\|u^*h\|^2}{4}\right)\sigma_u\right)\\ &= F'\left(\frac{\|u^*h\|^2}{4}\right)\operatorname{div}_g(\sigma_u) + \sigma_u\left(\operatorname{grad}\left(F'\left(\frac{\|u^*h\|^2}{4}\right)\right)\right),\end{aligned} \tag{4}$$

where $\sigma_u = \sum_j h(du(\cdot), du(e_j))du(e_j)$, which was defined in [1].

We need the following second variation formula for F-stationary maps (cf. [5]). Let $u : (M,g) \to (N,h)$ be an F-stationary map. Let $u_{s,t} : M \to N$ $(-\varepsilon < s, t < \varepsilon)$ be a compactly supported two-parameter variation such that $u_{0,0} = u$, and set $V = \frac{\partial}{\partial t}u_{s,t}|_{s,t=0}$, $W = \frac{\partial}{\partial s}u_{s,t}|_{s,t=0}$.

Then

$$\begin{aligned}
\frac{\partial^2}{\partial s\partial t}\Phi_F(u_{s,t})|_{s,t=0} &= \int_M F''\left(\frac{\|u^*h\|^2}{4}\right)\langle\widetilde{\nabla}V,\sigma_u\rangle\langle\widetilde{\nabla}W,\sigma_u\rangle\,dv_g \\
&\quad+\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i,j=1}^m h(\widetilde{\nabla}_{e_i}V,\widetilde{\nabla}_{e_j}W)h\big(du(e_i),du(e_j)\big)\,dv_g \\
&\quad+\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i,j=1}^m h\big(\widetilde{\nabla}_{e_i}V,du(e_j)\big)h\big(\widetilde{\nabla}_{e_i}W,du(e_j)\big)\,dv_g \\
&\quad+\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i,j=1}^m h\big(\widetilde{\nabla}_{e_i}V,du(e_j)\big)h\big(du(e_i),\widetilde{\nabla}_{e_j}W\big)\,dv_g \\
&\quad+\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)h\big(R^N\big(V,du(e_i)\big)W,du(e_j)\big)h\big(du(e_i),du(e_j)\big)\,dv_g,
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $T^*M\otimes u^{-1}TN$ and R^N is the curvature tensor of N.

We put

$$I(V,W)=\frac{\partial^2}{\partial s\partial t}\Phi_F(u_{s,t})|_{s,t=0}. \tag{5}$$

An F-stationary map u is called stable if $I(V,V)\geq 0$ for any compactly supported vector field V along u.

3 *F*-stationary maps from compact convex hypersurfaces

In this section, we obtain the following result.

Theorem 3.1 *Let $M\subset R^{m+1}$ be a compact convex hypersurface. Assume that the principal curvatures λ_i of M^m satisfy $0<\lambda_1\leq\cdots\leq\lambda_m$ and $3\lambda_m<\sum_{i=1}^{m-1}\lambda_i$. Then every nonconstant F-stationary map from M to any compact Riemannian manifold N is unstable if there exists a constant $c_F=\inf\{c\geq 0|F'(t)/t^c$ is nonincreasing$\}$ such that*

$$c_F<\frac{1}{4\lambda_m^2}\min_{1\leq i\leq m}\left\{\lambda_i\left(\sum_{k=1}^m\lambda_k-2\lambda_i-2\lambda_m\right)\right\} \tag{6}$$

or when $F''(t)=F'(t)$ (for example, $F(t)=\exp(t)$)

$$\|u^*h\|^2<\frac{1}{\lambda_m^2}\min_{1\leq i\leq m}\left\{\lambda_i\left(\sum_{k=1}^m\lambda_k-2\lambda_i-2\lambda_m\right)\right\}. \tag{7}$$

Proof In order to prove the instability of $u:M^m\to N$, we need to consider some special variational vector fields along u. To do this, we choose an orthonormal field $\{e_i,e_{m+1}\}$, $i=1,\ldots,m$, of R^{m+1} such that $\{e_i\}$ are tangent to $M^m\subset R^{m+1}$, e_{m+1} is normal to M^m and $\nabla_{e_i}e_j|_P=0$. Meanwhile, we take a fixed orthonormal basis E_A, $A=1,\ldots,m+1$, of R^{m+1} and set

$$V_A=\sum_{i=1}^m v_A^i e_i,\qquad v_A^i=\langle E_A,e_i\rangle, v_A^{m+1}=\langle E_A,e_{m+1}\rangle, \tag{8}$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical Euclidean inner product. Then $du(V_A) \in \Gamma(u^{-1}TN)$ and

$$\sum_A v_A^i v_A^j = \sum_A \langle E_A, e_i\rangle\langle E_A, e_j\rangle = \delta_{ij}, \tag{9}$$

$$\nabla_{e_i} V_A = v_A^{m+1} B_{ij} e_j, \tag{10}$$

$$\nabla_{e_i}(\nabla_{e_i} V_A) = -v_A^k B_{ik} B_{ij} e_j + v_A^{m+1}(\nabla_{e_i} h_{ij}) e_j, \tag{11}$$

$$\begin{aligned}\widetilde{\nabla}_{e_i}\big(du(\nabla_{e_i} V_A)\big) &= -v_A^k B_{ik} B_{ij} du(e_j)\\ &\quad + v_A^{m+1}(\nabla_{e_i} B_{ij}) du(e_j) + v_A^{m+1} B_{ij} \widetilde{\nabla}_{e_i} du(e_j),\end{aligned} \tag{12}$$

