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# Some new retarded nonlinear Volterra-Fredholm type integral inequalities with maxima in two variables and their applications

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## Abstract

In this paper, we establish some new retarded nonlinear Volterra-Fredholm type integral inequalities with maxima in two independent variables, and we present the applications to research the boundedness of solutions to retarded nonlinear Volterra-Fredholm type integral equations.

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## 1 Introduction

Gronwall-Bellman inequality [1, 2] and Bihari inequality [3] provided important devices in the study of existence, uniqueness, boundedness, oscillation, stability, invariant manifolds and other qualitative properties of solutions to differential equations, integral equations and integro-differential equations. In the past few decades, a number of studies have focused on generalizations of the Gronwall-Bellman inequality. For example, in [4–10], the Gronwall-Bellman-Gamidov type integral inequalities and their generalizations were studied; in [11–14], the Gronwall-like inequalities and their deformations were investigated; in [15–18], the Volterra type iterated inequalities were discussed; in [19–24], the Volterra-Fredholm type inequalities were examined.

The Gronwall-Bellman inequality can be stated as follows.

If  $u$  and  $f$  are nonnegative continuous functions on an interval  $[a, b]$ , and  $u$  satisfies the following inequality:

$$u(t) \leq c + \int_a^t f(s)u(s) ds, \quad t \in [a, b], \quad (1.1)$$

where  $c \geq 0$  is a constant. Then

$$u(t) \leq c \exp\left(\int_a^t f(s) ds\right). \quad (1.2)$$

In 2004, Pachpatte [6] investigated the retarded linear Volterra-Fredholm type integral inequality in two independent variables:

$$u(x, y) \leq c + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x, y, s, t) u(s, t) dt ds + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x, y, s, t) u(s, t) dt ds, \tag{1.3}$$

$$(x, y) \in [x_0, M] \times [y_0, N].$$

In 2010, Wang [14] investigated a retarded Volterra type integral inequality with two variables:

$$\begin{aligned} \psi(u(x, y)) \leq & a(x, y) + b(x, y) \int_{x_0}^x c(s, y) \psi(u(s, y)) ds \\ & + d(x, y) \left[ \int_{\alpha_1(x_0)}^{\alpha_1(x)} \int_{\beta_1(y_0)}^{\beta_1(y)} f_1(s, t) \varphi_1(u(s, t)) dt ds \right. \\ & \left. + \int_{\alpha_2(x_0)}^{\alpha_2(x)} \int_{\beta_2(y_0)}^{\beta_2(y)} f_2(s, t) \varphi_2(u(s, t)) dt ds \right], \end{aligned} \tag{1.4}$$

$$(x, y) \in [x_0, x_1] \times [y_0, y_1].$$

In 2014, Lu *et al.* [21] studied the nonlinear retarded Volterra-Fredholm type iterated integral inequality:

$$\begin{aligned} u(x, y) \leq & k + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h_1(s_1, t_1) \omega(u(s_1, t_1)) \\ & \times \left[ f_1(s_1, t_1) \omega_1(u(s_1, t_1)) + \int_{\alpha(x_0)}^{s_1} \int_{\beta(y_0)}^{t_1} h_2(s_2, t_2) \left[ f_2(s_2, t_2) \omega_2(u(s_2, t_2)) \right. \right. \\ & \left. \left. + \int_{\alpha(x_0)}^{s_2} \int_{\beta(y_0)}^{t_2} h_3(s_3, t_3) \omega_3(u(s_3, t_3)) dt_3 ds_3 \right] dt_2 ds_2 \right] dt_1 ds_1 \\ & + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} h_1(s_1, t_1) \omega(u(s_1, t_1)) \left[ f_1(s_1, t_1) \omega_1(u(s_1, t_1)) \right. \\ & \left. + \int_{\alpha(x_0)}^{s_1} \int_{\beta(y_0)}^{t_1} h_2(s_2, t_2) \left[ f_2(s_2, t_2) \omega_2(u(s_2, t_2)) \right. \right. \\ & \left. \left. + \int_{\alpha(x_0)}^{s_2} \int_{\beta(y_0)}^{t_2} h_3(s_3, t_3) \omega_3(u(s_3, t_3)) dt_3 ds_3 \right] dt_2 ds_2 \right] dt_1 ds_1, \end{aligned} \tag{1.5}$$

$$(x, y) \in \Delta.$$

In 2016, Huang and Wang [23] discussed the retarded nonlinear Volterra-Fredholm type integral inequality with maxima:

$$\begin{aligned} \varphi(v(t)) \leq & k + \int_{\alpha(t_0)}^{\alpha(t)} h_1(s) \left[ f_1(s) \phi_1(v(s)) + \int_{\alpha(t_0)}^s h_2(\tau) \left[ f_2(\tau) \phi_2(v(\tau)) \right. \right. \\ & \left. \left. + \int_{\alpha(t_0)}^{\tau} h_3(\xi) \phi_3 \left( \max_{\eta \in [\xi-h, \xi]} v(\eta) \right) d\xi \right] d\tau \right] ds \\ & + \int_{\alpha(t_0)}^{\alpha(T)} h_1(s) \left[ f_1(s) \phi_1(v(s)) + \int_{\alpha(t_0)}^s h_2(\tau) \left[ f_2(\tau) \phi_2(v(\tau)) \right. \right. \\ & \left. \left. + \int_{\alpha(t_0)}^{\tau} h_3(\xi) \phi_3 \left( \max_{\eta \in [\xi-h, \xi]} v(\eta) \right) d\xi \right] d\tau \right] ds, \end{aligned} \tag{1.6}$$

$$v(t) \leq k, \quad t \in [t_0 - h, t_0].$$

Motivated by the work presented in [14, 21, 23], we establish some new retarded nonlinear Volterra-Fredholm type integral inequality with maxima in two independent variables in this paper:

$$\begin{aligned} \psi(u(x, y)) \leq & k(x, y) + \int_{\alpha(x)}^{\infty} a(s, y) \psi(u(s, y)) ds + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ b_i(s, t, x, y) \varphi_1(u(s, t)) \right. \\ & \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y) \varphi_2\left(\max_{\sigma \in [\xi, h\xi]} u(\sigma, \eta)\right) d\xi d\eta \right] ds dt \\ & + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left[ d_j(s, t, x, y) \psi(u(s, t)) \right. \\ & \left. + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, x, y) \psi\left(\max_{\sigma \in [\xi, h\xi]} u(\sigma, \eta)\right) d\xi d\eta \right] ds dt, \quad (x, y) \in \Delta, \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} u^p(x, y) \leq & k(x, y) + \int_{\alpha(x)}^{\infty} a(s, y) u^p(s, y) ds + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ b_i(s, t, x, y) u^{q_i}(s, t) \right. \\ & \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y) \max_{\sigma \in [\xi, h\xi]} u^{r_i}(\sigma, \eta) d\xi d\eta \right] ds dt \\ & + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left[ d_j(s, t, x, y) u^{\varepsilon_j}(s, t) \right. \\ & \left. + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, x, y) \max_{\sigma \in [\xi, h\xi]} u^{\delta_j}(\sigma, \eta) d\xi d\eta \right] ds dt, \quad (x, y) \in \Delta. \end{aligned} \tag{1.8}$$

By the amplification method, differential and integration, and the inverse function, we obtain the lower bound estimation of the unknown function. The example is given to illustrate the application of our results.

### 2 Main results

In what follows,  $R$  denotes the set of real numbers,  $R_+ = [0, +\infty)$ ,  $I_1 = [M, +\infty)$ ,  $I_2 = [N, +\infty)$  are the given subsets of  $R$ ,  $\Delta = I_1 \times I_2$ .  $C^1(\Omega, S)$  denotes the class of all continuously differentiable functions defined on set  $\Omega$  with range in the set  $S$ ,  $C(\Omega, S)$  denotes the class of all continuous functions defined on set  $\Omega$  with range in the set  $S$ , and  $\alpha'(t)$  denotes the derived function of a function  $\alpha(t)$ . For convenience, we cite some useful lemmas in the discussion of our proof as follows:

**Lemma 2.1** (See [25]) *Let  $u(t)$ ,  $a(t)$ ,  $b(t)$  be nonnegative and continuous functions defined for  $t \in R_+$ . Assume that  $a(t)$  is non-increasing function for  $t \in R_+$ . If*

$$u(t) \leq a(t) + \int_t^{\infty} b(s)u(s) ds, \quad t \in R_+,$$

then

$$u(t) \leq a(t) \exp\left(\int_t^{\infty} b(s) ds\right), \quad t \in R_+.$$

From Lemma 2.1, we can get the generalization in two dimensions.

**Lemma 2.2** *Let  $u(x, y), a(x, y), b(x, y)$  be nonnegative and continuous functions defined for  $(x, y) \in \Delta$ . Assume that  $a(x, y)$  is a non-increasing function in the first variable. If*

$$u(x, y) \leq a(x, y) + \int_x^\infty b(s, y)u(s, y) ds, \quad (x, y) \in \Delta,$$

then

$$u(x, y) \leq a(x, y) \exp\left(\int_x^\infty b(s, y) ds\right), \quad (x, y) \in \Delta.$$

**Lemma 2.3** (See [26]) *Assume that  $a \geq 0, p \geq q \geq 0$ , and  $p \neq 0$ . Then, for any  $K > 0$ ,*

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}.$$

**Theorem 2.1** *Suppose that the following conditions hold:*

- (i)  $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$  is an increasing function and  $\psi(u) > 0, \forall u > 0, \psi(\infty) = \infty$ .  $\psi^{-1}$  is the inverse function of  $\psi$ .  $\varphi_1, \varphi_2, \varphi_2/\varphi_1 \in C(\mathbb{R}_+, \mathbb{R}_+)$  are increasing functions with  $\varphi_i(u) > 0 (i = 1, 2)$  for  $u > 0$ .  $\psi^{-1}, \varphi_i (i = 1, 2)$  are sub-multiplicative and sub-additive, that is,

$$\begin{aligned} \psi^{-1}(\alpha\beta) &\leq \psi^{-1}(\alpha)\psi^{-1}(\beta), & \psi^{-1}(\alpha + \beta) &\leq \psi^{-1}(\alpha) + \psi^{-1}(\beta), \\ \varphi_i(\alpha\beta) &\leq \varphi_i(\alpha)\varphi_i(\beta), & \varphi_i(\alpha + \beta) &\leq \varphi_i(\alpha) + \varphi_i(\beta), \quad \alpha, \beta \in \mathbb{R}_+; \end{aligned}$$

