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On a more accurate Hardy-Mulholland-type inequality

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Abstract
By using the way of weight coefficients, the technique of real analysis, and Hermite-Hadamard's inequality, a more accurate Hardy-Mulholland-type inequality with multi-parameters and a best possible constant factor is given. The equivalent forms, the reverses, the operator expressions and some particular cases are considered.

MSC: 26D15; 47A07

Keywords: Mulholland-type inequality; weight coefficient; equivalent form; reverse; operator

1 Introduction

Assuming that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$, $\|a\|_p = (\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}} > 0$, and $\|b\|_q > 0$, we have the following Hardy-Hilbert's inequality with the best possible constant $\frac{\pi}{\sin(\pi/p)}$ (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q. \tag{1}$$

A more accurate inequality of (1) is given as follows (cf. [1], Th. 323 and [2]):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n-\alpha} < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q \quad (0 \leq \alpha \leq 1), \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is still the best possible.

Also we have the following Mulholland's inequality similar to (1) with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [3] or [1], Th. 343, replacing $\frac{a_m}{m}$, $\frac{b_n}{n}$ by a_m, b_n):

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^{\infty}\frac{a_m^p}{m^{1-p}}\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\frac{b_n^q}{n^{1-q}}\right)^{\frac{1}{q}}. \tag{3}$$

Inequalities (1)-(3) are important in analysis and its applications (cf. [1, 2, 4–18]).

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Suppose that $\mu_i, \upsilon_j > 0$ $(i,j \in \mathbb{N} = \{1,2,\ldots\})$,

$$U_m = \sum_{i=1}^{m} \mu_i, \qquad V_n = \sum_{j=1}^{n} \upsilon_j \quad (m,n \in \mathbb{N}), \tag{4}$$

we have the following Hardy-Hilbert-type inequality (cf. [1], Theorem 321, replacing $\mu_m^{1/q} a_m$ and $\upsilon_n^{1/p} b_n$ by a_m and b_n): If $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} \frac{a_m^p}{m^{p-1}} < \infty$, $0 < \sum_{n=1}^{\infty} \frac{b_n^q}{n^{q-1}} < \infty$, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} \frac{a_m^p}{\mu_m^{p-1}} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{b_n^q}{\upsilon_n^{q-1}} \right)^{\frac{1}{q}}. \tag{5}$$

For $\mu_i = \upsilon_j = 1$ $(i,j \in \mathbb{N})$, inequality (5) reduces to (1).

In 2015, Yang [19] gave an extension of (5) as follows: If $0 < \lambda_1, \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = \lambda$, $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$ are positive and decreasing, with $U_\infty = V_\infty = \infty$, then we have the following inequality with the best possible constant factor $\pi / \sin(\frac{\pi\lambda_1}{\lambda})$:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{U_m^\lambda + V_n^\lambda} < \frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})} \left[\sum_{m=1}^{\infty} \frac{U_m^{p(1-\lambda_1)-1} a_m^p}{\mu_m^{p-1}} \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \frac{V_n^{q(1-\lambda_2)-1} b_n^q}{\upsilon_n^{q-1}} \right]^{\frac{1}{q}}. \tag{6}$$

In this paper, by using the way of weight coefficients, the technique of real analysis, and Hermite-Hadamard's inequality, a new Hardy-Mulholland-type inequality with a best possible constant factor is given as follows: If $\mu_1 = \upsilon_1 = 1$, $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$ are positive and decreasing, with $U_\infty = V_\infty = \infty$, we have the following inequality:

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{\ln U_m V_n} < \frac{\pi}{\sin(\frac{\pi}{p})} \left[\sum_{m=2}^{\infty} \left(\frac{U_m}{\mu_{m+1}} \right)^{p-1} a_m^p \right]^{\frac{1}{p}} \left[\sum_{n=2}^{\infty} \left(\frac{V_n}{\upsilon_{n+1}} \right)^{q-1} b_n^q \right]^{\frac{1}{q}}, \tag{7}$$

which is an extension of (3). Moreover, the more accurate inequality of (7) and its extension with multi-parameters and the best possible constant factors are obtained. The equivalent forms, the reverses, the operator expressions and some particular cases are considered.