where B_{ij} denotes the components of the second fundamental form of M^m in R^{m+1}. Suppose that $u : M^m \to N$ is a nonconstant F-stationary map. Then the condition $\tau_F(u) = -\delta(F'(\frac{\|u^*h\|^2}{4})\sigma_u) = 0$ implies that

$$\begin{aligned}&\sum_A \int_{M^m} F'\left(\frac{\|u^*h\|^2}{4}\right)\big\langle(\triangle du)(V_A), \sigma_u(V_A)\big\rangle\, dv_g\\ &\quad= \sum_A \int_{M^m} F'\left(\frac{\|u^*h\|^2}{4}\right) v_A^i v_A^j \big\langle(\triangle du)(e_i), \sigma_u(e_j)\big\rangle\, dv_g\\ &\quad= \sum_i \int_{M^m} F'\left(\frac{\|u^*h\|^2}{4}\right)\big\langle(\triangle du)(e_i), \sigma_u(e_i)\big\rangle\, dv_g\\ &\quad= \int_{M^m} F'\left(\frac{\|u^*h\|^2}{4}\right)\big\langle(\triangle du), \sigma_u\big\rangle\, dv_g\\ &\quad= \int_{M^m} \left\langle \delta du, \delta\left(F'\left(\frac{\|u^*h\|^2}{4}\right)\sigma_u\right)\right\rangle dv_g = 0.\end{aligned} \tag{13}$$

It follows from the Weitzenböck formula that

$$-\sum_{k=1}^m R^N\big(du(X), du(e_k)\big)du(e_k) + du\big(\mathrm{Ric}^M(X)\big) = \triangle du(X) + \widetilde{\nabla}^2 du(X), \tag{14}$$

where X is any smooth vector field on M^m. With respect to the variational vector field $du(V_A)$ along u, it follows from (13) and (14) that

$$\begin{aligned}&\sum_A I\big(du(V_A), du(V_A)\big)\\ &\quad= \int_M F''\left(\frac{\|u^*h\|^2}{4}\right)\sum_A \big\langle\widetilde{\nabla} du(V_A), \sigma_u\big\rangle^2\, dv_g\\ &\qquad+ \int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i,j,A} h\big(\widetilde{\nabla}_{e_i} du(V_A), \widetilde{\nabla}_{e_j} du(V_A)\big) h\big(du(e_i), du(e_j)\big)\, dv_g\\ &\qquad+ \int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i,j,A} h\big(\widetilde{\nabla}_{e_i} du(V_A), du(e_j)\big) h\big(\widetilde{\nabla}_{e_i} du(V_A), du(e_j)\big)\, dv_g\\ &\qquad+ \int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i,j,A} h\big(\widetilde{\nabla}_{e_i} du(V_A), du(e_j)\big) h\big(du(e_i), \widetilde{\nabla}_{e_j} du(V_A)\big)\, dv_g\end{aligned}$$

$$
\begin{aligned}
&-\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_A h\big(du\big(\mathrm{Ric}^{M^m}(V_A)\big),\sigma_u(V_A)\big)\,dv_g \\
&+\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_A h\big(\big(\widetilde{\nabla}^2 du\big)(V_A),\sigma_u(V_A)\big)\,dv_g.
\end{aligned}
\tag{15}
$$

For any fixed point $P \in M$, choose $\{e_i\}$ such that $\nabla_{e_i} e_j|_P = 0$. We have

$$
\widetilde{\nabla}^2 du(V_A) = \widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i}\big(du(V_A)\big) - 2\widetilde{\nabla}_{e_i}\big(du(\nabla_{e_i}V_A)\big) + du(\nabla_{e_i}\nabla_{e_i}V_A)
\tag{16}
$$

and

$$
\begin{aligned}
&\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{A,i} h\big(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} du(V_A),\sigma_u(V_A)\big)\,dv_g \\
&= -\int_M \sum_{A,i} h\left(\widetilde{\nabla}_{e_i} du(V_A),\widetilde{\nabla}_{e_i}\left[F'\left(\frac{\|u^*h\|^2}{4}\right)\sigma_u(V_A)\right]\right)dv_g \\
&= -\int_M \sum_{A,i} h\left(\widetilde{\nabla}_{e_i} du(V_A),\widetilde{\nabla}_{e_i}\left[F'\left(\frac{\|u^*h\|^2}{4}\right)\right]\sigma_u(V_A)\right)dv_g \\
&\quad -\int_M \sum_{A,i} h\left(\widetilde{\nabla}_{e_i} du(V_A),F'\left(\frac{\|u^*h\|^2}{4}\right)\widetilde{\nabla}_{e_i}\sigma_u(V_A)\right)dv_g \\
&= -\int_M \sum_{A,i} h\left(\widetilde{\nabla}_{e_i} du(V_A),\widetilde{\nabla}_{e_i}\left[F'\left(\frac{\|u^*h\|^2}{4}\right)\right]\sigma_u(V_A)\right)dv_g \\
&\quad -\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{A,i,j} h\big(\widetilde{\nabla}_{e_i} du(V_A),\widetilde{\nabla}_{e_i} du(e_j)\big)h\big(du(V_A),du(e_j)\big)\,dv_g \\
&\quad -\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{A,i,j} h\big(\widetilde{\nabla}_{e_i} du(V_A),du(e_j)\big)h\big(\widetilde{\nabla}_{e_i} du(V_A),du(e_j)\big)\,dv_g \\
&\quad -\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{A,i,j} h\big(\widetilde{\nabla}_{e_i} du(V_A),du(e_j)\big)h\big(du(V_A),\widetilde{\nabla}_{e_i} du(e_j)\big)\,dv_g.
\end{aligned}
\tag{17}
$$