- (ii)  $k(x, y), a(x, y) \in C(\Delta, \mathbb{R}_+)$  and  $k(x, y)$  is non-increasing in the first variable;
- (iii)  $b_i(s, t, x, y), c_i(s, t, x, y), d_j(s, t, x, y), e_j(s, t, x, y) \in C(\Delta^2, \mathbb{R}_+)$  for  $i = 1, 2, \dots, l_1; j = 1, 2, \dots, l_2; b_i, c_i, d_j, e_j$  are all non-increasing functions in the last two variables;
- (iv)  $\alpha, \alpha_i, \alpha_j \in C(I_1, I_1), \beta_i, \beta_j \in C(I_2, I_2)$  are non-decreasing functions with  $\alpha(x), \alpha_i(x), \alpha_j(x) \geq x, \beta_i(y), \beta_j(y) \geq y (i = 1, 2, \dots, l_1; j = 1, 2, \dots, l_2)$ .  $h \geq 1$  is a constant;
- (v) the function  $u \in C(\Delta, \mathbb{R}_+)$  satisfies the inequality

$$\begin{aligned} \psi(u(x, y)) &\leq k(x, y) + \int_{\alpha(x)}^\infty a(s, y)\psi(u(s, y)) ds \\ &\quad + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^\infty \int_{\beta_i(y)}^\infty \left[ b_i(s, t, x, y)\varphi_1(u(s, t)) \right. \\ &\quad \left. + \int_s^\infty \int_t^\infty c_i(\xi, \eta, x, y)\varphi_2\left(\max_{\sigma \in [\xi, h\xi]} u(\sigma, \eta)\right) d\xi d\eta \right] ds dt \\ &\quad + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^\infty \int_{\beta_j(N)}^\infty \left[ d_j(s, t, x, y)\psi(u(s, t)) \right. \\ &\quad \left. + \int_s^\infty \int_t^\infty e_j(\xi, \eta, x, y)\psi\left(\max_{\sigma \in [\xi, h\xi]} u(\sigma, \eta)\right) d\xi d\eta \right] ds dt, \end{aligned} \tag{2.1}$$

$(x, y) \in \Delta,$

then we have

$$u(x, y) \leq \psi^{-1} \{ [k(x, y) + W_1^{-1} \{ W_2^{-1} \{ W_2 [W_1(B(M, N) + G^{-1}(F(M, N))) + E(x, y)] + F(x, y) \} \} ] A(x, y) \}, \tag{2.2}$$

where

$$W_1(z) = \int_c^z \frac{ds}{\varphi_1(\psi^{-1}(s))}, \quad c > 0, z \in (0, +\infty), \tag{2.3}$$

$$W_2(z) = \int_c^z \frac{\varphi_1(\psi^{-1}(W_1^{-1}(s)))}{\varphi_2(\psi^{-1}(W_1^{-1}(s)))} ds, \quad c > 0, z \in (0, +\infty), \tag{2.4}$$

$$G(u) = W_2 \left( W_1 \left( \frac{u}{D(M, N)} \right) \right) - W_2(W_1(B(M, N) + u) + E(M, N)), \tag{2.5}$$

on condition that  $W_1(+\infty) = +\infty, W_2(+\infty) = +\infty$ , and  $G(u)$  is a strictly increasing function on  $R_+$ . We have

$$A(x, y) = \exp \left( \int_{\alpha(x)}^{\infty} a(s, y) ds \right), \tag{2.6}$$

$$\begin{aligned} B(M, N) = & \sum_{i=1}^{l_1} \int_{\alpha_i(M)}^{\infty} \int_{\beta_i(N)}^{\infty} \left[ b_i(s, t, M, N) \varphi_1(\psi^{-1}(k(s, t)A(s, t))) \right. \\ & \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, M, N) \varphi_2(\psi^{-1}(k(\xi, \eta)A(\xi, \eta))) d\xi d\eta \right] ds dt \\ & + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left[ d_j(s, t, M, N) k(s, t) A(s, t) \right. \\ & \left. + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, M, N) k(\xi, \eta) A(\xi, \eta) d\xi d\eta \right] ds dt, \end{aligned} \tag{2.7}$$

$$\begin{aligned} D(M, N) = & \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left[ d_j(s, t, M, N) A(s, t) \right. \\ & \left. + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, M, N) A(\xi, \eta) d\xi d\eta \right] ds dt, \end{aligned} \tag{2.8}$$

$$E(M, N) = \sum_{i=1}^{l_1} \int_{\alpha_i(M)}^{\infty} \int_{\beta_i(N)}^{\infty} b_i(s, t, M, N) \varphi_1(\psi^{-1}(A(s, t))) ds dt, \tag{2.9}$$

$$F(x, y) = \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y) \varphi_2(\psi^{-1}(A(\xi, \eta))) d\xi d\eta \right] ds dt. \tag{2.10}$$

*Proof* Let

$$\begin{aligned} z(x, y) = & \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ b_i(s, t, x, y) \varphi_1(u(s, t)) \right. \\ & \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y) \varphi_2 \left( \max_{\sigma \in [\xi, h\xi]} u(\sigma, \eta) \right) d\xi d\eta \right] ds dt \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left[ d_j(s, t, x, y) \psi(u(s, t)) \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, x, y) \psi\left(\max_{\sigma \in [\xi, h\xi]} u(\sigma, \eta)\right) d\xi d\eta \right] ds dt. \tag{2.11}
 \end{aligned}$$

Obviously,  $z(x, y)$  is non-increasing in each of the variables. From (2.1), we have

$$\psi(u(x, y)) \leq k(x, y) + z(x, y) + \int_{\alpha(x)}^{\infty} a(s, y) \psi(u(s, y)) ds. \tag{2.12}$$

Applying Lemma 2.2, we obtain

$$\psi(u(x, y)) \leq (k(x, y) + z(x, y))A(x, y), \tag{2.13}$$

*i.e.*

$$u(x, y) \leq \psi^{-1}[(k(x, y) + z(x, y))A(x, y)], \tag{2.14}$$

where  $A(x, y)$  is defined in (2.6), and  $A(x, y)$  is non-increasing in the first variable. So we have

$$\begin{aligned}
 \max_{\xi \in [x, hx]} u(\xi, y) & \leq \max_{\xi \in [x, hx]} \psi^{-1}[(k(\xi, y) + z(\xi, y))A(\xi, y)] \\
 & \leq \psi^{-1}\left[\max_{\xi \in [x, hx]} (k(\xi, y)A(\xi, y)) + \max_{\xi \in [x, hx]} (A(\xi, y)z(\xi, y))\right] \\
 & \leq \psi^{-1}[k(x, y)A(x, y) + A(x, y)z(x, y)]. \tag{2.15}
 \end{aligned}$$

By (2.11), (2.14), (2.15), and condition (i), we deduce

$$\begin{aligned}
 z(x, y) & \leq \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ b_i(s, t, x, y) \varphi_1(\psi^{-1}(k(s, t)A(s, t)) + \psi^{-1}(A(s, t))\psi^{-1}(z(s, t))) \right. \\
 & + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y) \varphi_2(\psi^{-1}(k(\xi, \eta)A(\xi, \eta)) \\
 & \left. + \psi^{-1}(A(\xi, \eta))\psi^{-1}(z(\xi, \eta))) d\xi d\eta \right] ds dt \\
 & + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left[ d_j(s, t, x, y) (k(s, t)A(s, t) + A(s, t)z(s, t)) \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, x, y) (k(\xi, \eta)A(\xi, \eta) + A(\xi, \eta)z(\xi, \eta)) d\xi d\eta \right] ds dt \\
 & \leq \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left\{ b_i(s, t, x, y) [\varphi_1(\psi^{-1}(k(s, t)A(s, t))) \right. \\
 & \left. + \varphi_1(\psi^{-1}(A(s, t)))\varphi_1(\psi^{-1}(z(s, t)))] \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y) [\varphi_2(\psi^{-1}(k(\xi, \eta)A(\xi, \eta))) \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \varphi_2(\psi^{-1}(A(\xi, \eta)))\varphi_2(\psi^{-1}(z(\xi, \eta))) d\xi d\eta \Big\} ds dt \\
 & + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left\{ d_j(s, t, x, y) [k(s, t)A(s, t) + A(s, t)z(s, t)] \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, x, y) [k(\xi, \eta)A(\xi, \eta) + A(\xi, \eta)z(\xi, \eta)] d\xi d\eta \right\} ds dt \\
 & \leq B(x, y) + C(x, y) \\
 & + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ b_i(s, t, x, y) \varphi_1(\psi^{-1}(A(s, t))) \varphi_1(\psi^{-1}(z(s, t))) \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y) \varphi_2(\psi^{-1}(A(\xi, \eta))) \varphi_2(\psi^{-1}(z(\xi, \eta))) d\xi d\eta \right] ds dt \\
 & \leq B(M, N) + C(M, N) \\
 & + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ b_i(s, t, x, y) \varphi_1(\psi^{-1}(A(s, t))) \varphi_1(\psi^{-1}(z(s, t))) \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y) \varphi_2(\psi^{-1}(A(\xi, \eta))) \varphi_2(\psi^{-1}(z(\xi, \eta))) d\xi d\eta \right] ds dt, \\
 & \forall (x, y) \in [X, \infty) \times [Y, \infty), \tag{2.16}
 \end{aligned}$$

where  $B(M, N)$  is defined in (2.7), and  $C(M, N)$  is defined as follows:

$$\begin{aligned}
 C(M, N) = & \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left[ d_j(s, t, M, N) A(s, t) z(s, t) \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, M, N) A(\xi, \eta) z(\xi, \eta) d\xi d\eta \right] ds dt. \tag{2.17}
 \end{aligned}$$

$\forall X \in I_1, Y \in I_2$ , for all  $(x, y) \in [X, \infty) \times [Y, \infty)$ , we have

$$\begin{aligned}
 z(x, y) \leq & B(M, N) + C(M, N) \\
 & + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ b_i(s, t, X, Y) \varphi_1(\psi^{-1}(A(s, t))) \varphi_1(\psi^{-1}(z(s, t))) \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, X, Y) \varphi_2(\psi^{-1}(A(\xi, \eta))) \varphi_2(\psi^{-1}(z(\xi, \eta))) d\xi d\eta \right] ds dt. \tag{2.18}
 \end{aligned}$$

Let  $z_1(x, y)$  denote the function on the right-hand side of (2.18), which is positive and non-increasing in each of the variables  $(x, y) \in [X, \infty) \times [Y, \infty)$ . From (2.18), we have

$$z(x, y) \leq z_1(x, y), \quad \forall (x, y) \in [X, \infty) \times [Y, \infty), \tag{2.19}$$

$$z_1(\infty, y) = B(M, N) + C(M, N). \tag{2.20}$$

Differentiating  $z_1(x, y)$  with respect to  $x$ , we have

$$\begin{aligned} \frac{\partial z_1(x, y)}{\partial x} &= - \sum_{i=1}^{l_1} \alpha'_i(x) \int_{\beta_i(y)}^{\infty} \left[ b_i(\alpha_i(x), t, X, Y) \varphi_1(\psi^{-1}(A(\alpha_i(x), t))) \varphi_1(\psi^{-1}(z(\alpha_i(x), t))) \right. \\ &\quad \left. + \int_{\alpha_i(x)}^{\infty} \int_t^{\infty} c_i(\xi, \eta, X, Y) \varphi_2(\psi^{-1}(A(\xi, \eta))) \varphi_2(\psi^{-1}(z(\xi, \eta))) d\xi d\eta \right] dt \\ &\geq - \sum_{i=1}^{l_1} \alpha'_i(x) \int_{\beta_i(y)}^{\infty} \left[ b_i(\alpha_i(x), t, X, Y) \varphi_1(\psi^{-1}(A(\alpha_i(x), t))) \varphi_1(\psi^{-1}(z_1(\alpha_i(x), t))) \right. \\ &\quad \left. + \int_{\alpha_i(x)}^{\infty} \int_t^{\infty} c_i(\xi, \eta, X, Y) \varphi_2(\psi^{-1}(A(\xi, \eta))) \varphi_2(\psi^{-1}(z_1(\xi, \eta))) d\xi d\eta \right] dt, \\ \forall (x, y) &\in [X, \infty) \times [Y, \infty). \end{aligned} \tag{2.21}$$

