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On some binomial $B^{(m)}$ -difference sequence spaces

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Abstract

In this paper, we introduce the binomial sequence spaces $b_0^{a,b}(B^{(m)})$, $b_c^{a,b}(B^{(m)})$ and $b_\infty^{a,b}(B^{(m)})$ by combining the binomial transformation and difference operator. We prove the BK -property and some inclusion relations. Furthermore, we obtain Schauder bases and compute the α -, β - and γ -duals of these sequence spaces. Finally, we characterize matrix transformations on the sequence space $b_c^{a,b}(B^{(m)})$.

Keywords: sequence space; matrix domain; Schauder basis; α -, β - and γ -duals

1 Introduction and preliminaries

Let w denote the space of all sequences. By ℓ_p , ℓ_∞ , c and c_0 , we denote the spaces of absolutely p -summable, bounded, convergent and null sequences, respectively, where $1 \leq p < \infty$. A Banach sequence space Z is called a BK -space [1] provided each of the maps $p_n : Z \rightarrow \mathbb{C}$ defined by $p_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$, which is of great importance in the characterization of matrix transformations between sequence spaces. One can prove that the sequence spaces ℓ_∞, c and c_0 are BK -spaces with their usual sup-norm.

Let Z be a sequence space, then Kizmaz [2] introduced the following difference sequence spaces:

$$Z(\Delta) = \{x_k \in w : (\Delta x_k) \in Z\}$$

for $Z \in \{\ell_\infty, c, c_0\}$, where $\Delta x_k = x_k - x_{k+1}$ for each $k \in \mathbb{N}$. Et and Colak [3] defined the generalization of the difference sequence spaces

$$Z(\Delta^m) = \{x_k \in w : (\Delta^m x_k) \in Z\}$$

for $Z \in \{\ell_\infty, c, c_0\}$, where $m \in \mathbb{N}$, $\Delta^0 x_k = x_k$ and $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$ for each $k \in \mathbb{N}$, which is equivalent to the binomial representation $\Delta^m x_k = \sum_{i=0}^m (-1)^i \binom{m}{i} x_{k+i}$. Since then, many authors have studied further generalization of the difference sequence spaces [4–7]. Moreover, Altay and Polat [8], Başarir and Kara [9–13], Başarir, Kara and Konca [14], Kara [15], Kara and İlkhān [16, 17], Polat and Başar [18], Song and Meng [19] and many others have studied new sequence spaces from matrix point of view that represent difference operators.

For an infinite matrix $A = (a_{n,k})$ and $x = (x_k) \in w$, the A -transform of x is defined by $Ax = \{(Ax)_n\}$ and is supposed to be convergent for all $n \in \mathbb{N}$, where $(Ax)_n = \sum_{k=0}^{\infty} a_{n,k} x_k$.

For two sequence spaces X and Y and an infinite matrix $A = (a_{n,k})$, the sequence space X_A is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\}, \tag{1.1}$$

which is called the domain of matrix A in the space X . By $(X : Y)$, we denote the class of all matrices such that $X \subseteq Y_A$.

The Euler means E^r of order r is defined by the matrix $E^r = (e^r_{n,k})$, where $0 < r < 1$ and

$$e^r_{n,k} = \begin{cases} \binom{n}{k} (1-r)^{n-k} r^k & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

The Euler sequence spaces e^r_0 , e^r_c and e^r_∞ were defined by Altay and Başar [20] and Altay, Başar and Mursaleen [21] as follows:

$$e^r_0 = \left\{ x = (x_k) \in w : \lim_{n \rightarrow \infty} \sum_{k=0}^n \binom{n}{k} (1-r)^{n-k} r^k x_k = 0 \right\},$$

$$e^r_c = \left\{ x = (x_k) \in w : \lim_{n \rightarrow \infty} \sum_{k=0}^n \binom{n}{k} (1-r)^{n-k} r^k x_k \text{ exists} \right\},$$

and

$$e^r_\infty = \left\{ x = (x_k) \in w : \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^n \binom{n}{k} (1-r)^{n-k} r^k x_k \right| < \infty \right\}.$$

Altay and Polat [8] defined further generalization of the Euler sequence spaces $e^r_0(\nabla)$, $e^r_c(\nabla)$ and $e^r_\infty(\nabla)$ by

$$Z(\nabla) = \{x = (x_k) \in w : (\nabla x_k) \in Z\}$$

for $Z \in \{e^r_0, e^r_c, e^r_\infty\}$, where $\nabla x_k = x_k - x_{k-1}$ for each $k \in \mathbb{N}$. Here any term with negative subscript is equal to naught.