2 Some lemmas and an example

In the following, we agree that $p \neq 0,1$, $\frac{1}{p} + \frac{1}{q} = 1$, $-1 < \gamma \leq 0$, $0 < \lambda_1, \lambda_2 < 1$, $\lambda_1 + \lambda_2 = \lambda$, $\mu_i, \upsilon_j > 0$ $(i,j \in \mathbb{N})$, with $\mu_1 = \upsilon_1 = 1$, U_m and V_n are defined by (4),

$$\frac{1}{1 + \frac{\mu_2}{2}} \leq \alpha \leq 1, \qquad \frac{1}{1 + \frac{\upsilon_2}{2}} \leq \beta \leq 1,$$

$a_m, b_n \geq 0$, $\|a\|_{p,\Phi_\lambda} := (\sum_{m=2}^{\infty} \Phi_\lambda(m) a_m^p)^{\frac{1}{p}}$ and $\|b\|_{q,\Psi_\lambda} := (\sum_{n=2}^{\infty} \Psi_\lambda(n) b_n^q)^{\frac{1}{q}}$, where

$$\begin{aligned} \Phi_\lambda(m) &:= \left(\frac{U_m}{\mu_{m+1}} \right)^{p-1} (\ln \alpha U_m)^{p(1-\lambda_1)-1}, \\ \Psi_\lambda(n) &:= \left(\frac{V_n}{\upsilon_{n+1}} \right)^{q-1} (\ln \beta V_n)^{q(1-\lambda_2)-1} \quad \bigl(m,n \in \mathbb{N} \backslash \{1\}\bigr). \end{aligned} \tag{8}$$

Lemma 1 *If $n \in \mathbb{N}\backslash\{1\}$, $a \in (n-\frac{1}{2}, n)$, $f(x)$ is continuous in $(n-\frac{1}{2}, n+\frac{1}{2})$, and $f'(x)$ is strictly increasing in the intervals $(n-\frac{1}{2}, a)$, (a, n) and $(n, n+\frac{1}{2})$, respectively, satisfying*

$$f'(a-0) \le f'(a+0), \qquad f'(n-0) \le f'(n+0), \tag{9}$$

then we have the following Hermite-Hadamard's inequality (cf. [20]).

Proof In view of $f'(n-0) \le f'(n+0) = \lim_{x\to n^+} f'(x)$ is finite, we set the linear function $g(x)$ as follows:

$$g(x) := f'(n-0)(x-n) + f(n), \quad x \in \left[n-\frac{1}{2}, n+\frac{1}{2}\right].$$

Since $f'(x)$ is strictly increasing in $[n-\frac{1}{2}, a)$ and (a, n), then for $x \in [n-\frac{1}{2}, a)$,

$$f'(x) < \lim_{x\to a^-} f'(x) = f'(a-0) \le f'(a+0) < f'(n-0);$$

for $x \in (a, n)$, $f'(x) < \lim_{x\to n^-} f'(x) = f'(n-0)$. Hence,

$$\bigl(f(x) - g(x)\bigr)' = f'(x) - f'(n-0) < 0, \quad x \in \left(n-\frac{1}{2}, a\right) \cup (a, n).$$

Since $f(x) - g(x)$ is continuous in $(n-\frac{1}{2}, n]$ with $f(n) - g(n) = 0$, it follows that

$$f(x) - g(x) > 0, \quad x \in \left(n-\frac{1}{2}, n\right).$$

In the same way, since $f'(x)$ is strictly increasing in $(n, n+\frac{1}{2})$, then for $x \in (n, n+\frac{1}{2})$, $f'(x) > f'(n+0) \ge f'(n-0)$. Hence,

$$\bigl(f(x) - g(x)\bigr)' = f'(x) - f'(n-0) > 0, \quad x \in \left(n, n+\frac{1}{2}\right).$$