By the monotonicity of  $\varphi_1, \varphi_2, z_1$  and the property of  $\alpha_i, \beta_i$ , from (2.21), we get

$$\begin{aligned} &\frac{(\partial/\partial x)z_1(x, y)}{\varphi_1(\psi^{-1}(z_1(x, y)))} \\ &\geq - \sum_{i=1}^{l_1} \alpha'_i(x) \int_{\beta_i(y)}^{\infty} \left[ b_i(\alpha_i(x), t, X, Y) \varphi_1(\psi^{-1}(A(\alpha_i(x), t))) \right. \\ &\quad \left. + \int_{\alpha_i(x)}^{\infty} \int_t^{\infty} c_i(\xi, \eta, X, Y) \varphi_2(\psi^{-1}(A(\xi, \eta))) \frac{\varphi_2(\psi^{-1}(z_1(\xi, \eta)))}{\varphi_1(\psi^{-1}(z_1(\xi, \eta)))} d\xi d\eta \right] dt, \\ \forall (x, y) &\in [X, \infty) \times [Y, \infty). \end{aligned} \tag{2.22}$$

Replacing  $x$  with  $s$ , and integrating it from  $x$  to  $\infty$ , we obtain

$$\begin{aligned} &W_1(z_1(\infty, y)) - W_1(z_1(x, y)) \\ &\geq - \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ b_i(s, t, X, Y) \varphi_1(\psi^{-1}(A(s, t))) \right. \\ &\quad \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, X, Y) \varphi_2(\psi^{-1}(A(\xi, \eta))) \right. \\ &\quad \left. \times \frac{\varphi_2(\psi^{-1}(z_1(\xi, \eta)))}{\varphi_1(\psi^{-1}(z_1(\xi, \eta)))} d\xi d\eta \right] ds dt, \quad \forall (x, y) \in [X, \infty) \times [Y, \infty), \end{aligned} \tag{2.23}$$

*i.e.*

$$\begin{aligned} W_1(z_1(x, y)) &\leq W_1(z_1(\infty, y)) + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ b_i(s, t, X, Y) \varphi_1(\psi^{-1}(A(s, t))) \right. \\ &\quad \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, X, Y) \varphi_2(\psi^{-1}(A(\xi, \eta))) \frac{\varphi_2(\psi^{-1}(z_1(\xi, \eta)))}{\varphi_1(\psi^{-1}(z_1(\xi, \eta)))} d\xi d\eta \right] ds dt \\ &\leq W_1(z_1(\infty, y)) + E(X, Y) + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, X, Y) \right. \\ &\quad \left. \times \varphi_2(\psi^{-1}(A(\xi, \eta))) \frac{\varphi_2(\psi^{-1}(z_1(\xi, \eta)))}{\varphi_1(\psi^{-1}(z_1(\xi, \eta)))} d\xi d\eta \right] ds dt, \end{aligned} \tag{2.24}$$



where  $E(X, Y)$  is defined in (2.9). Let  $z_2(x, y)$  denote the function on the right-hand side of (2.24), which is positive and non-increasing in each of the variables  $(x, y) \in [X, \infty) \times [Y, \infty)$ . From (2.24), we have

$$z_1(x, y) \leq W_1^{-1}(z_2(x, y)), \quad \forall (x, y) \in [X, \infty) \times [Y, \infty), \tag{2.25}$$

$$z_2(\infty, y) = W_1(z_1(\infty, y)) + E(X, Y). \tag{2.26}$$

Differentiating  $z_2(x, y)$  with respect to  $x$ , we have

$$\begin{aligned} \frac{\partial z_2(x, y)}{\partial x} &= - \sum_{i=1}^{l_1} \alpha'_i(x) \int_{\beta_i(y)}^{\infty} \left[ \int_{\alpha_i(x)}^{\infty} \int_t^{\infty} c_i(\xi, \eta, X, Y) \varphi_2(\psi^{-1}(A(\xi, \eta))) \right. \\ &\quad \left. \times \frac{\varphi_2(\psi^{-1}(z_1(\xi, \eta)))}{\varphi_1(\psi^{-1}(z_1(\xi, \eta)))} d\xi d\eta \right] dt \\ &\geq - \sum_{i=1}^{l_1} \alpha'_i(x) \int_{\beta_i(y)}^{\infty} \left[ \int_{\alpha_i(x)}^{\infty} \int_t^{\infty} c_i(\xi, \eta, X, Y) \varphi_2(\psi^{-1}(A(\xi, \eta))) \right. \\ &\quad \left. \times \frac{\varphi_2(\psi^{-1}(W_1^{-1}(z_2(\xi, \eta))))}{\varphi_1(\psi^{-1}(W_1^{-1}(z_2(\xi, \eta))))} d\xi d\eta \right] dt. \end{aligned} \tag{2.27}$$

By the monotonicity of  $\varphi_2/\varphi_1$  and  $z_2$ , from (2.27), we obtain

$$\begin{aligned} &\frac{\varphi_1(\psi^{-1}(W_1^{-1}(z_2(x, y))))(\partial/\partial x)z_2(x, y)}{\varphi_2(\psi^{-1}(W_1^{-1}(z_2(x, y))))} \\ &\geq - \sum_{i=1}^{l_1} \alpha'_i(x) \int_{\beta_i(y)}^{\infty} \left[ \int_{\alpha_i(x)}^{\infty} \int_t^{\infty} c_i(\xi, \eta, X, Y) \varphi_2(\psi^{-1}(A(\xi, \eta))) d\xi d\eta \right] dt. \end{aligned} \tag{2.28}$$

Replace  $x$  with  $s$ , and integrating it from  $x$  to  $\infty$ , we get

$$W_2(z_2(x, y)) \leq W_2(z_2(\infty, y)) + F(x, y, X, Y), \tag{2.29}$$

where

$$F(x, y, X, Y) = \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, X, Y) \varphi_2(\psi^{-1}(A(\xi, \eta))) d\xi d\eta \right] ds dt.$$

Obviously,  $F(x, y, x, y) = F(x, y)$ , which is defined in (2.10). From (2.19), (2.20), (2.25), (2.26) and (2.29), we have

$$\begin{aligned} z(x, y) &\leq z_1(x, y) \leq W_1^{-1}(z_2(x, y)) \\ &\leq W_1^{-1}\{W_2^{-1}\{W_2[W_1(B(M, N) + C(M, N)) + E(X, Y)] + F(x, y, X, Y)\}\}, \\ &\quad \forall (x, y) \in [X, \infty) \times [Y, \infty). \end{aligned} \tag{2.30}$$

Since  $X$  and  $Y$  are chosen arbitrarily, we have

$$\begin{aligned} z(x, y) &\leq z_1(x, y) \leq W_1^{-1}\{W_2^{-1}\{W_2[W_1(B(M, N) + C(M, N)) + E(x, y)] + F(x, y)\}\}, \\ &\quad \forall (x, y) \in [M, \infty) \times [N, \infty). \end{aligned} \tag{2.31}$$

By the definition of  $C(M, N)$  and (2.19), we get

$$\begin{aligned}
 C(M, N) &\leq z_1(M, N)D(M, N) \\
 &\leq W_1^{-1}\{W_2^{-1}\{W_2[W_1(B(M, N) + C(M, N)) + E(M, N)] \\
 &\quad + F(M, N)]\}D(M, N), \tag{2.32}
 \end{aligned}$$

or

$$W_2\left[W_1\left(\frac{C(M, N)}{D(M, N)}\right)\right] - W_2[W_1(B(M, N) + C(M, N)) + E(M, N)] \leq F(M, N), \tag{2.33}$$

where  $D(M, N)$  is defined in (2.8). By (2.5) and the hypothesis of  $G$ , we obtain

$$C(M, N) \leq G^{-1}(F(M, N)). \tag{2.34}$$

Combining (2.31), (2.34) and (2.14), we get the desired result. □

**Corollary 2.1** *Let the functions  $k, a, \alpha, b_i, c_i, \alpha_i, \beta_i$  ( $i = 1, 2, \dots, l_1$ ),  $d_j, e_j, \alpha_j, \beta_j$  ( $j = 1, 2, \dots, l_2$ ) and  $u$  be defined as in Theorem 2.1,  $p$  is a positive constant and  $p \geq 1$ . If the function  $u(x, y)$  satisfies the inequality,*

$$\begin{aligned}
 u^p(x, y) &\leq k(x, y) + \int_{\alpha(x)}^{\infty} a(s, y)u^p(s, y) ds + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} [b_i(s, t, x, y)u(s, t) \\
 &\quad + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y) \max_{\sigma \in [\xi, h\xi]} u(\sigma, \eta) d\xi d\eta] ds dt \\
 &\quad + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} [d_j(s, t, x, y)u^p(s, t) \\
 &\quad + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, x, y) \max_{\sigma \in [\xi, h\xi]} u^p(\sigma, \eta) d\xi d\eta] ds dt, \quad (x, y) \in \Delta, \tag{2.35}
 \end{aligned}$$

then: (i) if  $p > 1$ , we have

$$\begin{aligned}
 u(x, y) &\leq \left\{ \left\{ k(x, y) \right. \right. \\
 &\quad \left. \left. + \left[ \tilde{B}(M, N) + G_1^{-1}(\tilde{F}(M, N)) \right]^{\frac{p-1}{p}} + \frac{p-1}{p} \tilde{F}(x, y) \right]^{\frac{p}{p-1}} \right\}^{\frac{1}{p}}, \tag{2.36}
 \end{aligned}$$

where

$$\begin{aligned}
 \tilde{B}(M, N) &= \sum_{i=1}^{l_1} \int_{\alpha_i(M)}^{\infty} \int_{\beta_i(N)}^{\infty} [b_i(s, t, M, N)k^{\frac{1}{p}}(s, t)A^{\frac{1}{p}}(s, t) \\
 &\quad + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, M, N)k^{\frac{1}{p}}(\xi, \eta)A^{\frac{1}{p}}(\xi, \eta) d\xi d\eta] ds dt
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left[ d_j(s, t, M, N)k(s, t)A(s, t) \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, M, N)k(\xi, \eta)A(\xi, \eta) d\xi d\eta \right] ds dt, \tag{2.37}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{F}(x, y) & = \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ b_i(s, t, x, y)A^{\frac{1}{p}}(s, t) \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y)A^{\frac{1}{p}}(\xi, \eta) d\xi d\eta \right] ds dt, \tag{2.38}
 \end{aligned}$$

$$G_1(u) = \frac{p}{p-1} \left[ \left( \frac{u}{D(M, N)} \right)^{\frac{p-1}{p}} - (\tilde{B}(M, N) + u)^{\frac{p-1}{p}} \right], \tag{2.39}$$

on condition that  $G_1(u)$  is a strictly increasing function on  $R_+$ .