Polat and Başar [18] employed the matrix domain technique of the triangle limitation method for obtaining the following sequence spaces:

$$Z(\nabla^{(m)}) = \{x = (x_k) \in w : (\nabla^{(m)} x_k) \in Z\}$$

for $Z \in \{e^r_0, e^r_c, e^r_\infty\}$, where $\nabla^{(m)} = (\delta^{(m)}_{n,k})$ is a triangle matrix defined by

$$\delta^{(m)}_{n,k} = \begin{cases} (-1)^{n-k} \binom{m}{n-k} & \text{if } \max\{0, n-m\} \leq k \leq n, \\ 0 & \text{if } 0 \leq k < \max\{0, n-m\} \text{ or } k > n, \end{cases}$$

for all $k, n, m \in \mathbb{N}$. Also, Başarir and Kayikçi [22] defined the matrix $B^{(m)} = (b^{(m)}_{n,k})$ by

$$b^{(m)}_{n,k} = \begin{cases} \binom{m}{n-k} r^{m-n+k} s^{n-k} & \text{if } \max\{0, n-m\} \leq k \leq n, \\ 0 & \text{if } 0 \leq k < \max\{0, n-m\} \text{ or } k > n, \end{cases}$$

which is reduced to the matrix $\nabla^{(m)}$ in the case $r = 1, s = -1$. Kara and Başarir [23] introduced the spaces $e_0^r(B^{(m)})$, $e_c^r(B^{(m)})$ and $e_\infty^r(B^{(m)})$ of $B^{(m)}$ -difference sequences.

Recently Bişgin [24, 25] defined another generalization of the Euler sequence spaces and introduced the binomial sequence spaces $b_0^{a,b}$, $b_c^{a,b}$, $b_\infty^{a,b}$ and $b_p^{a,b}$. Let $a, b \in \mathbb{R}$ and $a, b \neq 0$. Then the binomial matrix $B^{a,b} = (b_{n,k}^{a,b})$ is defined by

$$b_{n,k}^{a,b} = \begin{cases} \frac{1}{(a+b)^n} \binom{n}{k} a^{n-k} b^k & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. For $ab > 0$ we have

- (i) $\|B^{a,b}\| < \infty$,
- (ii) $\lim_{n \rightarrow \infty} b_{n,k}^{a,b} = 0$ for each $k \in \mathbb{N}$,
- (iii) $\lim_{n \rightarrow \infty} \sum_k b_{n,k}^{a,b} = 1$.

Thus, the binomial matrix $B^{a,b}$ is regular for $ab > 0$. Unless stated otherwise, we assume that $ab > 0$. If we take $a + b = 1$, we obtain the Euler matrix E^r , so the binomial matrix generalizes the Euler matrix. Bişgin defined the following binomial sequence spaces:

$$b_0^{a,b} = \left\{ x = (x_k) \in w : \lim_{n \rightarrow \infty} \frac{1}{(a+b)^n} \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k x_k = 0 \right\},$$

$$b_c^{a,b} = \left\{ x = (x_k) \in w : \lim_{n \rightarrow \infty} \frac{1}{(a+b)^n} \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k x_k \text{ exists} \right\},$$

and

$$b_\infty^{a,b} = \left\{ x = (x_k) \in w : \sup_{n \in \mathbb{N}} \left| \frac{1}{(a+b)^n} \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k x_k \right| < \infty \right\}.$$

The purpose of the present paper is to study the binomial difference spaces $b_0^{a,b}(B^{(m)})$, $b_c^{a,b}(B^{(m)})$ and $b_\infty^{a,b}(B^{(m)})$ whose $B^{a,b}(B^{(m)})$ -transforms are in the spaces c_0, c and ℓ_∞ , respectively. These new sequence spaces are the generalization of the sequence spaces defined in [24, 25] and [23]. Also, we give some inclusion relations and compute the bases and α -, β - and γ -duals of these sequence spaces.

2 The binomial difference sequence spaces

In this section, we introduce the spaces $b_0^{a,b}(B^{(m)})$, $b_c^{a,b}(B^{(m)})$ and $b_\infty^{a,b}(B^{(m)})$ and prove the BK -property and inclusion relations.