Substituting (16) and (17) into (15), we have

$$
\begin{aligned}
&\sum_A I\big(du(V_A),du(V_A)\big) \\
&= \int_M \left\{F''\left(\frac{\|u^*h\|^2}{4}\right)\sum_A \big\langle\widetilde{\nabla} du(V_A),\sigma_u\big\rangle^2 \right.\\
&\quad \left.- h\left(\widetilde{\nabla}_{e_i} du(V_A),\widetilde{\nabla}_{e_i}\left[F'\left(\frac{\|u^*h\|^2}{4}\right)\right]\sigma_u(V_A)\right)\right\}dv_g \\
&\quad +\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)h\big(-2\widetilde{\nabla}_{e_i}\big(du(\nabla_{e_i}V_A)\big) \\
&\quad + du(\nabla_{e_i}\nabla_{e_i}V_A) - du\big(\mathrm{Ric}^{M^m}(V_A)\big),\sigma_u(V_A)\big)\,dv_g \\
&\quad +\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i,j,A} h\big(\widetilde{\nabla}_{e_i} du(V_A),\widetilde{\nabla}_{e_j} du(V_A)\big)h\big(du(e_i),du(e_j)\big)\,dv_g
\end{aligned}
$$

$$
\begin{aligned}
&+\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i,j,A} h\big(\widetilde{\nabla}_{e_i}du(V_A),du(e_j)\big)h\big(du(e_i),\widetilde{\nabla}_{e_j}du(V_A)\big)\,dv_g\\
&-\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{A,i,j} h\big(\widetilde{\nabla}_{e_i}du(V_A),\widetilde{\nabla}_{e_i}du(e_j)\big)h\big(du(V_A),du(e_j)\big)\,dv_g\\
&-\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{A,i,j} h\big(\widetilde{\nabla}_{e_i}du(V_A),du(e_j)\big)h\big(du(V_A),\widetilde{\nabla}_{e_i}du(e_j)\big)\,dv_g.
\end{aligned}
\tag{18}
$$

In the following, we shall estimate each term in (18). Because trace is independent of the choice of orthonormal basis, we can take pointwisely $\{e_i, e_{m+1}\}$ such that $B_{ij} = \lambda_i\delta_{ij}$.

A straightforward computation shows

$$
\begin{aligned}
&\sum_A h\left(\widetilde{\nabla}_{e_i}du(V_A),\widetilde{\nabla}_{e_i}\left[F'\left(\frac{\|u^*h\|^2}{4}\right)\right]\sigma_u(V_A)\right)\\
&\quad=\sum_A F''\left(\frac{\|u^*h\|^2}{4}\right)\widetilde{\nabla}_{e_i}\left(\frac{\|u^*h\|^2}{4}\right)h\big(v_A^{m+1}B_{ik}du(e_k)+v_A^k\widetilde{\nabla}_{e_i}du(e_k),v_A^l\sigma_u(e_l)\big)\\
&\quad=F''\left(\frac{\|u^*h\|^2}{4}\right)\widetilde{\nabla}_{e_i}\left(\frac{\|u^*h\|^2}{4}\right)h\big(\widetilde{\nabla}_{e_i}du(e_k),\sigma_u(e_k)\big)\\
&\quad=F''\left(\frac{\|u^*h\|^2}{4}\right)\langle\widetilde{\nabla}_{e_i}du,\sigma_u\rangle^2
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
&\sum_A F''\left(\frac{\|u^*h\|^2}{4}\right)\big\langle\widetilde{\nabla}du(V_A),\sigma_u\big\rangle^2\\
&\quad=\sum_A F''\left(\frac{\|u^*h\|^2}{4}\right)\big\langle v_A^{m+1}B_{ik}du(e_k)+v_A^k\widetilde{\nabla}_{e_i}du(e_k),\sigma_u(e_i)\big\rangle^2\\
&\quad=\sum_A F''\left(\frac{\|u^*h\|^2}{4}\right)\big\{B_{ik}B_{jl}h\big(du(e_k),\sigma_u(e_i)\big)h\big(du(e_l),\sigma_u(e_j)\big)\\
&\qquad+h\big(\widetilde{\nabla}_{e_i}du(e_k),\sigma_u(e_i)\big)h\big(\widetilde{\nabla}_{e_j}du(e_k),\sigma_u(e_j)\big)\big\}\\
&\quad=\sum_A F''\left(\frac{\|u^*h\|^2}{4}\right)\big\{\lambda_i\lambda_j h\big(du(e_i),\sigma_u(e_i)\big)h\big(du(e_j),\sigma_u(e_j)\big)+\langle\widetilde{\nabla}_{e_i}du,\sigma_u\rangle^2\big\}.
\end{aligned}
\tag{20}
$$