Since $f(x) - g(x)$ is continuous in $[n, n+\frac{1}{2})$ with $f(n) - g(n) = 0$, it follows that

$$f(x) - g(x) > 0, \quad x \in \left(n, n+\frac{1}{2}\right).$$

Therefore, we have $f(x) - g(x) > 0$, $x \in (n-\frac{1}{2}, n+\frac{1}{2})\backslash\{n\}$. Then we find

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(x)\,dx > \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} g(x)\,dx = f(n),$$

namely, (9) follows. The lemma is proved. □

Note With the assumptions of Lemma 1, if (i) $a \in (n, n+\frac{1}{2})$, $f'(x)$ is strictly increasing in the intervals $(n-\frac{1}{2}, n)$, (n, a) and $(a, n+\frac{1}{2})$, respectively, or (ii) $a = n$, $f'(x)$ is strictly increasing in the intervals $(n-\frac{1}{2}, n)$ and $(n, n+\frac{1}{2})$, respectively, then in the same way, we still can obtain (9).

Example 1 $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$ are decreasing, we set functions $\mu(t) := \mu_m$, $t \in (m-1, m]$ ($m \in \mathbb{N}$), $\upsilon(t) := \upsilon_n$, $t \in (n-1, n]$ ($n \in \mathbb{N}$), and

$$U(x) := \int_0^x \mu(t)\,dt \quad (x \geq 0), \qquad V(y) := \int_0^y \upsilon(t)\,dt \quad (y \geq 0). \tag{10}$$

Then it follows that $U(m) = U_m$, $V(n) = V_n$, $U(\infty) = U_\infty$, $V(\infty) = V_\infty$ and

$$U'(x) = \mu(x) = \mu_m, \quad x \in (m-1, m),$$
$$V'(y) = \upsilon(y) = \upsilon_n, \quad y \in (n-1, n)\ (x, y \in \mathbb{N}).$$

For $0 < \lambda \leq 1$, $-1 < \gamma \leq 0$, we set

$$k_\lambda(x,y) := \frac{1}{x^\lambda + y^\lambda + \gamma|x^\lambda - y^\lambda|} \quad (x, y > 0). \tag{11}$$

We find

$$\begin{aligned} 0 < K_\gamma(\lambda_1) &:= \int_0^\infty k_\lambda(1,t)t^{\lambda_2 - 1}\,dt = \int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}\,dt \\ &= \int_0^\infty \frac{t^{\lambda_1-1}}{t^\lambda + 1 + \gamma|t^\lambda - 1|}\,dt = \int_0^1 \frac{t^{\lambda_1-1} + t^{\lambda_2-1}}{1+\gamma+(1-\gamma)t^\lambda}\,dt \\ &\leq \int_0^1 \frac{t^{\lambda_1-1}+t^{\lambda_2-1}}{1+\gamma}\,dt = \frac{1}{1+\gamma}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) < \infty, \end{aligned} \tag{12}$$

namely, $K_\gamma(\lambda_1) \in \mathbb{R}_+$. In the following, we express $K_\gamma(\lambda_1)$ in other forms.

(i) For $\gamma = 0$, we obtain

$$K_0(\lambda_1) = \int_0^\infty \frac{t^{\lambda_1-1}}{t^\lambda+1}\,dt = \frac{1}{\lambda}\int_0^\infty \frac{\nu^{(\lambda_1/\lambda)-1}}{\nu+1}\,d\nu = \frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}; \tag{13}$$

(ii) for $-1 < \gamma < 0$, $0 < \frac{1+\gamma}{1-\gamma} < 1$, by the Lebesgue term by term integration theorem (cf. [21]), we find