We first define the binomial difference sequence spaces $b_0^{a,b}(B^{(m)})$, $b_c^{a,b}(B^{(m)})$ and $b_\infty^{a,b}(B^{(m)})$ by

$$Z(B^{(m)}) = \{x = (x_k) \in w : (B^{(m)}x_k) \in Z\}$$

for $Z \in \{b_0^{a,b}, b_c^{a,b}, b_\infty^{a,b}\}$. By using the notion of (1.1), the sequence spaces $b_0^{a,b}(B^{(m)})$, $b_c^{a,b}(B^{(m)})$ and $b_\infty^{a,b}(B^{(m)})$ can be redefined by

$$b_0^{a,b}(B^{(m)}) = (b_0^{a,b})_{B^{(m)}}, \quad b_c^{a,b}(B^{(m)}) = (b_c^{a,b})_{B^{(m)}}, \quad b_\infty^{a,b}(B^{(m)}) = (b_\infty^{a,b})_{B^{(m)}}. \tag{2.1}$$

It is obvious that the sequence spaces $b_0^{a,b}(B^{(m)})$, $b_c^{a,b}(B^{(m)})$ and $b_\infty^{a,b}(B^{(m)})$ may be reduced to some sequence spaces in the special cases of a, b, s, r and $m \in \mathbb{N}$. For instance, if we take $a + b = 1$, then we obtain the spaces $e_0^r(B^{(m)})$, $e_c^r(B^{(m)})$ and $e_\infty^r(B^{(m)})$, defined by Kara and Başarir [23]. If we take $a + b = 1$, $r = 1$ and $s = -1$, then we obtain the spaces $e_0^r(\nabla^{(m)})$, $e_c^r(\nabla^{(m)})$ and $e_\infty^r(\nabla^{(m)})$, defined by Polat and Başar [18]. Especially, taking $r = 1$ and $s = -1$, we obtain the new binomial difference sequence spaces $b_0^{a,b}(\nabla^{(m)})$, $b_c^{a,b}(\nabla^{(m)})$ and $b_\infty^{a,b}(\nabla^{(m)})$.

Let us define the sequence $y = (y_n)$ as the $B^{a,b}(B^{(m)})$ -transform of a sequence $x = (x_k)$, that is,

$$y_n = [B^{a,b}(B^{(m)})x_k]_n = \frac{1}{(a+b)^n} \sum_{k=0}^n \sum_{i=k}^n \binom{m}{i-k} \binom{n}{i} a^{n-i} b^i r^{m+k-i} s^{i-k} x_k \tag{2.2}$$

for each $n \in \mathbb{N}$. Then the sequence spaces $b_0^{a,b}(B^{(m)})$, $b_c^{a,b}(B^{(m)})$ and $b_\infty^{a,b}(B^{(m)})$ can be redefined by all sequences whose $B^{a,b}(B^{(m)})$ -transforms are in the spaces c_0 , c and ℓ_∞ .

Theorem 2.1 *The sequence spaces $b_0^{a,b}(B^{(m)})$, $b_c^{a,b}(B^{(m)})$ and $b_\infty^{a,b}(B^{(m)})$ are BK-spaces with their sup-norm defined by*

$$\|x\|_{b_0^{a,b}(B^{(m)})} = \|x\|_{b_c^{a,b}(B^{(m)})} = \|x\|_{b_\infty^{a,b}(B^{(m)})} = \sup_{n \in \mathbb{N}} |[B^{a,b}(B^{(m)})x_k]_n|.$$

Proof The sequence spaces $b_0^{a,b}$, $b_c^{a,b}$ and $b_\infty^{a,b}$ are BK-spaces with their sup-norm (see [24], Theorem 2.1 and [25], Theorem 2.1). Moreover, $B^{(m)}$ is a triangle matrix and (2.1) holds. By using Theorem 4.3.12 of Wilansky [26], we deduce that the binomial sequence spaces $b_0^{a,b}(B^{(m)})$, $b_c^{a,b}(B^{(m)})$ and $b_\infty^{a,b}(B^{(m)})$ are BK-spaces. □

Theorem 2.2 *The sequence spaces $b_0^{a,b}(B^{(m)})$, $b_c^{a,b}(B^{(m)})$ and $b_\infty^{a,b}(B^{(m)})$ are linearly isomorphic to the spaces c_0 , c and ℓ_∞ , respectively.*

Proof Similarly, we prove the theorem only for the space $b_0^{a,b}(B^{(m)})$. To prove $b_0^{a,b}(B^{(m)}) \cong c_0$, we must show the existence of a linear bijection between the spaces $b_0^{a,b}(B^{(m)})$ and c_0 .

Consider $b_0^{a,b}(B^{(m)}) \rightarrow c_0$ by $T(x) = B^{a,b}(B^{(m)})x_k$. The linearity of T is obvious and $x = 0$ whenever $T(x) = 0$. Therefore, T is injective.