(ii) If  $p = 1$ , we have

$$u(x, y) \leq \left[ k(x, y) + \frac{\bar{B}(M, N) \exp(\bar{F}(x, y))}{1 - D(M, N) \exp(\bar{F}(M, N))} \right] A(x, y), \tag{2.40}$$

where

$$D(M, N) \exp(\bar{F}(M, N)) < 1 \tag{2.41}$$

and

$$\begin{aligned}
 \bar{B}(M, N) & = \sum_{i=1}^{l_1} \int_{\alpha_i(M)}^{\infty} \int_{\beta_i(N)}^{\infty} \left[ b_i(s, t, M, N)k(s, t)A(s, t) \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, M, N)k(\xi, \eta)A(\xi, \eta) d\xi d\eta \right] ds dt \\
 & + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left[ d_j(s, t, M, N)k(s, t)A(s, t) \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, M, N)k(\xi, \eta)A(\xi, \eta) d\xi d\eta \right] ds dt, \tag{2.42}
 \end{aligned}$$

$$\begin{aligned}
 \bar{F}(x, y) & = \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ b_i(s, t, x, y)A(s, t) \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y)A(\xi, \eta) d\xi d\eta \right] ds dt. \tag{2.43}
 \end{aligned}$$

*Proof* Inequality (2.35) followed by letting  $\psi(u(x, y)) = u^p(x, y)$ ,  $\varphi_1(u(x, y)) = \varphi_2(u(x, y)) = u(x, y)$  in Theorem 2.1. Then  $\psi^{-1}(u(x, y)) = u^{\frac{1}{p}}(x, y)$  and  $(u + v)^{\frac{1}{p}} \leq u^{\frac{1}{p}} + v^{\frac{1}{p}}$ ,  $(uv)^{\frac{1}{p}} = u^{\frac{1}{p}} v^{\frac{1}{p}}$ .

If  $p > 1$ , we have

$$W_1(z) = \int_c^z \frac{du}{u^{1/p}} = \frac{p}{p-1} z^{\frac{p-1}{p}} - \frac{p}{p-1} c^{\frac{p-1}{p}},$$

$$W_1^{-1}(z) = \left( \frac{p-1}{p} z + c^{\frac{p-1}{p}} \right)^{\frac{p}{p-1}}.$$

Applying Theorem 2.1, we can easily get (2.36).

If  $p = 1$ , we have

$$W_1(z) = \int_c^z \frac{du}{u} = \ln z - \ln c, \quad W_1^{-1}(z) = c \exp z,$$

$$G_2(u) = W_1\left(\frac{u}{D(M,N)}\right) - W_1(\bar{B}(M,N) + u) = \ln \frac{u}{D(M,N)(\bar{B}(M,N) + u)}.$$

Obviously,  $G_2(u)$  is a strictly increasing function on  $R_+$ ,  $G_2^{-1}(u)$  is the inverse of  $G_2(u)$ , we get

$$G_2^{-1}(u) = \frac{\bar{B}(M,N)D(M,N) \exp(u)}{1 - D(M,N) \exp(u)}, \quad D(M,N) \exp(u) < 1,$$

where  $\bar{B}(M,N)$  is defined in (2.42). Applying Theorem 2.1, we can easily get (2.40). Details are omitted here. □

**Theorem 2.2** *Suppose that the following conditions hold:*

- (i) (ii)-(iv) of Theorem 2.1 are satisfied;
- (ii)  $q_i, r_i$  are nonnegative constants with  $p \geq q_i, p \geq r_i, i = 1, 2, \dots, l_1$ , and  $\varepsilon_j, \delta_j$  are nonnegative constants with  $p \geq \varepsilon_j, p \geq \delta_j, j = 1, 2, \dots, l_2$ .

If  $(x, y) \in \Delta, u(x, y)$  satisfies the following inequality:

$$u^p(x, y) \leq k(x, y) + \int_{\alpha(x)}^\infty a(s, y) u^p(s, y) ds + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^\infty \int_{\beta_i(y)}^\infty \left[ b_i(s, t, x, y) u^{q_i}(s, t) \right. \\ \left. + \int_s^\infty \int_t^\infty c_i(\xi, \eta, x, y) \max_{\sigma \in [\xi, h\xi]} u^{r_i}(\sigma, \eta) d\xi d\eta \right] ds dt \\ + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^\infty \int_{\beta_j(N)}^\infty \left[ d_j(s, t, x, y) u^{\varepsilon_j}(s, t) \right. \\ \left. + \int_s^\infty \int_t^\infty e_j(\xi, \eta, x, y) \max_{\sigma \in [\xi, h\xi]} u^{\delta_j}(\sigma, \eta) d\xi d\eta \right] ds dt, \quad (x, y) \in \Delta, \quad (2.44)$$

then we have

$$u(x, y) \leq \left\{ \left[ k(x, y) + \frac{B_1(M,N)}{1 - D_1(M,N)} \exp(F_1(x, y)) \right] A(x, y) \right\}^{\frac{1}{p}}, \quad (x, y) \in \Delta, \quad (2.45)$$

where

$$\begin{aligned}
 B_1(M, N) = & \sum_{i=1}^{l_1} \int_{\alpha_i(M)}^{\infty} \int_{\beta_i(N)}^{\infty} \left\{ b_i(s, t, M, N) A^{\frac{q_i}{p}}(s, t) \left[ \frac{q_i}{p} K_1^{\frac{q_i-p}{p}} k(s, t) + \frac{p-q_i}{p} K_1^{\frac{q_i}{p}} \right] \right. \\
 & + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, M, N) A^{\frac{r_i}{p}}(\xi, \eta) \\
 & \times \left. \left[ \frac{r_i}{p} K_2^{\frac{r_i-p}{p}} k(\xi, \eta) + \frac{p-r_i}{p} K_2^{\frac{r_i}{p}} \right] d\xi d\eta \right\} ds dt \\
 & + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left\{ d_j(s, t, M, N) A^{\frac{\varepsilon_j}{p}}(s, t) \left[ \frac{\varepsilon_j}{p} K_3^{\frac{\varepsilon_j-p}{p}} k(s, t) + \frac{p-\varepsilon_j}{p} K_3^{\frac{\varepsilon_j}{p}} \right] \right. \\
 & + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, M, N) A^{\frac{\delta_j}{p}}(\xi, \eta) \\
 & \times \left. \left[ \frac{\delta_j}{p} K_4^{\frac{\delta_j-p}{p}} k(\xi, \eta) + \frac{p-\delta_j}{p} K_4^{\frac{\delta_j}{p}} \right] d\xi d\eta \right\} ds dt, \tag{2.46}
 \end{aligned}$$

$$\begin{aligned}
 F_1(x, y) = & \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ b_i(s, t, x, y) A^{\frac{q_i}{p}}(s, t) \frac{q_i}{p} K_1^{\frac{q_i-p}{p}} \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y) A^{\frac{r_i}{p}}(\xi, \eta) \frac{r_i}{p} K_2^{\frac{r_i-p}{p}} d\xi d\eta \right] ds dt, \tag{2.47}
 \end{aligned}$$

$$\begin{aligned}
 D_1(M, N) = & \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left[ d_j(s, t, M, N) A^{\frac{\varepsilon_j}{p}}(s, t) \frac{\varepsilon_j}{p} K_3^{\frac{\varepsilon_j-p}{p}} \exp(F_1(s, t)) \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, M, N) A^{\frac{\delta_j}{p}}(\xi, \eta) \frac{\delta_j}{p} K_4^{\frac{\delta_j-p}{p}} \exp(F_1(\xi, \eta)) d\xi d\eta \right] ds dt \\
 < 1. \tag{2.48}
 \end{aligned}$$

*Proof* Let

$$\begin{aligned}
 z(x, y) = & \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ b_i(s, t, x, y) u^{q_i}(s, t) \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y) \max_{\sigma \in [\xi, h\xi]} u^{r_i}(\sigma, \eta) d\xi d\eta \right] ds dt \\
 & + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left[ d_j(s, t, x, y) u^{\varepsilon_j}(s, t) \right. \\
 & \left. + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, x, y) \max_{\sigma \in [\xi, h\xi]} u^{\delta_j}(\sigma, \eta) d\xi d\eta \right] ds dt. \tag{2.49}
 \end{aligned}$$

Obviously,  $z(x, y)$  is non-increasing in every variable. From (2.44) and (2.49), we have

$$u^p(x, y) \leq k(x, y) + z(x, y) + \int_{\alpha(x)}^{\infty} a(s, y) u^p(s, y) ds. \tag{2.50}$$

By Lemma 2.2, we obtain

$$u^p(x, y) \leq [k(x, y) + z(x, y)]A(x, y), \quad (x, y) \in [M, \infty) \times [N, \infty), \tag{2.51}$$

where  $A(x, y)$  is defined in (2.6). Then we get

$$u(x, y) \leq [(k(x, y) + z(x, y))A(x, y)]^{\frac{1}{p}}, \quad (x, y) \in [M, \infty) \times [N, \infty). \tag{2.52}$$

By Lemma 2.3, we have

$$\begin{aligned} u^{q_i}(x, y) &\leq [(k(x, y) + z(x, y))A(x, y)]^{\frac{q_i}{p}} \\ &\leq A^{\frac{q_i}{p}}(x, y) \left[ \frac{q_i}{p} K_1^{\frac{q_i-p}{p}} (k(x, y) + z(x, y)) + \frac{p-q_i}{p} K_1^{\frac{q_i}{p}} \right] \\ &= A^{\frac{q_i}{p}}(x, y) \left[ \frac{q_i}{p} K_1^{\frac{q_i-p}{p}} k(x, y) + \frac{p-q_i}{p} K_1^{\frac{q_i}{p}} + \frac{q_i}{p} K_1^{\frac{q_i-p}{p}} z(x, y) \right], \\ \max_{\xi \in [x, hx]} u^{r_i}(\xi, y) &\leq \max_{\xi \in [x, hx]} [(k(\xi, y) + z(\xi, y))A(\xi, y)]^{\frac{r_i}{p}} \\ &\leq \left[ \left( \max_{\xi \in [x, hx]} k(\xi, y) + \max_{\xi \in [x, hx]} z(\xi, y) \right) \max_{\xi \in [x, hx]} A(\xi, y) \right]^{\frac{r_i}{p}} \\ &\leq A^{\frac{r_i}{p}}(x, y) \left[ \frac{r_i}{p} K_2^{\frac{r_i-p}{p}} (k(x, y) + z(x, y)) + \frac{p-r_i}{p} K_2^{\frac{r_i}{p}} \right] \\ &= A^{\frac{r_i}{p}}(x, y) \left[ \frac{r_i}{p} K_2^{\frac{r_i-p}{p}} k(x, y) + \frac{p-r_i}{p} K_2^{\frac{r_i}{p}} + \frac{r_i}{p} K_2^{\frac{r_i-p}{p}} z(x, y) \right], \\ u^{\varepsilon_j}(x, y) &\leq A^{\frac{\varepsilon_j}{p}}(x, y) \left[ \frac{\varepsilon_j}{p} K_3^{\frac{\varepsilon_j-p}{p}} k(x, y) + \frac{p-\varepsilon_j}{p} K_3^{\frac{\varepsilon_j}{p}} + \frac{\varepsilon_j}{p} K_3^{\frac{\varepsilon_j-p}{p}} z(x, y) \right], \\ \max_{\xi \in [x, hx]} u^{\delta_j}(\xi, y) &\leq A^{\frac{\delta_j}{p}}(x, y) \left[ \frac{\delta_j}{p} K_4^{\frac{\delta_j-p}{p}} k(x, y) + \frac{p-\delta_j}{p} K_4^{\frac{\delta_j}{p}} + \frac{\delta_j}{p} K_4^{\frac{\delta_j-p}{p}} z(x, y) \right]. \end{aligned} \tag{2.53}$$