Then it follows from (19) and (20) that

$$
\begin{aligned}
&\int_M\left\{F''\left(\frac{\|u^*h\|^2}{4}\right)\sum_A\big\langle\widetilde{\nabla}du(V_A),\sigma_u\big\rangle^2\right.\\
&\qquad\left.-h\left(\widetilde{\nabla}_{e_i}du(V_A),\widetilde{\nabla}_{e_i}\left[F'\left(\frac{\|u^*h\|^2}{4}\right)\right]\sigma_u(V_A)\right)\right\}dv_g\\
&=\int_M F''\left(\frac{\|u^*h\|^2}{4}\right)\lambda_i\lambda_j h\big(du(e_i),\sigma_u(e_i)\big)h\big(du(e_j),\sigma_u(e_j)\big)\,dv_g.
\end{aligned}
\tag{21}
$$

From the Gauss equation it follows that

$$
\mathrm{Ric}^M(V_A)=v_A^i(B_{kk}B_{ij}-B_{ik}B_{jk})e_j.
\tag{22}
$$

Using (10), (11),(12) and (22), we have

$$\begin{aligned}
&\int_M F'\left(\frac{\|u^*h\|^2}{4}\right) h\big(-2\widetilde{\nabla}_{e_i}\big(du(\nabla_{e_i}V_A)\big)\\
&\quad + du(\nabla_{e_i}\nabla_{e_i}V_A) - du\big(\mathrm{Ric}^{M^m}(V_A)\big), \sigma_u(V_A)\big)\, dv_g\\
&= \int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\big\{\big[h\big(2v_A^k B_{ik}B_{ij}du(e_j) - v_A^{m+1}\nabla_{e_i}(B_{ij})du(e_j)\\
&\quad - v_A^{m+1}B_{ij}\widetilde{\nabla}_{e_i}du(e_j), v_A^l\sigma_u(e_l)\big)\big]\\
&\quad + h\big(-v_A^k B_{ik}B_{ij}du(e_j) + v_A^{m+1}(\nabla_{e_i}B_{ij})du(e_j), v_A^l\sigma_u(e_l)\big)\\
&\quad + h\big(v_A^k B_{ik}B_{ij}du(e_j) - v_A^i B_{kk}B_{ij}du(e_j), v_A^l\sigma_u(e_l)\big)\big\}\, dv_g\\
&= \int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\big\{h\big(2v_A^k B_{ik}B_{ij}du(e_j) - v_A^i B_{kk}B_{ij}du(e_j), v_A^l\sigma_u(e_l)\big)\big\}\, dv_g\\
&= \int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_i\left\{\left[2\lambda_i - \left(\sum_k \lambda_k\right)\right]\lambda_i h\big(du(e_i), \sigma_u(e_i)\big)\right\} dv_g.
\end{aligned}\tag{23}$$

A straightforward computation shows

$$\begin{aligned}
&\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i,j,A} h\big(\widetilde{\nabla}_{e_i}du(V_A), \widetilde{\nabla}_{e_j}du(V_A)\big)h\big(du(e_i), du(e_j)\big)\, dv_g\\
&= \int_M F'\left(\frac{\|u^*h\|^2}{4}\right) h\big(v_A^{m+1}B_{ik}du(e_k) + v_A^k\widetilde{\nabla}_{e_i}du(e_k),\\
&\quad v_A^{m+1}B_{jl}du(e_l) + v_A^l\widetilde{\nabla}_{e_j}du(e_l)\big)h\big(du(e_i), du(e_j)\big)\, dv_g\\
&= \int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\big\{B_{ik}B_{jl}h\big(du(e_k), du(e_l)\big)h\big(du(e_i), du(e_j)\big)\\
&\quad + h\big(\widetilde{\nabla}_{e_i}du(e_k), \widetilde{\nabla}_{e_j}du(e_k)\big)h\big(du(e_i), du(e_j)\big)\big\}\, dv_g\\
&= \int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\big\{\lambda_i\lambda_j h\big(du(e_i), du(e_j)\big)h\big(du(e_i), du(e_j)\big)\\
&\quad + h\big(\widetilde{\nabla}_{e_i}du(e_k), \widetilde{\nabla}_{e_j}du(e_k)\big)h\big(du(e_i), du(e_j)\big)\big\}\, dv_g
\end{aligned}\tag{24}$$

and

$$\begin{aligned}
&\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i,j,A} h\big(\widetilde{\nabla}_{e_i}du(V_A), du(e_j)\big)h\big(du(e_i), \widetilde{\nabla}_{e_j}du(V_A)\big)\, dv_g\\
&= \int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\big\{h\big(v_A^{m+1}B_{ik}du(e_k) + v_A^k\widetilde{\nabla}_{e_i}du(e_k), du(e_j)\big)\\
&\quad \times h\big(du(e_i), v_A^{m+1}B_{jk}du(e_k) + v_A^k\widetilde{\nabla}_{e_j}du(e_k)\big)\big\}\, dv_g\\
&= \int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\big\{\lambda_i\lambda_j h\big(du(e_i), du(e_j)\big)h\big(du(e_i), du(e_j)\big)\\
&\quad + h\big(\widetilde{\nabla}_{e_i}du(e_k), du(e_j)\big)h\big(\widetilde{\nabla}_{e_j}du(e_k), du(e_i)\big)\big\}\, dv_g
\end{aligned}\tag{25}$$