Let $y = (y_n) \in c_0$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{i=0}^k (a+b)^i \sum_{j=i}^k \binom{m+k-j-1}{k-j} \binom{j}{i} \frac{(-s)^{k-j}}{r^{m+k-j}} (-a)^{j-i} b^{-j} y_j \tag{2.3}$$

for each $k \in \mathbb{N}$. Then we have

$$\lim_{n \rightarrow \infty} [B^{a,b}(B^{(m)})x_k]_n = \lim_{n \rightarrow \infty} \frac{1}{(a+b)^n} \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k (B^{(m)})x_k = \lim_{n \rightarrow \infty} y_n = 0,$$

which implies that $x \in b_0^{a,b}(B^{(m)})$ and $T(x) = y$. Consequently, T is surjective and is norm preserving. Thus, $b_0^{a,b}(B^{(m)}) \cong c_0$. □

The following theorems give some inclusion relations for this class of sequence spaces. We have the well known inclusion $c_0 \subseteq c \subseteq \ell_\infty$, then the corresponding extended versions also preserve this inclusion.

Theorem 2.3 *The inclusion $b_0^{a,b}(B^{(m)}) \subseteq b_c^{a,b}(B^{(m)}) \subseteq b_\infty^{a,b}(B^{(m)})$ holds.*

Theorem 2.4 *The inclusions $e_0^a(B^{(m)}) \subseteq b_0^{a,b}(B^{(m)})$, $e_c^a(B^{(m)}) \subseteq b_c^{a,b}(B^{(m)})$ and $e_\infty^a(B^{(m)}) \subseteq b_\infty^{a,b}(B^{(m)})$ strictly hold.*

Proof Similarly, we only prove the inclusion $e_0^a(B^{(m)}) \subseteq b_0^{a,b}(B^{(m)})$. If $a + b = 1$, we have $E^a = B^{a,b}$. So $e_0^a(B^{(m)}) \subseteq b_0^{a,b}(B^{(m)})$ holds. Let $0 < a < 1$ and $b = 4$. We define a sequence $x = (x_k)$ by $x_k = (-\frac{3}{a})^k$ for each $k \in \mathbb{N}$. It is clear that

$$E^a(B^{(m)}x_k) = \left(\sum_{i=0}^m \binom{m}{i} s^i r^{m-i} \left(-\frac{a}{3}\right)^i (-2-a)^n \right) \notin c_0$$

and

$$B^{a,b}(B^{(m)}x_k) = \left(\sum_{i=0}^m \binom{m}{i} s^i r^{m-i} \left(-\frac{a}{3}\right)^i \left(\frac{1}{4+a}\right)^n \right) \in c_0.$$

So, we have $x \in b_0^{a,b}(B^{(m)}) \setminus e_0^a(B^{(m)})$. This shows that the inclusion $e_0^a(B^{(m)}) \subseteq b_0^{a,b}(B^{(m)})$ strictly holds. □

3 The Schauder basis and α -, β - and γ -duals

For a normed space $(X, \|\cdot\|)$, a sequence $\{x_k : x_k \in X\}_{k \in \mathbb{N}}$ is called a *Schauder basis* [1] if for every $x \in X$, there is a unique scalar sequence (λ_k) such that $\|x - \sum_{k=0}^n \lambda_k x_k\| \rightarrow 0$ as $n \rightarrow \infty$. We shall construct Schauder bases for the sequence spaces $b_0^{a,b}(B^{(m)})$ and $b_c^{a,b}(B^{(m)})$.

We define the sequence $g^{(k)}(a, b) = \{g_i^{(k)}(a, b)\}_{i \in \mathbb{N}}$ by

$$g_i^{(k)}(a, b) = \begin{cases} 0 & \text{if } 0 \leq i < k, \\ (a+b)^k \sum_{j=k}^i \binom{m+i-j-1}{i-j} \binom{j}{k} \frac{(-s)^{i-j}}{r^{m+i-j}} b^{-j} (-a)^{j-k} & \text{if } i \geq k, \end{cases}$$

for each $k \in \mathbb{N}$.

Theorem 3.1 *The sequence $(g^{(k)}(a, b))_{k \in \mathbb{N}}$ is a Schauder basis for the binomial sequence space $b_0^{a,b}(B^{(m)})$ and every $x = (x_i) \in b_0^{a,b}(B^{(m)})$ has a unique representation by*

$$x = \sum_k \lambda_k(a, b) g^{(k)}(a, b), \tag{3.1}$$

where $\lambda_k(a, b) = [B^{a,b}(B^{(m)}x_i)]_k$ for each $k \in \mathbb{N}$.

Proof Obviously, $B^{a,b}(B^{(m)}g_i^{(k)}(a, b)) = e_k \in c_0$, where e_k is the sequence with 1 in the k th place and zeros elsewhere for each $k \in \mathbb{N}$. This implies that $g^{(k)}(a, b) \in b_0^{a,b}(B^{(m)})$ for each $k \in \mathbb{N}$.