$$\begin{aligned} K_\gamma(\lambda_1) &= \frac{1}{1-\gamma}\int_0^1 \frac{t^{-\lambda_2-1}+t^{-\lambda_1-1}}{\frac{1+\gamma}{1-\gamma}t^{-\lambda}+1}\,dt \\ &\overset{\nu=\frac{1+\gamma}{1-\gamma}t^{-\lambda}}{=} \frac{1}{\lambda(1-\gamma)}\int_{\frac{1+\gamma}{1-\gamma}}^\infty \frac{1}{\nu+1}\left[\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_1}{\lambda}-1}\nu^{\frac{\lambda_2}{\lambda}-1} + \left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_2}{\lambda}-1}\nu^{\frac{\lambda_1}{\lambda}-1}\right]d\nu \\ &= \frac{1}{\lambda(1-\gamma)}\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_1}{\lambda}-1}\int_0^\infty \frac{\nu^{\frac{\lambda_2}{\lambda}-1}}{\nu+1}\,d\nu + \frac{1}{\lambda(1-\gamma)}\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_2}{\lambda}-1}\int_0^\infty \frac{\nu^{\frac{\lambda_1}{\lambda}-1}}{\nu+1}\,d\nu \\ &\quad - \frac{1}{\lambda(1-\gamma)}\int_0^{\frac{1+\gamma}{1-\gamma}} \frac{1}{\nu+1}\left[\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_1}{\lambda}-1}\nu^{\frac{\lambda_2}{\lambda}-1} + \left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_2}{\lambda}-1}\nu^{\frac{\lambda_1}{\lambda}-1}\right]d\nu \\ &= \frac{1}{\lambda(1-\gamma)}\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_1}{\lambda}-1}\frac{\pi}{\sin(\frac{\pi\lambda_2}{\lambda})} + \frac{1}{\lambda(1-\gamma)}\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_2}{\lambda}-1}\frac{\pi}{\sin(\frac{\pi\lambda_1}{\lambda})} \\ &\quad - \frac{1}{\lambda(1-\gamma)}\int_0^{\frac{1+\gamma}{1-\gamma}} \sum_{k=0}^\infty (-1)^k\nu^k\left[\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_1}{\lambda}-1}\nu^{\frac{\lambda_2}{\lambda}-1} + \left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_2}{\lambda}-1}\nu^{\frac{\lambda_1}{\lambda}-1}\right]d\nu \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\lambda(1-\gamma)}\left[\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_1}{\lambda}-1} + \left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_2}{\lambda}-1}\right]\frac{\pi}{\sin(\frac{\pi\lambda_1}{\lambda})} \\
&\quad - \frac{1}{\lambda(1-\gamma)}\int_0^{\frac{1+\gamma}{1-\gamma}}\sum_{k=0}^{\infty}\left(v^{2k}-v^{2k+1}\right)\left[\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_1}{\lambda}-1} v^{\frac{\lambda_2}{\lambda}-1} \right.\\
&\quad \left. + \left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_2}{\lambda}-1} v^{\frac{\lambda_1}{\lambda}-1}\right]dv \\
&= \frac{1}{1+\gamma}\left[\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_1}{\lambda}} + \left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_2}{\lambda}}\right]\frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})} \\
&\quad - \frac{1}{\lambda(1+\gamma)}\int_0^{\frac{1+\gamma}{1-\gamma}}\sum_{k=0}^{\infty}\left(v^{2k}-v^{2k+1}\right)\left[\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_1}{\lambda}} v^{\frac{\lambda_2}{\lambda}-1} + \left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_2}{\lambda}} v^{\frac{\lambda_1}{\lambda}-1}\right]dv \\
&= \frac{1}{1+\gamma}\left[\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_1}{\lambda}} + \left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_2}{\lambda}}\right]\frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})} \\
&\quad - \frac{1}{\lambda(1+\gamma)}\sum_{k=0}^{\infty}\int_0^{\frac{1+\gamma}{1-\gamma}}(-1)^k v^k\left[\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_1}{\lambda}} v^{\frac{\lambda_2}{\lambda}-1} + \left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_2}{\lambda}-1} v^{\frac{\lambda_1}{\lambda}-1}\right]dv \\
&= \frac{1}{1+\gamma}\left[\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_1}{\lambda}} + \left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{\lambda_2}{\lambda}}\right]\frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})} \\
&\quad - \frac{1}{1+\gamma}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1+\gamma}{1-\gamma}\right)^{k+1}\left(\frac{1}{\lambda k+\lambda_2}+\frac{1}{\lambda k+\lambda_1}\right);
\end{aligned}
\tag{14}
$$