and

$$\begin{aligned}
&\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{A,i,j} h\big(\widetilde{\nabla}_{e_i}du(V_A),\widetilde{\nabla}_{e_i}du(e_j)\big)h\big(du(V_A),du(e_j)\big)\,dv_g \\
&\quad=\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\big\{h\big(v_A^{m+1}B_{ik}du(e_k)+v_A^k\widetilde{\nabla}_{e_i}du(e_k),\widetilde{\nabla}_{e_i}du(e_j)\big) \\
&\qquad\times h\big(v_A^l du(e_l),du(e_j)\big)\big\}\,dv_g \\
&\quad=\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)h\big(\widetilde{\nabla}_{e_i}du(e_k),\widetilde{\nabla}_{e_i}du(e_j)\big)h\big(du(e_k),du(e_j)\big)\,dv_g
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
&\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{A,i,j} h\big(\widetilde{\nabla}_{e_i}du(V_A),du(e_j)\big)h\big(du(V_A),\widetilde{\nabla}_{e_i}du(e_j)\big)\,dv_g \\
&\quad=\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\big\{h\big(v_A^{m+1}B_{ik}du(e_k)+v_A^k\widetilde{\nabla}_{e_i}du(e_k),du(e_j)\big) \\
&\qquad\times h\big(v_A^l du(e_l),\widetilde{\nabla}_{e_i}du(e_j)\big)\big\}\,dv_g \\
&\quad=\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)h\big(\widetilde{\nabla}_{e_i}du(e_k),du(e_j)\big)h\big(du(e_k),\widetilde{\nabla}_{e_i}du(e_j)\big)\,dv_g.
\end{aligned} \tag{27}$$

From (18), (21), (23), (24), (25), (26), (27) and $\widetilde{\nabla}_{e_i}du(e_j)=\widetilde{\nabla}_{e_j}du(e_i)$, we obtain

$$\begin{aligned}
&\sum_A I\big(du(V_A),du(V_A)\big) \\
&\quad=\int_M F''\left(\frac{\|u^*h\|^2}{4}\right)\lambda_i\lambda_j h\big(du(e_i),\sigma_u(e_i)\big)h\big(du(e_j),\sigma_u(e_j)\big)\,dv_g \\
&\qquad+\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_i\left\{\left[2\lambda_i-\left(\sum_k\lambda_k\right)\right]\lambda_i h\big(du(e_i),\sigma_u(e_i)\big)\right\}dv_g \\
&\qquad+2\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\lambda_i\lambda_j h\big(du(e_i),du(e_j)\big)h\big(du(e_i),du(e_j)\big)\,dv_g \\
&\quad\le\int_M F''\left(\frac{\|u^*h\|^2}{4}\right)\lambda_i\lambda_j h\big(du(e_i),\sigma_u(e_i)\big)h\big(du(e_j),\sigma_u(e_j)\big)\,dv_g \\
&\qquad+\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_i\left\{\left[2\lambda_i-\left(\sum_k\lambda_k\right)\right]\lambda_i h\big(du(e_i),\sigma_u(e_i)\big)\right\}dv_g \\
&\qquad+2\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\lambda_i\lambda_m h\big(du(e_i),\sigma_u(e_i)\big)\,dv_g \\
&\quad=\int_M F''\left(\frac{\|u^*h\|^2}{4}\right)\lambda_i\lambda_j h\big(du(e_i),\sigma_u(e_i)\big)h\big(du(e_j),\sigma_u(e_j)\big)\,dv_g \\
&\qquad+\int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_i\left\{\left[2\lambda_i+2\lambda_m-\left(\sum_k\lambda_k\right)\right]\lambda_i h\big(du(e_i),\sigma_u(e_i)\big)\right\}dv_g.
\end{aligned} \tag{28}$$

If $F''(t) = F'(t)$, then (28) leads to the following inequality:

$$\begin{aligned}
&\sum_A I\bigl(du(V_A), du(V_A)\bigr) \\
&\quad \le \int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\lambda_m^2 \left\|u^*h\right\|^4 \, dv_g \\
&\qquad + \int_M F'\left(\frac{\|u^*h\|^2}{4}\right) \max_{1\le i\le m}\left\{\left[2\lambda_i + 2\lambda_m - \left(\sum_k \lambda_k\right)\right]\lambda_i\right\} \left\|u^*h\right\|^2 \, dv_g \\
&\quad = \int_M F'\left(\frac{\|u^*h\|^2}{4}\right) \left\|u^*h\right\|^2 \left\{\lambda_m^2 \left\|u^*h\right\|^2\right. \\
&\qquad \left. + \max_{1\le i\le m}\left\{\left[2\lambda_i + 2\lambda_m - \left(\sum_k \lambda_k\right)\right]\lambda_i\right\}\right\} dv_g. \qquad (29)
\end{aligned}$$

If there exists a constant c_F such that $\frac{F'(t)}{t^{c_F}}$ is nonincreasing, it follows that $F''(t)t \le c_F F'(t)$ on $t \in (0,\infty)$, thus (28) implies

$$\begin{aligned}
&\sum_A I\bigl(du(V_A), du(V_A)\bigr) \\
&\quad \le \int_M 4c_F F'\left(\frac{\|u^*h\|^2}{4}\right)\lambda_m^2 \left\|u^*h\right\|^2 \, dv_g \\
&\qquad + \int_M F'\left(\frac{\|u^*h\|^2}{4}\right) \max_{1\le i\le m}\left\{\left[2\lambda_i + 2\lambda_m - \left(\sum_k \lambda_k\right)\right]\lambda_i\right\} \left\|u^*h\right\|^2 \, dv_g \\
&\quad = \int_M F'\left(\frac{\|u^*h\|^2}{4}\right) \left\|u^*h\right\|^2 \left\{4c_F\lambda_m^2\right. \\
&\qquad \left. + \max_{1\le i\le m}\left\{\left[2\lambda_i + 2\lambda_m - \left(\sum_k \lambda_k\right)\right]\lambda_i\right\}\right\} dv_g. \qquad (30)
\end{aligned}$$