For $x \in b_0^{a,b}(B^{(m)})$ and $n \in \mathbb{N}$, we put

$$x^{(n)} = \sum_{k=0}^n \lambda_k(a, b) g^{(k)}(a, b).$$

By the linearity of $B^{a,b}(B^{(m)})$, we have

$$B^{a,b}(B^{(m)}x_i^{(n)}) = \sum_{k=0}^n \lambda_k(a,b)B^{a,b}(B^{(m)}g_i^{(k)}(a,b)) = \sum_{k=0}^n \lambda_k(a,b)e_k$$

and

$$[B^{a,b}(B^{(m)}(x_i - x_i^{(n)}))]_k = \begin{cases} 0 & \text{if } 0 \leq k < n, \\ [B^{a,b}(B^{(m)}x_i)]_k & \text{if } k \geq n, \end{cases}$$

for each $k \in \mathbb{N}$.

For every $\varepsilon > 0$, there is a positive integer n_0 such that

$$|[B^{a,b}(B^{(m)}x_i)]_k| < \frac{\varepsilon}{2}$$

for all $k \geq n_0$. Then we have

$$\|x - x^{(n)}\|_{b_0^{a,b}(B^{(m)})} = \sup_{k \geq n} |[B^{a,b}(B^{(m)}x_i)]_k| \leq \sup_{k \geq n_0} |[B^{a,b}(B^{(m)}x_i)]_k| < \frac{\varepsilon}{2} < \varepsilon,$$

which implies that $x \in b_0^{a,b}(B^{(m)})$ is represented as in (3.1).

To show the uniqueness of this representation, we assume that

$$x = \sum_k \mu_k(a,b)g^{(k)}(a,b).$$

Then we have

$$[B^{a,b}(B^{(m)}x_i)]_k = \sum_k \mu_k(a,b)[B^{a,b}(B^{(m)}g_i^{(k)}(a,b))]_k = \sum_k \mu_k(a,b)(e_k)_k = \mu_k(a,b),$$

which is a contradiction with the assumption that $\lambda_k(a,b) = [B^{a,b}(B^{(m)}x_i)]_k$ for each $k \in \mathbb{N}$.

This shows the uniqueness of this representation. □

Theorem 3.2 *Let $g = (1, 1, 1, 1, \dots)$ and $\lim_{k \rightarrow \infty} \lambda_k(a,b) = l$. The set $\{g, g^{(0)}(a,b), g^{(1)}(a,b), \dots, g^{(k)}(a,b), \dots\}$ is a Schauder basis for the space $b_c^{a,b}(B^{(m)})$ and every $x \in b_c^{a,b}(B^{(m)})$ has a unique representation by*

$$x = lg + \sum_k [\lambda_k(a,b) - l]g^{(k)}(a,b). \tag{3.2}$$

Proof Obviously, $B^{a,b}(B^{(m)}g_i^k(a,b)) = e_k \in c$ and $g \in b_c^{a,b}(B^{(m)})$. For $x \in b_c^{a,b}(B^{(m)})$, we put $y = x - lg$ and we have $y \in b_0^{a,b}(B^{(m)})$. Hence, we deduce that y has a unique representation by (3.1), which implies that x has a unique representation by (3.2). Thus, we complete the proof. □

Corollary 3.3 *The sequence spaces $b_0^{a,b}(B^{(m)})$ and $b_c^{a,b}(B^{(m)})$ are separable.*

Köthe and Toeplitz [27] first computed the dual whose elements can be represented as sequences and defined the α -dual (or Köthe-Toeplitz dual). Next, we compute the α -, β - and γ -duals of the sequence spaces $b_0^{a,b}(B^{(m)})$, $b_c^{a,b}(B^{(m)})$ and $b_\infty^{a,b}(B^{(m)})$.

For the sequence spaces X and Y , define multiplier space $M(X, Y)$ by

$$M(X, Y) = \{u = (u_k) \in w : ux = (u_k x_k) \in Y \text{ for all } x = (x_k) \in X\}.$$

Then the α -, β - and γ -duals of a sequence space X are defined by

$$X^\alpha = M(X, \ell_1), \quad X^\beta = M(X, c) \quad \text{and} \quad X^\gamma = M(X, \ell_\infty),$$

respectively.

Let us give the following properties:

$$\sup_{K \in \Gamma} \sum_n \left| \sum_{k \in K} a_{n,k} \right| < \infty, \tag{3.3}$$

$$\sup_{n \in \mathbb{N}} \sum_k |a_{n,k}| < \infty, \tag{3.4}$$

$$\lim_{n \rightarrow \infty} a_{n,k} = a_k \quad \text{for each } k \in \mathbb{N}, \tag{3.5}$$

$$\lim_{n \rightarrow \infty} \sum_k a_{n,k} = a, \tag{3.6}$$

$$\lim_{n \rightarrow \infty} \sum_k |a_{n,k}| = \sum_k \left| \lim_{n \rightarrow \infty} a_{n,k} \right|, \tag{3.7}$$

where Γ is the collection of all finite subsets of \mathbb{N} .