Combining (2.53) and (2.49), we have

$$\begin{aligned} z(x, y) &\leq \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left\{ b_i(s, t, x, y) A^{\frac{q_i}{p}}(s, t) \left[ \frac{q_i}{p} K_1^{\frac{q_i-p}{p}} k(s, t) \right. \right. \\ &\quad \left. \left. + \frac{p-q_i}{p} K_1^{\frac{q_i}{p}} + \frac{q_i}{p} K_1^{\frac{q_i-p}{p}} z(s, t) \right] \right. \\ &\quad \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y) A^{\frac{r_i}{p}}(\xi, \eta) \left[ \frac{r_i}{p} K_2^{\frac{r_i-p}{p}} k(\xi, \eta) \right. \right. \\ &\quad \left. \left. + \frac{p-r_i}{p} K_2^{\frac{r_i}{p}} + \frac{r_i}{p} K_2^{\frac{r_i-p}{p}} z(\xi, \eta) \right] d\xi d\eta \right\} ds dt \\ &\quad + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left\{ d_j(s, t, x, y) A^{\frac{\varepsilon_j}{p}}(s, t) \left[ \frac{\varepsilon_j}{p} K_3^{\frac{\varepsilon_j-p}{p}} k(s, t) \right. \right. \\ &\quad \left. \left. + \frac{p-\varepsilon_j}{p} K_3^{\frac{\varepsilon_j}{p}} + \frac{\varepsilon_j}{p} K_3^{\frac{\varepsilon_j-p}{p}} z(s, t) \right] \right\} \end{aligned}$$

$$\begin{aligned}
 & + \int_s^\infty \int_t^\infty e_j(\xi, \eta, x, y) A^{\frac{\delta_j}{p}}(\xi, \eta) \left[ \frac{\delta_j}{p} K_4^{\frac{\delta_j-p}{p}} k(\xi, \eta) \right. \\
 & \left. + \frac{p - \delta_j}{p} K_4^{\frac{\delta_j}{p}} + \frac{\delta_j}{p} K_4^{\frac{\delta_j-p}{p}} z(\xi, \eta) \right] d\xi d\eta \Big\} ds dt \\
 & = B_1(x, y) + C_1(x, y) + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^\infty \int_{\beta_i(y)}^\infty \left[ b_i(s, t, x, y) A^{\frac{q_i}{p}}(s, t) \frac{q_i}{p} K_1^{\frac{q_i-p}{p}} z(s, t) \right. \\
 & \left. + \int_s^\infty \int_t^\infty c_i(\xi, \eta, x, y) A^{\frac{r_i}{p}}(\xi, \eta) \frac{r_i}{p} K_2^{\frac{r_i-p}{p}} z(\xi, \eta) d\xi d\eta \right] ds dt \\
 & \leq B_1(M, N) + C_1(M, N) \\
 & + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^\infty \int_{\beta_i(y)}^\infty \left[ b_i(s, t, x, y) A^{\frac{q_i}{p}}(s, t) \frac{q_i}{p} K_1^{\frac{q_i-p}{p}} z(s, t) \right. \\
 & \left. + \int_s^\infty \int_t^\infty c_i(\xi, \eta, x, y) A^{\frac{r_i}{p}}(\xi, \eta) \frac{r_i}{p} K_2^{\frac{r_i-p}{p}} z(\xi, \eta) d\xi d\eta \right] ds dt, \\
 & (x, y) \in [M, \infty) \times [N, \infty), \tag{2.54}
 \end{aligned}$$

where  $B_1(M, N)$  is defined in (2.46),  $C_1(M, N)$  is defined as follows:

$$\begin{aligned}
 C_1(M, N) & = \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^\infty \int_{\beta_j(N)}^\infty \left[ d_j(s, t, M, N) A^{\frac{\varepsilon_j}{p}}(s, t) \frac{\varepsilon_j}{p} K_3^{\frac{\varepsilon_j-p}{p}} z(s, t) \right. \\
 & \left. + \int_s^\infty \int_t^\infty e_j(\xi, \eta, M, N) A^{\frac{\delta_j}{p}}(\xi, \eta) \frac{\delta_j}{p} K_4^{\frac{\delta_j-p}{p}} z(\xi, \eta) d\xi d\eta \right] ds dt. \tag{2.55}
 \end{aligned}$$

$\forall X \in I_1, \forall Y \in I_2$ , for all  $(x, y) \in [X, \infty) \times [Y, \infty)$ , we have

$$\begin{aligned}
 z(x, y) & \leq B_1(M, N) + C_1(M, N) + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^\infty \int_{\beta_i(y)}^\infty \left[ b_i(s, t, X, Y) A^{\frac{q_i}{p}}(s, t) \frac{q_i}{p} K_1^{\frac{q_i-p}{p}} z(s, t) \right. \\
 & \left. + \int_s^\infty \int_t^\infty c_i(\xi, \eta, X, Y) A^{\frac{r_i}{p}}(\xi, \eta) \frac{r_i}{p} K_2^{\frac{r_i-p}{p}} z(\xi, \eta) d\xi d\eta \right] ds dt. \tag{2.56}
 \end{aligned}$$

Let  $z_1(x, y)$  denote the function on the right-hand side of (2.56), which is positive and non-increasing in each of the variables  $(x, y) \in [X, \infty) \times [Y, \infty)$ . From (2.56), we have

$$z(x, y) \leq z_1(x, y), \quad (x, y) \in [X, \infty) \times [Y, \infty), \tag{2.57}$$

$$z_1(\infty, y) = B_1(M, N) + C_1(M, N). \tag{2.58}$$

Differentiating  $z_1(x, y)$  with respect to  $x$ , we have

$$\begin{aligned}
 \frac{\partial z_1(x, y)}{\partial x} & = - \sum_{i=1}^{l_1} \alpha_i'(x) \int_{\beta_i(y)}^\infty \left[ b_i(\alpha_i(x), t, X, Y) A^{\frac{q_i}{p}}(\alpha_i(x), t) \frac{q_i}{p} K_1^{\frac{q_i-p}{p}} z(\alpha_i(x), t) \right. \\
 & \left. + \int_{\alpha_i(x)}^\infty \int_t^\infty c_i(\xi, \eta, X, Y) A^{\frac{r_i}{p}}(\xi, \eta) \frac{r_i}{p} K_2^{\frac{r_i-p}{p}} z(\xi, \eta) d\xi d\eta \right] dt
 \end{aligned}$$

$$\begin{aligned} &\geq - \sum_{i=1}^{l_1} \alpha'_i(x) \int_{\beta_i(y)}^\infty \left[ b_i(\alpha_i(x), t, X, Y) A^{\frac{q_i}{p}}(\alpha_i(x), t) \frac{q_i}{p} K_1^{\frac{q_i-p}{p}} z_1(\alpha_i(x), t) \right. \\ &\quad \left. + \int_{\alpha_i(x)}^\infty \int_t^\infty c_i(\xi, \eta, X, Y) A^{\frac{r_i}{p}}(\xi, \eta) \frac{r_i}{p} K_2^{\frac{r_i-p}{p}} z_1(\xi, \eta) d\xi d\eta \right] dt. \end{aligned} \tag{2.59}$$

Dividing both sides of (2.59) by  $z_1(x, y)$ , noticing that  $z_1(x, y)$  is non-increasing in each variable, we have

$$\begin{aligned} \frac{(\partial/\partial x)z_1(x, y)}{z_1(x, y)} &\geq - \sum_{i=1}^{l_1} \alpha'_i(x) \int_{\beta_i(y)}^\infty \left[ b_i(\alpha_i(x), t, X, Y) A^{\frac{q_i}{p}}(\alpha_i(x), t) \frac{q_i}{p} K_1^{\frac{q_i-p}{p}} \right. \\ &\quad \left. + \int_{\alpha_i(x)}^\infty \int_t^\infty c_i(\xi, \eta, X, Y) A^{\frac{r_i}{p}}(\xi, \eta) \frac{r_i}{p} K_2^{\frac{r_i-p}{p}} d\xi d\eta \right] dt, \\ &(x, y) \in [X, \infty) \times [Y, \infty). \end{aligned} \tag{2.60}$$

Replace  $x$  with  $s$ , and integrate it from  $x$  to  $\infty$ , we get

$$z_1(x, y) \leq z_1(\infty, y) \exp(F_1(x, y, X, Y)), \quad (x, y) \in [X, \infty) \times [Y, \infty), \tag{2.61}$$

where

$$\begin{aligned} F_1(x, y, X, Y) &= \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^\infty \int_{\beta_i(y)}^\infty \left[ b_i(s, t, X, Y) A^{\frac{q_i}{p}}(s, t) \frac{q_i}{p} K_1^{\frac{q_i-p}{p}} \right. \\ &\quad \left. + \int_s^\infty \int_t^\infty c_i(\xi, \eta, X, Y) A^{\frac{r_i}{p}}(\xi, \eta) \frac{r_i}{p} K_2^{\frac{r_i-p}{p}} d\xi d\eta \right] ds dt. \end{aligned} \tag{2.62}$$

It is obvious that  $F_1(x, y, x, y) = F_1(x, y)$ , which is defined in (2.47). From (2.57), (2.58) and (2.61), we get

$$z(x, y) \leq [B_1(M, N) + C_1(M, N)] \exp(F_1(x, y, X, Y)), \quad (x, y) \in [X, \infty) \times [Y, \infty). \tag{2.63}$$

Due to the fact that  $X, Y$  are chosen arbitrarily, we have

$$z(x, y) \leq [B_1(M, N) + C_1(M, N)] \exp(F_1(x, y)), \quad (x, y) \in [M, \infty) \times [N, \infty). \tag{2.64}$$

By the definition of  $C_1(M, N)$ , we have

$$\begin{aligned} B_1(M, N) + C_1(M, N) &\leq B_1(M, N) + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^\infty \int_{\beta_j(N)}^\infty \left[ d_j(s, t, M, N) A^{\frac{\varepsilon_j}{p}}(s, t) \frac{\varepsilon_j}{p} K_3^{\frac{\varepsilon_j-p}{p}} \right. \\ &\quad \times [B_1(M, N) + C_1(M, N)] \exp(F_1(s, t)) \\ &\quad \left. + \int_s^\infty \int_t^\infty e_j(\xi, \eta, M, N) A^{\frac{\delta_j}{p}}(\xi, \eta) \frac{\delta_j}{p} K_4^{\frac{\delta_j-p}{p}} \right. \\ &\quad \left. \times [B_1(M, N) + C_1(M, N)] \exp(F_1(\xi, \eta)) d\xi d\eta \right] ds dt \\ &\leq B_1(M, N) + [B_1(M, N) + C_1(M, N)] D_1(M, N), \end{aligned} \tag{2.65}$$



where  $D_1(M, N)$  is defined in (2.48). Then, according to  $D_1(M, N) < 1$ , we have

$$B_1(M, N) + C_1(M, N) \leq \frac{B_1(M, N)}{1 - D_1(M, N)}. \tag{2.66}$$

From (2.64) and (2.66), we get

$$z(x, y) \leq \frac{B_1(M, N)}{1 - D_1(M, N)} \exp(F_1(x, y)), \quad (x, y) \in [M, \infty) \times [N, \infty). \tag{2.67}$$

Combining (2.52) and (2.67), we obtain the desired result. □

**Remark 2.1** If  $q_i = r_i = 1$  ( $i = 1, 2, \dots, l_1$ ),  $\varepsilon_j = \delta_j = p$  ( $j = 1, 2, \dots, l_2$ ), the inequality (2.44) becomes (2.35), but the proof of Theorem 2.2 is different from that of Corollary 2.1.