If u is nonconstant and (6) or (7) holds, we have

$$\sum_A I\bigl(du(V_A), du(V_A)\bigr) < 0 \qquad (31)$$

and u is unstable. □

Corollary 3.2 *Let $u : S^m \to N$ be a nonconstant F-stationary map and $m > 4$. If $c_F < \frac{m}{4} - 1$ or $\|u^*h\|^2 < m - 4$, then u is unstable.*

4 *F*-stationary maps into compact convex hypersurfaces

In this section, we obtain the following result.

Theorem 4.1 *With the same assumption on M^m as in Theorem* 3.1, *every nonconstant F-stationary map from any compact Riemannian manifold N to M^m is unstable if* (6) *or* (7) *holds.*

Proof In order to prove the instability of $u : N^n \to M^m$, we need to consider some special variational vector fields along u. To do this, we choose an orthonormal field $\{\epsilon_\alpha, \epsilon_{m+1}\}$,

$\alpha = 1, \dots, m$, of R^{m+1} such that $\{\epsilon_\alpha\}$ are tangent to $M^m \subset R^{m+1}$, ϵ_{m+1} is normal to M^m, ${}^{M^m}\nabla_{\epsilon_\alpha}\epsilon_\beta|_P = 0$ and $B_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}$, where $B_{\alpha\beta}$ denotes the components of the second fundamental form of M^m in R^{m+1}. Meanwhile, take a fixed orthonormal basis E_A, $A = 1, \dots, m+1$, of R^{m+1} and set

$$V_A = \sum_{\alpha=1}^{m} v_A^\alpha \epsilon_\alpha, \quad v_A^\alpha = \langle E_A, \epsilon_\alpha\rangle, v_A^{m+1} = \langle E_A, \epsilon_{m+1}\rangle, \tag{32}$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical Euclidean inner product. We shall consider the second variation

$$\begin{aligned}
\sum_A I(V_A, V_A) = {} & \int_N F''\left(\frac{\|u^*h\|^2}{4}\right)\langle\widetilde{\nabla} V_A, \sigma_u\rangle\langle\widetilde{\nabla} V_A, \sigma_u\rangle\, dv_g \\
& + \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i,j=1}^{m} h(\widetilde{\nabla}_{e_i} V_A, \widetilde{\nabla}_{e_j} V_A) h\big(du(e_i), du(e_j)\big)\, dv_g \\
& + \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i,j=1}^{m} h\big(\widetilde{\nabla}_{e_i} V_A, du(e_j)\big) h\big(\widetilde{\nabla}_{e_i} V_A, du(e_j)\big)\, dv_g \\
& + \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i,j=1}^{m} h\big(\widetilde{\nabla}_{e_i} V_A, du(e_j)\big) h\big(du(e_i), \widetilde{\nabla}_{e_j} V_A\big)\, dv_g \\
& + \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i} h\big(R^{M^m}\big(V_A, du(e_i)\big) V_A, \sigma_u(e_i)\big)\, dv_g,
\end{aligned} \tag{33}$$

where $\{e_1, \dots, e_n\}$ is the local orthonormal frame of N^n.

Firstly, we compute the first term of (33)

$$\begin{aligned}
& \sum_A \int_N F''\left(\frac{\|u^*h\|^2}{4}\right)\langle\widetilde{\nabla} V_A, \sigma_u\rangle\langle\widetilde{\nabla} V_A, \sigma_u\rangle\, dv_g \\
& = \int_N F''\left(\frac{\|u^*h\|^2}{4}\right)\left[\sum_i h\big(\widetilde{\nabla}_{e_i} V_A, \sigma_u(e_i)\big)\right]^2 dv_g \\
& = \int_N F''\left(\frac{\|u^*h\|^2}{4}\right)\left[\sum_i h\big({}^{M^m}\nabla_{du(e_i)} V_A, \sigma_u(e_i)\big)\right]^2 dv_g \\
& = \int_N F''\left(\frac{\|u^*h\|^2}{4}\right)\left[\sum_i u_i^\alpha h\big({}^{M^m}\nabla_{\epsilon_\alpha} V_A, \sigma_u(e_i)\big)\right]^2 dv_g \\
& = \int_N F''\left(\frac{\|u^*h\|^2}{4}\right)\left[\sum_i v_A^{m+1} u_i^\alpha B_{\alpha\beta} h\big(\epsilon_\beta, \sigma_u(e_i)\big)\right]^2 dv_g \\
& = \int_N F''\left(\frac{\|u^*h\|^2}{4}\right)\left[\sum_i v_A^{m+1} u_i^\alpha \lambda_\alpha h\big(\epsilon_\alpha, \sigma_u(e_i)\big)\right]^2 dv_g \\
& = \int_N F''\left(\frac{\|u^*h\|^2}{4}\right)\lambda_\alpha\lambda_\beta h\big(u_i^\alpha \epsilon_\alpha, \sigma_u(e_i)\big) h\big(u_j^\beta \epsilon_\beta, \sigma_u(e_j)\big)\, dv_g.
\end{aligned} \tag{34}$$