Lemma 3.4 ([28]) *Let $A = (a_{n,k})$ be an infinite matrix. Then the following statements hold:*

- (i) $A \in (c_0 : \ell_1) = (c : \ell_1) = (\ell_\infty : \ell_1)$ if and only if (3.3) holds.
- (ii) $A \in (c_0 : c)$ if and only if (3.4) and (3.5) hold.
- (iii) $A \in (c : c)$ if and only if (3.4), (3.5) and (3.6) hold.
- (iv) $A \in (\ell_\infty : c)$ if and only if (3.5) and (3.7) hold.
- (v) $A \in (c_0 : \ell_\infty) = (c : \ell_\infty) = (\ell_\infty : \ell_\infty)$ if and only if (3.4) holds.

Theorem 3.5 *The α -dual of the spaces $b_0^{a,b}(B^{(m)})$, $b_c^{a,b}(B^{(m)})$ and $b_\infty^{a,b}(B^{(m)})$ is the set*

$$U_1^{a,b} = \left\{ u = (u_k) \in w : \sup_{K \in \Gamma} \sum_k \left| \sum_{i \in K} (a+b)^i \sum_{j=i}^k \binom{m+k-j-1}{k-j} \binom{j}{i} \right. \right. \\ \left. \left. \times \frac{(-s)^{k-j}}{r^{m+k-j}} b^{-j} (-a)^{j-i} u_k \right| < \infty \right\}.$$

Proof Let $u = (u_k) \in w$ and $x = (x_k)$ be defined by (2.3), then we have

$$u_k x_k = \sum_{i=0}^k (a+b)^i \sum_{j=i}^k \binom{m+k-j-1}{k-j} \binom{j}{i} \frac{(-s)^{k-j}}{r^{m+k-j}} b^{-j} (-a)^{j-i} u_k y_i = (G^{a,b}y)_k$$

for each $k \in \mathbb{N}$, where $G^{a,b} = (g_{k,i}^{a,b})$ is defined by

$$g_{k,i}^{a,b} = \begin{cases} (a+b)^i \sum_{j=i}^k \binom{m+k-j-1}{k-j} \binom{j}{i} \frac{(-s)^{k-j}}{r^{m+k-j}} b^{-j} (-a)^{j-i} u_k & \text{if } 0 \leq i \leq k, \\ 0 & \text{if } i > k. \end{cases}$$

Therefore, we deduce that $ux = (u_k x_k) \in \ell_1$ whenever $x \in b_0^{a,b}(B^{(m)})$, $b_c^{a,b}(B^{(m)})$ or $b_\infty^{a,b}(B^{(m)})$, if and only if $G^{a,b}y \in \ell_1$, whenever $y \in c_0, c$ or ℓ_∞ . This implies that $u = (u_k) \in [b_0^{a,b}(B^{(m)})]^\alpha$, $[b_c^{a,b}(B^{(m)})]^\alpha$ or $[b_\infty^{a,b}(B^{(m)})]^\alpha$ if and only if $G^{a,b} \in (c_0 : \ell_1)$, $G^{a,b} \in (c : \ell_1)$ or $G^{a,b} \in (\ell_\infty : \ell_1)$ by Parts (i) of Lemma 3.4. So we obtain

$$u = (u_k) \in [b_0^{a,b}(B^{(m)})]^\alpha = [b_c^{a,b}(B^{(m)})]^\alpha = [b_\infty^{a,b}(B^{(m)})]^\alpha$$

if and only if

$$\sup_{K \in \Gamma} \sum_k \left| \sum_{i \in K} (a+b)^i \sum_{j=i}^k \binom{m+k-j-1}{k-j} \binom{j}{i} \frac{(-s)^{k-j}}{r^{m+k-j}} b^{-j} (-a)^{j-i} u_k \right| < \infty.$$

Thus, we have $[b_0^{a,b}(B^{(m)})]^\alpha = [b_c^{a,b}(B^{(m)})]^\alpha = [b_\infty^{a,b}(B^{(m)})]^\alpha = U_1^{a,b}$. □