(iii) for $\lambda_1 = \lambda_2 = \frac{\lambda}{2}$, $-1 < \gamma < 0$, we find

$$
\begin{aligned}
K_\gamma\left(\frac{\lambda}{2}\right) &= 2\int_0^1 \frac{t^{(\lambda/2)-1}}{1+\gamma+(1-\gamma)t^\lambda}\,dt \overset{u=(\frac{1-\gamma}{1+\gamma}t^\lambda)^{\frac{1}{2}}}{=} \frac{4}{\lambda(1+\gamma)}\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{1}{2}}\int_0^{(\frac{1-\gamma}{1+\gamma})^{\frac{1}{2}}}\frac{du}{1+u^2} \\
&= \frac{4}{\lambda(1+\gamma)}\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{1}{2}}\arctan\left(\frac{1-\gamma}{1+\gamma}\right)^{\frac{1}{2}}.
\end{aligned}
\tag{15}
$$

For fixed $m \in \mathbb{N}\backslash\{1\}$, we define the function $f(y)$ as follows:

$$
f(y) := k_\lambda\left(\ln\alpha U_m, \ln\beta V(y)\right), \quad y \in \left(n-\frac{1}{2}, n+\frac{1}{2}\right)\ \left(n \in \mathbb{N}\backslash\{1\}\right).
$$

Then $f(y)$ is continuous in $(n-\frac{1}{2}, n+\frac{1}{2})$ $(n \in \mathbb{N}\backslash\{1\})$. There exists a unified number $y_0 > \frac{3}{2}$ satisfying $V(y_0) = \frac{\alpha}{\beta}U_m$.

(i) If $y_0 \in (n-\frac{1}{2}, n+\frac{1}{2})$, we find

$$
f(y) = \begin{cases} \frac{1}{(1+\gamma)\ln^\lambda\alpha U_m + (1-\gamma)\ln^\lambda\beta V(y)}, & n-\frac{1}{2} < y < y_0, \\ \frac{1}{(1-\gamma)\ln^\lambda\alpha U_m + (1+\gamma)\ln^\lambda\beta V(y)}, & y_0 < y < n+\frac{1}{2}. \end{cases}
$$

For $y_0 \neq n$, we obtain for $y \neq n$ that

$$f'(y) = \begin{cases} \frac{-\lambda(1-\gamma)V'(y)\ln^{\lambda-1}\beta V(y)}{V(y)[(1+\gamma)\ln^{\lambda}\alpha U_m+(1-\gamma)\ln^{\lambda}\beta V(y)]^2}, & n-\frac{1}{2}<y<y_0, \\ \frac{-\lambda(1+\gamma)V'(y)\ln^{\lambda-1}\beta V(y)}{V(y)[(1-\gamma)\ln^{\lambda}\alpha U_m+(1+\gamma)\ln^{\lambda}\beta V(y)]^2}, & y_0<y<n+\frac{1}{2}; \end{cases}$$

for $y_0 \neq n$, we obtain for $y = n$ that

$$f'(n-0) = \begin{cases} \frac{-\lambda(1-\gamma)\upsilon_n\ln^{\lambda-1}\beta V_n}{V_n[(1+\gamma)\ln^{\lambda}\alpha U_m+(1-\gamma)\ln^{\lambda}\beta V_n]^2}, & n-\frac{1}{2}<y<y_0, \\ \frac{-\lambda(1+\gamma)\upsilon_n\ln^{\lambda-1}\beta V_n}{V_n[(1-\gamma)\ln^{\lambda}\alpha U_m+(1+\gamma)\ln^{\lambda}\beta V_n]^2}, & y_0<y<n+\frac{1}{2}, \end{cases}$$

$$f'(n+0) = \begin{cases} \frac{-\lambda(1-\gamma)\upsilon_{n+1}\ln^{\lambda-1}\beta V_n}{V_n[(1+\gamma)\ln^{\lambda}\alpha U_m+(1-\gamma)\ln^{\lambda}\beta V_n]^2}, & n-\frac{1}{2}<y<y_0, \\ \frac{-\lambda(1+\gamma)\upsilon_{n+1}\ln^{\lambda-1}\beta V_n}{V_n[(1-\gamma)\ln^{\lambda}\alpha U_m+(1+\gamma)\ln^{\lambda}\beta V_n]^2}, & y_0<y<n+\frac{1}{2}. \end{cases}$$