The second term of (33)

$$\begin{aligned}
&\sum_A \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i,j=1}^m h(\widetilde{\nabla}_{e_i} V_A, \widetilde{\nabla}_{e_j} V_A) h\big(du(e_i), du(e_j)\big)\, dv_g \\
&\quad = \int_N F'\left(\frac{\|u^*h\|^2}{4}\right) h\big({}^{M^m}\nabla_{du(e_i)} V_A, {}^{M^m}\nabla_{du(e_j)} V_A\big) h\big(du(e_i), du(e_j)\big)\, dv_g \\
&\quad = \int_N F'\left(\frac{\|u^*h\|^2}{4}\right) u_i^\alpha u_j^\beta h\big({}^{M^m}\nabla_{\epsilon_\alpha} V_A, {}^{M^m}\nabla_{\epsilon_\beta} V_A\big) h\big(du(e_i), du(e_j)\big)\, dv_g \\
&\quad = \int_N F'\left(\frac{\|u^*h\|^2}{4}\right) u_i^\alpha u_j^\beta B_{\alpha\gamma} B_{\beta\delta} h(\epsilon_\gamma, \epsilon_\delta) h\big(du(e_i), du(e_j)\big)\, dv_g \\
&\quad = \int_N F'\left(\frac{\|u^*h\|^2}{4}\right) \lambda_\alpha \lambda_\beta h\big(u_i^\alpha \epsilon_\alpha, u_j^\beta \epsilon_\beta\big) h\big(du(e_i), du(e_j)\big)\, dv_g \\
&\quad = \int_N F'\left(\frac{\|u^*h\|^2}{4}\right) \lambda_\alpha^2 h\big(u_i^\alpha \epsilon_\alpha, du(e_j)\big) h\big(du(e_i), du(e_j)\big)\, dv_g.
\end{aligned} \tag{35}$$

The third term of (33)

$$\begin{aligned}
&\sum_A \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i,j=1}^m h\big(\widetilde{\nabla}_{e_i} V_A, du(e_j)\big) h\big(\widetilde{\nabla}_{e_i} V_A, du(e_j)\big)\, dv_g \\
&\quad = \int_N F'\left(\frac{\|u^*h\|^2}{4}\right) \lambda_\alpha \lambda_\beta h\big(u_i^\alpha \epsilon_\alpha, du(e_j)\big) h\big(u_i^\beta \epsilon_\beta, du(e_j)\big)\, dv_g.
\end{aligned} \tag{36}$$

The fourth term of (33)

$$\begin{aligned}
&\sum_A \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_{i,j=1}^m h\big(\widetilde{\nabla}_{e_i} V_A, du(e_j)\big) h\big(du(e_i), \widetilde{\nabla}_{e_j} V_A\big)\, dv_g \\
&\quad = \int_N F'\left(\frac{\|u^*h\|^2}{4}\right) \lambda_\alpha \lambda_\beta h\big(u_i^\alpha \epsilon_\alpha, du(e_j)\big) h\big(du(e_i), u_j^\beta \epsilon_\beta\big)\, dv_g.
\end{aligned} \tag{37}$$

The fifth term of (33)

$$\begin{aligned}
&\sum_A \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)\sum_i h\big(R^{M^m}\big(V_A, du(e_i)\big) V_A, \sigma_u(e_i)\big)\, dv_g \\
&\quad = \int_N F'\left(\frac{\|u^*h\|^2}{4}\right) v_A^\alpha v_A^\beta h\big(R^{M^m}\big(\epsilon_\alpha, du(e_i)\big)\epsilon_\beta, \sigma_u(e_i)\big)\, dv_g \\
&\quad = \int_N F'\left(\frac{\|u^*h\|^2}{4}\right) u_i^\gamma u_j^\delta h\big(R^{M^m}(\epsilon_\alpha, \epsilon_\gamma)\epsilon_\alpha, \epsilon_\delta\big) h\big(du(e_i), du(e_j)\big)\, dv_g \\
&\quad = \int_N F'\left(\frac{\|u^*h\|^2}{4}\right) u_i^\gamma u_j^\delta [B_{\alpha\delta} B_{\gamma\alpha} - B_{\alpha\alpha} B_{\gamma\delta}] h\big(du(e_i), du(e_j)\big)\, dv_g \\
&\quad = \int_N F'\left(\frac{\|u^*h\|^2}{4}\right) u_i^\alpha u_j^\alpha \left[\lambda_\alpha^2 - \left(\sum_\beta \lambda_\beta\right)\lambda_\alpha\right] h\big(du(e_i), du(e_j)\big)\, dv_g
\end{aligned}$$

$$
\begin{aligned}
&= \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)\left[\lambda_\alpha^2 - \left(\sum_\beta \lambda_\beta\right)\lambda_\alpha\right] h\big(u_i^\alpha \epsilon_\alpha, u_j^\gamma \epsilon_\gamma\big) h\big(du(e_i), du(e_j)\big)\, dv_g \\
&= \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)\left[\lambda_\alpha^2 - \left(\sum_\beta \lambda_\beta\right)\lambda_\alpha\right] h\big(u_i^\alpha \epsilon_\alpha, du(e_j)\big) h\big(du(e_i), du(e_j)\big)\, dv_g. \qquad (38)
\end{aligned}
$$