**Corollary 2.2** Let  $k, a, \alpha, \alpha_i, \beta_i, b_i, c_i$  ( $i = 1, 2, \dots, l_1$ ),  $\alpha_j, \beta_j, d_j, e_j$  ( $j = 1, 2, \dots, l_2$ ) be defined as in Theorem 2.1, then  $q, r$  are nonnegative constants with  $0 \leq q \leq 2, 0 \leq r \leq 2$ . For  $(x, y) \in \Delta$ ,  $u(x, y)$  satisfies the following inequality:

$$\begin{aligned} u^2(x, y) &\leq k(x, y) + \int_{\alpha(x)}^{\infty} a(s, y) u^2(s, y) ds \\ &\quad + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ b_i(s, t, x, y) u^q(s, t) \right. \\ &\quad \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y) \max_{\sigma \in [\xi, h\xi]} u^r(\sigma, \eta) d\xi d\eta \right] ds dt \\ &\quad + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left[ d_j(s, t, x, y) u(s, t) \right. \\ &\quad \left. + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, x, y) \max_{\sigma \in [\xi, h\xi]} u(\sigma, \eta) d\xi d\eta \right] ds dt, \quad (x, y) \in \Delta, \end{aligned} \tag{2.68}$$

then we have

$$u(x, y) \leq \left\{ \left[ k(x, y) + \frac{B_2(M, N)}{1 - D_2(M, N)} \exp(F_2(x, y)) \right] A(x, y) \right\}^{\frac{1}{2}}, \tag{2.69}$$

where

$$\begin{aligned} F_2(x, y) &= \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ b_i(s, t, x, y) A^{\frac{q}{2}}(s, t) \frac{q}{2} K_1^{\frac{q-2}{2}} \right. \\ &\quad \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y) A^{\frac{r}{2}}(\xi, \eta) \frac{r}{2} K_2^{\frac{r-2}{2}} d\xi d\eta \right] ds dt, \end{aligned} \tag{2.70}$$

$$\begin{aligned}
 & B_2(M, N) \\
 &= \sum_{i=1}^{l_1} \int_{\alpha_i(M)}^{\infty} \int_{\beta_i(N)}^{\infty} \left\{ b_i(s, t, M, N) A^{\frac{q}{2}}(s, t) \left[ \frac{q}{2} K_1^{\frac{q-2}{2}} k(s, t) + \frac{2-q}{2} K_1^{\frac{q}{2}} \right] \right. \\
 &\quad \left. + \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, M, N) A^{\frac{r}{2}}(\xi, \eta) \left[ \frac{r}{2} K_2^{\frac{r-2}{2}} k(\xi, \eta) + \frac{2-r}{2} K_2^{\frac{r}{2}} \right] d\xi d\eta \right\} ds dt \\
 &\quad + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left\{ d_j(s, t, M, N) A^{\frac{1}{2}}(s, t) \left[ \frac{1}{2} K_3^{-\frac{1}{2}} k(s, t) + \frac{1}{2} K_3^{\frac{1}{2}} \right] \right. \\
 &\quad \left. + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, M, N) A^{\frac{1}{2}}(\xi, \eta) \left[ \frac{1}{2} K_4^{-\frac{1}{2}} k(\xi, \eta) + \frac{1}{2} K_4^{\frac{1}{2}} \right] d\xi d\eta \right\} ds dt, \tag{2.71}
 \end{aligned}$$

$$\begin{aligned}
 D_2(M, N) &= \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left[ d_j(s, t, M, N) A^{\frac{1}{2}}(s, t) \frac{1}{2} K_3^{-\frac{1}{2}} \exp(F_2(s, t)) \right. \\
 &\quad \left. + \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, M, N) A^{\frac{1}{2}}(\xi, \eta) \frac{1}{2} K_4^{-\frac{1}{2}} \exp(F_2(\xi, \eta)) d\xi d\eta \right] ds dt \\
 &< 1. \tag{2.72}
 \end{aligned}$$

*Proof* Inequality (2.68) follows by inequality (2.44) with  $p = 2, q_i = q, r_i = r (i = 1, 2, \dots, l_1), \varepsilon_j = \delta_j = 1 (j = 1, 2, \dots, l_2)$ . Then, applying Theorem 2.2, we can easily get (2.69). Details are omitted here.  $\square$

**Remark 2.2** As one can see, the established results above mainly deal with Volterra-Fredholm type integral inequalities with maxima in two variables. And they are different from the results presented in [14, 21, 23]. In Theorem 2.1, in the case of one variable, if we take  $k(x, y) = k, a(x, y) = 0, l_1 = l_2 = 1, b_1(s, t, x, y) = d_1(s, t, x, y) = h_1(s), c_1(\xi, \eta, x, y) = e_1(\xi, \eta, x, y) = h_2(\xi), \psi(u) = \varphi_1(u), \psi(u) = \varphi_2(u)$  in the second iterated integral, orderly, we will get the inequality that is similar to inequality (1.5). If the above conditions are satisfied in two dimensions and  $\varphi_2(\max_{\sigma \in [\xi, h\xi]} u(\sigma, \eta)) = \varphi_2(u(\xi, \eta))$ , we get analogs of the inequality (1.4). And if we take  $l_1 = 2, l_2 = 0, b_i(s, t, x, y) = f_i(s, t), c_i(\xi, \eta, x, y) = 0$  in Theorem 2.1, inequality (2.1) reduces to (1.3).

### 3 Applications in the integral equation

In this section, we apply our results in Theorem 2.1 and Theorem 2.2 to study the retarded Volterra-Fredholm type integral equations with maxima in two variables. Some results on the boundedness of their solutions are presented, which demonstrate that our results can be used to investigate the qualitative properties of solutions of some integral equations.

**Example** We consider the retarded Volterra-Fredholm type integral equation of the form

$$\begin{aligned}
 \psi(v(x, y)) &= g_1(x, y) + \int_x^{\infty} g_2(s, y) \psi(v(s + \rho(s), y)) ds \\
 &\quad + \sum_{i=1}^{l_1} \int_x^{\infty} \int_y^{\infty} F_{1i} \left( s, t, x, y, v(s + \rho_i(s), t + \gamma_i(t)), \right. \\
 &\quad \left. \int_s^{\infty} \int_t^{\infty} F_{2i} \left( s, t, x, y, \max_{\sigma \in [\xi + \rho_i(\xi), h(\xi + \rho_i(\xi))]} v(\sigma, \eta + \gamma_i(\eta)) \right) d\xi d\eta \right) ds dt
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^{l_2} \int_M^\infty \int_N^\infty G_{1j} \left( s, t, x, y, v(s + \rho_j(s), t + \gamma_j(t)), \right. \\
 & \left. \int_s^\infty \int_t^\infty G_{2j} \left( s, t, x, y, \max_{\sigma \in [\xi + \rho_j(\xi), \mu(\xi + \rho_j(\xi))]} v(\sigma, \eta + \gamma_j(\eta)) \right) d\xi d\eta \right) ds dt, \\
 & (x, y) \in \Delta.
 \end{aligned} \tag{3.1}$$

Suppose that the following conditions hold:

- (i)  $g_1(x, y), g_2(x, y), v(x, y) \in C(\Delta, R)$ ;
- (ii)  $x + \rho(x), x + \rho_i(x), x + \rho_j(x) \in C^1(I_1, I_1)$  and  $y + \gamma_i(y), y + \gamma_j(y) \in C^1(I_2, I_2)$  are strictly increasing with

$$\begin{aligned}
 & \rho(M) = \rho_i(M) = \rho_j(M) = 0, \quad \gamma_i(N) = \gamma_j(N) = 0, \\
 & \rho(x) \geq 0, \quad \rho_i(x) \geq 0, \quad \rho_j(x) \geq 0 \quad \text{for } x \geq M, \\
 & \gamma_i(y) \geq 0, \quad \gamma_j(y) \geq 0 \quad \text{for } y \geq N, \\
 & \rho'(x) > -1, \quad \rho_i'(x) > -1, \quad \rho_j'(x) > -1, \\
 & \gamma_i'(y) > -1, \quad \gamma_j'(y) > -1 \quad (i = 1, 2, \dots, l_1; j = 1, 2, \dots, l_2);
 \end{aligned}$$

- (iii)  $F_{1i}, G_{1j} \in C(\Delta^2 \times R^2, R), F_{2i}, G_{2j} \in C(\Delta^2 \times R, R)$  ( $i = 1, 2, \dots, l_1; j = 1, 2, \dots, l_2$ ).

Let  $\alpha(x) = x + \rho(x), \alpha_i(x) = x + \rho_i(x), \alpha_j(x) = x + \rho_j(x), \beta_i(y) = y + \gamma_i(y), \beta_j(y) = y + \gamma_j(y)$ .

Then  $\alpha, \alpha_i, \alpha_j, \beta_i, \beta_j$  satisfy the condition (iv) of Theorem 2.1.