Now, we define the sets $U_2^{a,b}$, $U_3^{a,b}$, $U_4^{a,b}$ and $U_5^{a,b}$ by

$$\begin{aligned} U_2^{a,b} &= \left\{ u = (u_k) \in w : \sup_{n \in \mathbb{N}} \sum_k |u_{n,k}| < \infty \right\}, \\ U_3^{a,b} &= \left\{ u = (u_k) \in w : \lim_{n \rightarrow \infty} u_{n,k} \text{ exists for each } k \in \mathbb{N} \right\}, \\ U_4^{a,b} &= \left\{ u = (u_k) \in w : \lim_{n \rightarrow \infty} \sum_k |u_{n,k}| = \sum_k \left| \lim_{n \rightarrow \infty} u_{n,k} \right| \right\}, \end{aligned}$$

and

$$U_5^{a,b} = \left\{ u = (u_k) \in w : \lim_{n \rightarrow \infty} \sum_k u_{n,k} \text{ exists} \right\},$$

where

$$u_{n,k} = (a+b)^k \sum_{i=k}^n \sum_{j=k}^i \binom{m+i-j-1}{i-j} \binom{j}{k} \frac{(-s)^{i-j}}{r^{m+i-j}} (-a)^{j-k} (b)^{-j} u_i.$$

Theorem 3.6 *We have the following relations:*

- (i) $[b_0^{a,b}(B^{(m)})]^\beta = U_2^{a,b} \cap U_3^{a,b}$,
- (ii) $[b_c^{a,b}(B^{(m)})]^\beta = U_2^{a,b} \cap U_3^{a,b} \cap U_5^{a,b}$,
- (iii) $[b_\infty^{a,b}(B^{(m)})]^\beta = U_3^{a,b} \cap U_4^{a,b}$.

Proof Let $u = (u_k) \in w$ and $x = (x_k)$ be defined by (2.3), then we consider the following equation:

$$\begin{aligned} \sum_{k=0}^n u_k x_k &= \sum_{k=0}^n u_k \left[\sum_{i=0}^k (a+b)^i \sum_{j=i}^k \binom{m+k-j-1}{k-j} \binom{j}{i} \frac{(-s)^{k-j}}{r^{m+k-j}} (-a)^{j-i} b^{-j} y_i \right] \\ &= \sum_{k=0}^n \left[(a+b)^k \sum_{i=k}^n \sum_{j=k}^i \binom{m+i-j-1}{i-j} \binom{j}{k} \frac{(-s)^{i-j}}{r^{m+i-j}} (-a)^{j-k} b^{-j} u_i \right] y_k \\ &= (U^{a,b}y)_n, \end{aligned}$$

where $U^{a,b} = (u_{n,k}^{a,b})$ is defined by

$$u_{n,k}^{a,b} = \begin{cases} (a+b)^k \sum_{i=k}^n \sum_{j=k}^i \binom{m+i-j-1}{i-j} \binom{j}{k} \frac{(-s)^{i-j}}{r^{m+i-j}} (-a)^{j-k} b^{-j} u_i & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

Therefore, we deduce that $ux = (u_k x_k) \in c$ whenever $x \in b_0^{a,b}(B^{(m)})$ if and only if $U^{a,b}y \in c$ whenever $y \in c_0$, which implies that $u = (u_k) \in [b_0^{a,b}(B^{(m)})]^\beta$ if and only if $U^{a,b} \in (c_0 : c)$ by Part (ii) of Lemma 3.4. So we obtain $[b_0^{a,b}(B^{(m)})]^\beta = U_2^{a,b} \cap U_3^{a,b}$. Using Parts (iii), (iv) instead of Part (ii) of Lemma 3.4, the proof can be completed in a similar way. \square

Similarly, we give the following theorem without proof.

Theorem 3.7 *The γ -dual of the spaces $b_0^{a,b}(B^{(m)})$, $b_c^{a,b}(B^{(m)})$ and $b_\infty^{a,b}(B^{(m)})$ is the set $U_2^{a,b}$.*

4 Certain matrix mappings on the space $b_c^{a,b}(B^{(m)})$

In this section, we characterize matrix transformations from $b_c^{a,b}(B^{(m)})$ into ℓ_p , ℓ_∞ and c . Let us define the matrix $\Theta = (\theta_{n,k})$ via an infinite matrix $\Lambda = (\lambda_{n,k})$ by $\Theta = \Lambda(B^{a,b}(B^{(m)}))^{-1}$, that is,

$$\theta_{n,k} = (a+b)^k \sum_{j=k}^\infty \binom{m+n-j-1}{n-j} \binom{j}{k} \frac{(-s)^{n-j}}{r^{m+n-j}} (-a)^{j-k} b^{-j} \lambda_{n,j}, \tag{4.1}$$

where $(B^{a,b}(B^{(m)}))^{-1}$ is the inverse of the $B^{a,b}(B^{(m)})$ -transform. We now give the following lemmas.