From (33)-(38), we have

$$
\begin{aligned}
&\sum_A I(V_A, V_A) \\
&\quad= \int_N F''\left(\frac{\|u^*h\|^2}{4}\right)\lambda_\alpha\lambda_\beta h\big(u_i^\alpha \epsilon_\alpha, \sigma_u(e_i)\big) h\big(u_j^\alpha \epsilon_\alpha, \sigma_u(e_j)\big)\, dv_g \\
&\qquad+ \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)\left[2\lambda_\alpha^2 - \left(\sum_\beta \lambda_\beta\right)\lambda_\alpha\right] h\big(u_i^\alpha \epsilon_\alpha, du(e_j)\big) h\big(du(e_i), du(e_j)\big)\, dv_g \\
&\qquad+ \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)\lambda_\alpha\lambda_\beta h\big(u_i^\alpha \epsilon_\alpha, du(e_j)\big) h\big(u_i^\beta \epsilon_\beta, du(e_j)\big)\, dv_g \\
&\qquad+ \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)\lambda_\alpha\lambda_\beta h\big(u_i^\alpha \epsilon_\alpha, du(e_j)\big) h\big(du(e_i), u_j^\beta \epsilon_\beta\big)\, dv_g \\
&\quad\le \int_N F''\left(\frac{\|u^*h\|^2}{4}\right)\lambda_\alpha\lambda_\beta h\big(u_i^\alpha \epsilon_\alpha, \sigma_u(e_i)\big) h\big(u_j^\alpha \epsilon_\alpha, \sigma_u(e_j)\big)\, dv_g \\
&\qquad+ \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)\left[2\lambda_\alpha^2 - \left(\sum_\beta \lambda_\beta\right)\lambda_\alpha\right] h\big(u_i^\alpha \epsilon_\alpha, du(e_j)\big) h\big(du(e_i), du(e_j)\big)\, dv_g \\
&\qquad+ \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)2\lambda_\alpha\lambda_m h\big(u_i^\alpha \epsilon_\alpha, du(e_j)\big) h\big(du(e_i), du(e_j)\big)\, dv_g \\
&\quad\le \int_N F''\left(\frac{\|u^*h\|^2}{4}\right)\lambda_\alpha\lambda_\beta h\big(u_i^\alpha \epsilon_\alpha, \sigma_u(e_i)\big) h\big(u_j^\alpha \epsilon_\alpha, \sigma_u(e_j)\big)\, dv_g \\
&\qquad+ \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)\left[2\lambda_\alpha^2 + 2\lambda_\alpha\lambda_m - \left(\sum_\beta \lambda_\beta\right)\lambda_\alpha\right] h\big(u_i^\alpha \epsilon_\alpha, \sigma_u(e_i)\big)\, dv_g. \qquad (39)
\end{aligned}
$$

If $F''(t) = F'(t)$, then (39) leads to the following inequality:

$$
\begin{aligned}
\sum_A I(V_A, V_A) \le{}& \int_N F'\left(\frac{\|u^*h\|^2}{4}\right)\|u^*h\|^2\Bigg\{\|u^*h\|^2\lambda_m^2 \\
&+ \max_{1\le\alpha\le m}\left[2\lambda_\alpha^2 + 2\lambda_\alpha\lambda_m - \left(\sum_\beta \lambda_\beta\right)\lambda_\alpha\right]\Bigg\}\, dv_g. \qquad (40)
\end{aligned}
$$

If there exists a constant c_F such that $\frac{F'(t)}{t^{c_F}}$ is nonincreasing, it follows that $F''(t)t \le c_F F'(t)$ on $t \in (0,\infty)$, thus (39) implies

$$
\begin{aligned}
\sum_A I(V_A, V_A) \le{}& \int_M F'\left(\frac{\|u^*h\|^2}{4}\right)\|u^*h\|^2\Bigg\{4c_F\lambda_m^2 \\
&+ \max_{1\le\alpha\le m}\left\{\left[2\lambda_\alpha + 2\lambda_m - \left(\sum_\beta \lambda_\beta\right)\right]\lambda_\alpha\right\}\Bigg\}\, dv_g. \qquad (41)
\end{aligned}
$$

Now, if $u : N \to M^m$ is a nonconstant F-stationary map and (6) or (7) holds, then, from (41) or (40), we know that $\sum_A I(V_A, V_A) < 0$ and u is unstable. □

Corollary 4.2 *Let $u : N \to S^m$ be a nonconstant F-stationary map with $m > 4$, where N is any compact Riemannian manifold. If $c_F < \frac{m}{4} - 1$ or $\|u^*h\|^2 < m - 4$, then u is unstable.*

5 Conclusions

In this paper, we investigate F-stationary maps between the compact convex hypersurface M^m and any compact Riemannian manifold N. Assume that the principal curvatures λ_i of M^m satisfy $0 < \lambda_1 \leq \cdots \leq \lambda_m$ and $3\lambda_m < \sum_{i=1}^{m-1} \lambda_i$, then every nonconstant F-stationary map from M^m to N or from N to M^m is unstable if (6) or (7) holds. We mainly use the second variation formula for F-stationary maps (cf. [5]) to get the instability. In particular, we consider S^m as a special case of compact convex hypersurfaces and obtain similar inferences.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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