**Theorem 3.1** *In Eq. (3.1), suppose that the following conditions hold:*

$$\begin{aligned}
 & \psi(v(x, y)) = v(x, y), \quad |g_1(x, y)| \leq k(x, y), \quad |g_2(x, y)| \leq a(x, y), \\
 & |F_{1i}(s, t, x, y, u, v)| \leq b_i(s, t, x, y)\varphi_1(|u|) + |v|, \\
 & |F_{2i}(s, t, x, y, u)| \leq c_i(s, t, x, y)\varphi_2(|u|), \quad i = 1, 2, \dots, l_1, \\
 & |G_{1j}(s, t, x, y, u, v)| \leq d_j(s, t, x, y)|u| + |v|, \\
 & |G_{2j}(s, t, x, y, u)| \leq e_j(s, t, x, y)|u|, \quad j = 1, 2, \dots, l_2,
 \end{aligned} \tag{3.2}$$

where  $k, a, b_i, c_i, d_j, e_j, \varphi_1, \varphi_2$  are defined in Theorem 2.1. Assume that the function  $G_3(u) = W_2(W_1(\frac{u}{D_3(M, N)})) - W_2(W_1(B_3(M, N) + u) + E_3(M, N))$  is increasing. Then we have the following estimate:

$$\begin{aligned}
 |v(x, y)| \leq & [k(x, y) + W_1^{-1}\{W_2^{-1}\{W_2[W_1(B_3(M, N) + G_3^{-1}(F_3(M, N))) \\
 & + E_3(x, y)] + F_3(x, y)\}\}\}A_1(x, y), \quad (x, y) \in \Delta,
 \end{aligned} \tag{3.3}$$

where

$$A_1(x, y) = \exp\left(\int_{\alpha(x)}^\infty M_1 a(\alpha^{-1}(s), y) ds\right), \tag{3.4}$$

$$\begin{aligned}
 & B_3(M, N) \\
 &= \sum_{i=1}^{l_1} \int_{\alpha_i(M)}^{\infty} \int_{\beta_i(N)}^{\infty} M_{1i} M_{2i} \left[ b_i(\alpha_i^{-1}(s), \beta_i^{-1}(t), M, N) \varphi_1(k(s, t) A_1(s, t)) \right. \\
 &+ \left. \int_s^{\infty} \int_t^{\infty} M_{1i} M_{2i} c_i(\alpha_i^{-1}(\xi), \beta_i^{-1}(\eta), M, N) \varphi_2(k(\xi, \eta) A_1(\xi, \eta)) d\xi d\eta \right] ds dt \\
 &+ \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} M_{1j} M_{2j} \left[ d_j(\alpha_j^{-1}(s), \beta_j^{-1}(t), M, N) k(s, t) A_1(s, t) \right. \\
 &+ \left. \int_s^{\infty} \int_t^{\infty} M_{1j} M_{2j} e_j(\alpha_j^{-1}(\xi), \beta_j^{-1}(\eta), M, N) k(\xi, \eta) A_1(\xi, \eta) d\xi d\eta \right] ds dt, \tag{3.5}
 \end{aligned}$$

$$\begin{aligned}
 D_3(M, N) &= \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} M_{1j} M_{2j} \left[ d_j(\alpha_j^{-1}(s), \beta_j^{-1}(t), M, N) A_1(s, t) \right. \\
 &+ \left. \int_s^{\infty} \int_t^{\infty} M_{1j} M_{2j} e_j(\alpha_j^{-1}(\xi), \beta_j^{-1}(\eta), M, N) A_1(\xi, \eta) d\xi d\eta \right] ds dt, \tag{3.6}
 \end{aligned}$$

$$E_3(M, N) = \sum_{i=1}^{l_1} \int_{\alpha_i(M)}^{\infty} \int_{\beta_i(N)}^{\infty} M_{1i} M_{2i} b_i(\alpha_i^{-1}(s), \beta_i^{-1}(t), M, N) \varphi_1(A_1(s, t)) ds dt, \tag{3.7}$$

$$\begin{aligned}
 F_3(x, y) &= \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} M_{1i}^2 M_{2i}^2 \\
 &\times \left[ \int_s^{\infty} \int_t^{\infty} c_i(\alpha_i^{-1}(\xi), \beta_i^{-1}(\eta), x, y) \varphi_2(A_1(\xi, \eta)) d\xi d\eta \right] ds dt, \tag{3.8}
 \end{aligned}$$

$$\begin{aligned}
 M_1 &= \max_{x \in I_1} \frac{1}{\alpha'(\alpha^{-1}(x))} < \infty, \\
 M_{1i} &= \max_{x \in I_1} \frac{1}{\alpha'_i(\alpha_i^{-1}(x))} < \infty, & M_{2i} &= \max_{y \in I_2} \frac{1}{\beta'_i(\beta_i^{-1}(y))} < \infty, & i &= 1, 2, \dots, l_1; \\
 M_{1j} &= \max_{x \in I_1} \frac{1}{\alpha'_j(\alpha_j^{-1}(x))} < \infty, & M_{2j} &= \max_{y \in I_2} \frac{1}{\beta'_j(\beta_j^{-1}(y))} < \infty, & j &= 1, 2, \dots, l_2;
 \end{aligned} \tag{3.9}$$

$W_1, W_2$  are defined in Theorem 2.1.

*Proof* By applying the conditions (3.2) to (3.1), we have

$$\begin{aligned}
 |v(x, y)| &\leq k(x, y) + \int_x^{\infty} a(s, y) \cdot |v(s + \rho(s), y)| ds \\
 &+ \sum_{i=1}^{l_1} \int_x^{\infty} \int_y^{\infty} \left[ b_i(s, t, x, y) \varphi_1(|v(s + \rho_i(s), t + \gamma_i(t))|) \right. \\
 &+ \left. \int_s^{\infty} \int_t^{\infty} c_i(\xi, \eta, x, y) \varphi_2 \left( \left| \max_{\sigma \in [\xi + \rho_i(\xi), \eta(\xi + \rho_i(\xi))]} v(\sigma, \eta + \gamma_i(\eta)) \right| \right) d\xi d\eta \right] ds dt \\
 &+ \sum_{j=1}^{l_2} \int_M^{\infty} \int_N^{\infty} \left[ d_j(s, t, x, y) |v(s + \rho_j(s), t + \gamma_j(t))| \right. \\
 &+ \left. \int_s^{\infty} \int_t^{\infty} e_j(\xi, \eta, x, y) \left| \max_{\sigma \in [\xi + \rho_j(\xi), \eta(\xi + \rho_j(\xi))]} v(\sigma, \eta + \gamma_j(\eta)) \right| d\xi d\eta \right] ds dt
 \end{aligned}$$

$$\begin{aligned}
 &\leq k(x, y) + \int_{\alpha(x)}^{\infty} a(\alpha^{-1}(s), y) |v(s, y)| \frac{1}{\alpha'(\alpha^{-1}(s))} ds \\
 &\quad + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ b_i(\alpha_i^{-1}(s), \beta_i^{-1}(t), x, y) \varphi_1(|v(s, t)|) \right. \\
 &\quad + \int_s^{\infty} \int_t^{\infty} c_i(\alpha_i^{-1}(\xi), \beta_i^{-1}(\eta), x, y) \varphi_2\left(\max_{\sigma \in [\xi, h\xi]} |v(\sigma, \eta)|\right) \\
 &\quad \times \frac{1}{\alpha'_i(\alpha_i^{-1}(\xi))} \frac{1}{\beta'_i(\beta_i^{-1}(\eta))} d\xi d\eta \left. \right] \frac{1}{\alpha'_i(\alpha_i^{-1}(s))} \frac{1}{\beta'_i(\beta_i^{-1}(t))} ds dt \\
 &\quad + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left[ d_j(\alpha_j^{-1}(s), \beta_j^{-1}(t), x, y) |v(s, t)| \right. \\
 &\quad + \int_s^{\infty} \int_t^{\infty} e_j(\alpha_j^{-1}(\xi), \beta_j^{-1}(\eta), x, y) \\
 &\quad \times \max_{\sigma \in [\xi, h\xi]} |v(\sigma, \eta)| \frac{1}{\alpha'_j(\alpha_j^{-1}(\xi))} \frac{1}{\beta'_j(\beta_j^{-1}(\eta))} d\xi d\eta \left. \right] \frac{1}{\alpha'_j(\alpha_j^{-1}(s))} \frac{1}{\beta'_j(\beta_j^{-1}(t))} ds dt \\
 &\leq k(x, y) + \int_{\alpha(x)}^{\infty} M_1 a(\alpha^{-1}(s), y) |v(s, y)| ds \\
 &\quad + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \left[ M_{1i} M_{2i} b_i(\alpha_i^{-1}(s), \beta_i^{-1}(t), x, y) \varphi_1(|v(s, t)|) \right. \\
 &\quad + \int_s^{\infty} \int_t^{\infty} M_{1i}^2 M_{2i}^2 c_i(\alpha_i^{-1}(\xi), \beta_i^{-1}(\eta), x, y) \varphi_2\left(\max_{\sigma \in [\xi, h\xi]} |v(\sigma, \eta)|\right) d\xi d\eta \left. \right] ds dt \\
 &\quad + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} \left[ M_{1j} M_{2j} d_j(\alpha_j^{-1}(s), \beta_j^{-1}(t), x, y) |v(s, t)| \right. \\
 &\quad + \int_s^{\infty} \int_t^{\infty} M_{1j}^2 M_{2j}^2 e_j(\alpha_j^{-1}(\xi), \beta_j^{-1}(\eta), x, y) \\
 &\quad \times \max_{\sigma \in [\xi, h\xi]} |v(\sigma, \eta)| d\xi d\eta \left. \right] ds dt, \tag{3.10}
 \end{aligned}$$

for  $(x, y) \in \Delta$ , where  $M_1, M_{1i}, M_{2i}$  ( $i = 1, 2, \dots, l_1$ ),  $M_{1j}, M_{2j}$  ( $j = 1, 2, \dots, l_2$ ) are defined in (3.9). Applying the results of Theorem 2.1 to (3.10) with  $\psi(u) = u$ ,  $a(s, y) = M_1 a(\alpha^{-1}(s), y)$ ,  $b_i(s, t, x, y) = M_{1i} M_{2i} b_i(\alpha_i^{-1}(s), \beta_i^{-1}(t), x, y)$ ,  $c_i(\xi, \eta, x, y) = M_{1i}^2 M_{2i}^2 c_i(\alpha_i^{-1}(\xi), \beta_i^{-1}(\eta), x, y)$ ,  $d_j(s, t, x, y) = M_{1j} M_{2j} d_j(\alpha_j^{-1}(s), \beta_j^{-1}(t), x, y)$ ,  $e_j(\xi, \eta, x, y) = M_{1j}^2 M_{2j}^2 e_j(\alpha_j^{-1}(\xi), \beta_j^{-1}(\eta), x, y)$ , we obtain the desired estimation (3.3).  $\square$