Lemma 4.1 *Let Z be any given sequence space and the entries of the matrices $\Lambda = (\lambda_{n,k})$ and $\Theta = (\theta_{n,k})$ are connected with equation (4.1). If $(\lambda_{n,k})_k \in [b_c^{a,b}(B^{(m)})]^\beta$ for all $n \in \mathbb{N}$, then $\Lambda \in (b_c^{a,b}(B^{(m)}) : Z)$ if and only if $\Theta \in (c : Z)$.*

Proof Let $\Lambda \in (b_c^{a,b}(B^{(m)}) : Z)$ and $y = (y_n) \in c$. For every $x = (x_k) \in b_c^{a,b}(B^{(m)})$, we have $x_k = [(B^{a,b}(B^{(m)}))^{-1}y_n]_k$. Since $(\lambda_{n,k})_k \in [b_c^{a,b}(B^{(m)})]^\beta$ for all $n \in \mathbb{N}$, this implies the existence of the Λ -transform of x , i.e. Λx exists. So we obtain $\Lambda x = \Lambda(B^{a,b}(B^{(m)}))^{-1}y = \Theta y$, which implies that $\Theta \in (c : Z)$.

Conversely, let $\Theta \in (c : Z)$ and $x \in b_c^{a,b}(B^{(m)})$. For every $y \in c$, we have $y_n = [B^{a,b}(B^{(m)})x_k]_n$. Since $(\lambda_{n,k})_k \in [b_c^{a,b}(B^{(m)})]^\beta$ for all $n \in \mathbb{N}$, this implies that Θy exists, which can be proved

in a similar way to the proof of Theorem 3.6. So we have $\Theta y = \Theta B^{a,b}(B^{(m)})x = \Lambda x$, which shows that $\Lambda \in (b_c^{a,b}(B^{(m)}): Z)$. □

Lemma 4.2 ([28]) *Let $A = (a_{n,k})$ be an infinite matrix. Then the following statement holds: $A \in (c: \ell_p)$ if and only if*

$$\sup_{K \in \Gamma} \sum_n \left| \sum_{k \in K} a_{n,k} \right|^p < \infty, \quad 1 \leq p < \infty. \tag{4.2}$$

For brevity of notation, we write

$$t_{n,k} = (a+b)^k \sum_{i=k}^n \sum_{j=k}^i \binom{m+i-j-1}{i-j} \binom{j}{k} \frac{(-s)^{i-j}}{r^{m+i-j}} (-a)^{j-k} (b)^{-j} a_{n,j},$$

$$t_{n,k}^l = (a+b)^k \sum_{i=k}^l \sum_{j=k}^i \binom{m+i-j-1}{i-j} \binom{j}{k} \frac{(-s)^{i-j}}{r^{m+i-j}} (-a)^{j-k} (b)^{-j} a_{n,j}$$

for all $n, k \in \mathbb{N}$.

By using Lemma 4.1, there are some immediate consequences with $t_{n,k}$ or $t_{n,k}^l$ in place of $a_{n,k}$ in Lemma 3.4 and Lemma 4.2.

Theorem 4.3 *$A \in (b_c^{a,b}(B^{(m)}): \ell_p)$ if and only if*

$$\sup_{K \in \Gamma} \sum_n \left| \sum_{k \in K} t_{n,k} \right|^p < \infty, \tag{4.3}$$

$$t_{n,k} \text{ exists for each } k, n \in \mathbb{N}, \tag{4.4}$$

$$\sum_k t_{n,k} \text{ converges for each } n \in \mathbb{N}, \tag{4.5}$$

$$\sup_{l \in \mathbb{N}} \sum_{k=0}^l |t_{n,k}^l| < \infty, \quad n \in \mathbb{N}. \tag{4.6}$$

Theorem 4.4 *$A \in (b_c^{a,b}(B^{(m)}): \ell_\infty)$ if and only if (4.4) and (4.6) hold, and*

$$\sup_{n \in \mathbb{N}} \sum_k |t_{n,k}| < \infty. \tag{4.7}$$

Theorem 4.5 *$A \in (b_c^{a,b}(B^{(m)}): c)$ if and only if (4.4), (4.6) and (4.7) hold, and*

$$\lim_{n \rightarrow \infty} t_{n,k} \text{ exists for each } k \in \mathbb{N}, \tag{4.8}$$

$$\lim_{n \rightarrow \infty} \sum_k t_{n,k} \text{ exists.} \tag{4.9}$$

5 Conclusion

By considering the definitions of the binomial matrix $B^{a,b} = (b_{n,k}^{a,b})$ and the difference operator, we introduce the sequence spaces $b_0^{a,b}(B^{(m)})$, $b_c^{a,b}(B^{(m)})$ and $b_\infty^{a,b}(B^{(m)})$. These spaces are the natural continuation of [18, 23–25]. Our results are the generalization of the matrix

domain of the Euler matrix. In order to give full knowledge to the reader on related topics with applications and a possible line of further investigation, the e-book [29] is added to the list of references.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

JM came up with the main ideas and drafted the manuscript. MS revised the paper. All authors read and approved the final manuscript.

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