**Theorem 3.2** *In equation (3.1), suppose that the following conditions hold:*

$$\begin{aligned}
 &\psi(v(x, y)) = v^p(x, y), \quad |g_1(x, y)| \leq k(x, y), \quad |g_2(x, y)| \leq a(x, y), \\
 &|F_{1i}(s, t, x, y, u, v)| \leq b_i(s, t, x, y) |u|^{q_i} + |v|, \\
 &|F_{2i}(s, t, x, y, u)| \leq c_i(s, t, x, y) |u|^{r_i}, \quad i = 1, 2, \dots, l_1, \\
 &|G_{1j}(s, t, x, y, u, v)| \leq d_j(s, t, x, y) |u|^{e_j} + |v|, \\
 &|G_{2j}(s, t, x, y, u)| \leq e_j(s, t, x, y) |u|^{\delta_j}, \quad j = 1, 2, \dots, l_2,
 \end{aligned} \tag{3.11}$$

where  $p, q_i, r_i, \varepsilon_j, \delta_j, b_i, c_i, d_j, e_j$  ( $i = 1, 2, \dots, l_1; j = 1, 2, \dots, l_2$ ) are defined as in Theorem 2.2. Then we have the following estimate:

$$|v(x, y)| \leq \left\{ \left[ k(x, y) + \frac{B_4(M, N)}{1 - D_4(M, N)} \exp(F_4(x, y)) \right] A_1(x, y) \right\}^{\frac{1}{p}}, \tag{3.12}$$

where

$$\begin{aligned} B_4(M, N) = & \sum_{i=1}^{l_1} \int_{\alpha_i(M)}^{\infty} \int_{\beta_i(N)}^{\infty} M_{1i} M_{2i} \left\{ b_i(\alpha_i^{-1}(s), \beta_i^{-1}(t), M, N) A_1^{\frac{q_i}{p}}(s, t) \right. \\ & \times \left[ \frac{q_i}{p} K_1^{\frac{q_i-p}{p}} k(s, t) + \frac{p - q_i}{p} K_1^{\frac{q_i}{p}} \right] \\ & + \int_s^{\infty} \int_t^{\infty} M_{1i} M_{2i} c_i(\alpha_i^{-1}(\xi), \beta_i^{-1}(\eta), M, N) \\ & \times A_1^{\frac{r_i}{p}}(\xi, \eta) \left[ \frac{r_i}{p} K_2^{\frac{r_i-p}{p}} k(\xi, \eta) + \frac{p - r_i}{p} K_2^{\frac{r_i}{p}} \right] d\xi d\eta \left. \right\} ds dt \\ & + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} M_{1j} M_{2j} \left\{ d_j(\alpha_j^{-1}(s), \beta_j^{-1}(t), M, N) A_1^{\frac{\varepsilon_j}{p}}(s, t) \right. \\ & \times \left[ \frac{\varepsilon_j}{p} K_3^{\frac{\varepsilon_j-p}{p}} k(s, t) + \frac{p - \varepsilon_j}{p} K_3^{\frac{\varepsilon_j}{p}} \right] \\ & + \int_s^{\infty} \int_t^{\infty} M_{1j} M_{2j} e_j(\alpha_j^{-1}(\xi), \beta_j^{-1}(\eta), M, N) \\ & \times A_1^{\frac{\delta_j}{p}}(\xi, \eta) \left[ \frac{\delta_j}{p} K_4^{\frac{\delta_j-p}{p}} k(\xi, \eta) + \frac{p - \delta_j}{p} K_4^{\frac{\delta_j}{p}} \right] d\xi d\eta \left. \right\} ds dt, \tag{3.13} \end{aligned}$$

$$\begin{aligned} D_4(M, N) = & \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^{\infty} \int_{\beta_j(N)}^{\infty} M_{1j} M_{2j} \\ & \times \left[ d_j(\alpha_j^{-1}(s), \beta_j^{-1}(t), M, N) A_1^{\frac{\varepsilon_j}{p}}(s, t) \frac{\varepsilon_j}{p} K_3^{\frac{\varepsilon_j-p}{p}} \exp(F_4(s, t)) \right. \\ & + \int_s^{\infty} \int_t^{\infty} M_{1j} M_{2j} e_j(\alpha_j^{-1}(\xi), \beta_j^{-1}(\eta), M, N) \\ & \times A_1^{\frac{\delta_j}{p}}(\xi, \eta) \frac{\delta_j}{p} K_4^{\frac{\delta_j-p}{p}} \exp(F_4(\xi, \eta)) d\xi d\eta \left. \right] ds dt \\ & < 1, \tag{3.14} \end{aligned}$$

$$\begin{aligned} F_4(x, y) = & \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} M_{1i} M_{2i} \left[ b_i(\alpha_i^{-1}(s), \beta_i^{-1}(t), x, y) A_1^{\frac{q_i}{p}}(s, t) \frac{q_i}{p} K_1^{\frac{q_i-p}{p}} \right. \\ & + \int_s^{\infty} \int_t^{\infty} M_{1i} M_{2i} c_i(\alpha_i^{-1}(\xi), \beta_i^{-1}(\eta), x, y) \\ & \times A_1^{\frac{r_i}{p}}(\xi, \eta) \frac{r_i}{p} K_2^{\frac{r_i-p}{p}} d\xi d\eta \left. \right] ds dt. \tag{3.15} \end{aligned}$$

*Proof* Applying the conditions of (3.11) to (3.1), we have

$$\begin{aligned}
 |v(x, y)|^p &\leq k(x, y) + \int_x^\infty a(s, y) |v(s + \rho(s), y)|^p ds \\
 &\quad + \sum_{i=1}^{l_1} \int_x^\infty \int_y^\infty \left[ b_i(s, t, x, y) |v(s + \rho_i(s), t + \gamma_i(t))|^{q_i} \right. \\
 &\quad \left. + \int_s^\infty \int_t^\infty c_i(\xi, \eta, x, y) \Big|_{\sigma \in [\xi + \rho_i(\xi), h(\xi + \rho_i(\xi))]}^{\max} v(\sigma, \eta + \gamma_i(\eta)) \Big|^{r_i} d\xi d\eta \right] ds dt \\
 &\quad + \sum_{j=1}^{l_2} \int_M^\infty \int_N^\infty \left[ d_j(s, t, x, y) |v(s + \rho_j(s), t + \gamma_j(t))|^{\varepsilon_j} \right. \\
 &\quad \left. + \int_s^\infty \int_t^\infty e_j(\xi, \eta, x, y) \Big|_{\sigma \in [\xi + \rho_j(\xi), h(\xi + \rho_j(\xi))]}^{\max} v(\sigma, \eta + \gamma_j(\eta)) \Big|^{\delta_j} d\xi d\eta \right] ds dt \\
 &\leq k(x, y) + \int_{\alpha(x)}^\infty a(\alpha^{-1}(s), y) |v(s, y)|^p \frac{1}{\alpha'(\alpha^{-1}(s))} ds \\
 &\quad + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^\infty \int_{\beta_i(y)}^\infty \left[ b_i(\alpha_i^{-1}(s), \beta_i^{-1}(t), x, y) |v(s, t)|^{q_i} \right. \\
 &\quad \left. + \int_s^\infty \int_t^\infty c_i(\alpha_i^{-1}(\xi), \beta_i^{-1}(\eta), x, y) \right. \\
 &\quad \left. \times \max_{\sigma \in [\xi, h\xi]} |v(\sigma, \eta)|^{r_i} \frac{1}{\alpha'_i(\alpha_i^{-1}(\xi))} \frac{1}{\beta'_i(\beta_i^{-1}(\eta))} d\xi d\eta \right] \frac{1}{\alpha'_i(\alpha_i^{-1}(s))} \frac{1}{\beta'_i(\beta_i^{-1}(t))} ds dt \\
 &\quad + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^\infty \int_{\beta_j(N)}^\infty \left[ d_j(\alpha_j^{-1}(s), \beta_j^{-1}(t), x, y) |v(s, t)|^{\varepsilon_j} \right. \\
 &\quad \left. + \int_s^\infty \int_t^\infty e_j(\alpha_j^{-1}(\xi), \beta_j^{-1}(\eta), x, y) \right. \\
 &\quad \left. \times \max_{\sigma \in [\xi, h\xi]} |v(\sigma, \eta)|^{\delta_j} \frac{1}{\alpha'_j(\alpha_j^{-1}(\xi))} \frac{1}{\beta'_j(\beta_j^{-1}(\eta))} d\xi d\eta \right] \\
 &\quad \times \frac{1}{\alpha'_j(\alpha_j^{-1}(s))} \frac{1}{\beta'_j(\beta_j^{-1}(t))} ds dt \\
 &\leq k(x, y) + \int_{\alpha(x)}^\infty M_1 a(\alpha^{-1}(s), y) |v(s, y)|^p ds \\
 &\quad + \sum_{i=1}^{l_1} \int_{\alpha_i(x)}^\infty \int_{\beta_i(y)}^\infty \left[ M_{1i} M_{2i} b_i(\alpha_i^{-1}(s), \beta_i^{-1}(t), x, y) |v(s, t)|^{q_i} \right. \\
 &\quad \left. + \int_s^\infty \int_t^\infty M_{1i}^2 M_{2i}^2 c_i(\alpha_i^{-1}(\xi), \beta_i^{-1}(\eta), x, y) \max_{\sigma \in [\xi, h\xi]} |v(\sigma, \eta)|^{r_i} d\xi d\eta \right] ds dt \\
 &\quad + \sum_{j=1}^{l_2} \int_{\alpha_j(M)}^\infty \int_{\beta_j(N)}^\infty \left[ M_{1j} M_{2j} d_j(\alpha_j^{-1}(s), \beta_j^{-1}(t), x, y) |v(s, t)|^{\varepsilon_j} \right. \\
 &\quad \left. + \int_s^\infty \int_t^\infty M_{1j}^2 M_{2j}^2 e_j(\alpha_j^{-1}(\xi), \beta_j^{-1}(\eta), x, y) \right. \\
 &\quad \left. \times \max_{\sigma \in [\xi, h\xi]} |v(\sigma, \eta)|^{\delta_j} d\xi d\eta \right] ds dt, \tag{3.16}
 \end{aligned}$$

for  $(x, y) \in \Delta$ , where  $M_1, M_{1i}, M_{2i}$  ( $i = 1, 2, \dots, l_1$ ),  $M_{1j}, M_{2j}$  ( $j = 1, 2, \dots, l_2$ ) are defined in (3.9). Applying the results of Theorem 2.2 to (3.16) with  $a(s, y) = M_1 a(\alpha^{-1}(s), y)$ ,  $b_i(s, t, x, y) = M_{1i} M_{2i} b_i(\alpha_i^{-1}(s), \beta_i^{-1}(t), x, y)$ ,  $c_i(\xi, \eta, x, y) = M_{1i}^2 M_{2i}^2 c_i(\alpha_i^{-1}(\xi), \beta_i^{-1}(\eta), x, y)$ ,  $d_j(s, t, x, y) = M_{1j} M_{2j} d_j(\alpha_j^{-1}(s), \beta_j^{-1}(t), x, y)$ ,  $e_j(\xi, \eta, x, y) = M_{1j}^2 M_{2j}^2 e_j(\alpha_j^{-1}(\xi), \beta_j^{-1}(\eta), x, y)$ , we obtain the desired estimation (3.12).  $\square$

#### 4 Conclusion

In this paper, we established several new retarded nonlinear Volterra-Fredholm type integral inequalities with maxima in two independent variables in Theorem 2.1 and Theorem 2.2, and gave their specific cases in Corollary 2.1 and Corollary 2.2, respectively, which can be used in the analysis of the qualitative properties to solutions of integral equations with maxima. In Theorem 3.1 and Theorem 3.2, we also presented the applications to research the boundedness of solutions of retarded nonlinear Volterra-Fredholm type integral equations.

Using our method, one can further study the integral inequality with more dimensions.

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#### Competing interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

#### Authors' contributions

RX proved parts of the results in Section 2 and participated in Section 3 - Applications. XM carried out the generalized weakly singular integral inequalities and completed part of the proof. All authors read and approved the final manuscript.

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