Since $0<\lambda\leq 1$, $-1<\gamma\leq 0$, $(1-\gamma)\upsilon_n \geq (1+\gamma)\upsilon_{n+1}$, in view of the above results, we find $f'(n-0)\leq f'(n+0)$ $(n\neq y_0)$, and $f'(y)$ (<0) is strictly increasing in $(n-\frac{1}{2},y_0)$, (y_0,n) and $(n,n+\frac{1}{2})$ for $y_0<n$ or in $(n-\frac{1}{2},n)$, (n,y_0) and $(y_0,n+\frac{1}{2})$ for $y_0>n$.

We obtain

$$\begin{aligned} f'(y_0-0) &= \frac{-\lambda(1-\gamma)V'(y_0-0)\ln^{\lambda-1}\beta V(y_0)}{V(y_0)[(1+\gamma)\ln^{\lambda}\alpha U_m+(1-\gamma)\ln^{\lambda}\beta V(y_0)]^2} \\ &= \frac{-\lambda(1-\gamma)V'(y_0-0)\ln^{\lambda-1}\beta V(y_0)}{V(y_0)(2\ln^{\lambda}\alpha U_m)^2}, \\ f'(y_0+0) &= \frac{-\lambda(1+\gamma)V'(y_0+0)\ln^{\lambda-1}\beta V(y_0)}{V(y_0)[(1-\gamma)\ln^{\lambda}\alpha U_m+(1+\gamma)\ln^{\lambda}\beta V(y_0)]^2} \\ &= \frac{-\lambda(1+\gamma)V'(y_0+0)\ln^{\lambda-1}\beta V(y_0)}{V(y_0)(2\ln^{\lambda}\alpha U_m)^2}. \end{aligned}$$

Since for $y_0=n$, $V'(y_0-0)=\upsilon_n$, $V'(y_0+0)=\upsilon_{n+1}$ and for $y_0\neq n$, $V'(y_0-0)=V'(y_0)$, then we have $\lambda(1-\gamma)V'(y_0-0)\geq\lambda(1+\gamma)V'(y_0+0)$, namely, $f'(y_0-0)\leq f'(y_0+0)$.

(ii) If $y_0\notin(n-\frac{1}{2},n+\frac{1}{2})$, then it follows that $f'(y)=\frac{V'(y)}{V(y)}\frac{d}{dy}k_\lambda(\ln\alpha U_m,\ln\beta V(y))<0$, $y\in(n-\frac{1}{2},n+\frac{1}{2})\backslash\{n\}$. We still can find that

$$\begin{aligned} \frac{\upsilon_n}{V_n}\frac{d}{dy}k_\lambda\bigl(\ln\alpha U_m,\ln\beta V(y)\bigr)\Big|_{y=n} &= f'(n-0) \\ &\leq f'(n+0)=\frac{\upsilon_{n+1}}{V_n}\frac{d}{dy}k_\lambda\bigl(\ln\alpha U_m,\ln\beta V(y)\bigr)\Big|_{y=n}, \end{aligned}$$

and $f'(y)$ (<0) is strictly increasing in $(n-\frac{1}{2},n)$ and $(n,n+\frac{1}{2})$.

Therefore, $f(y)$ satisfies the conditions of Lemma 1 with Note. So does $g(y)=\frac{f(y)}{V(y)\ln^{1-\lambda_2}\beta V(y)}$. Hence, by (9), we have

$$\frac{k_\lambda(\ln\alpha U_m,\ln\beta V_n)}{V_n\ln^{1-\lambda_2}\beta V_n}<\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\frac{k_\lambda(\ln\alpha U_m,\ln\beta V(y))}{V(y)\ln^{1-\lambda_2}\beta V(y)}\,dy \quad \bigl(n\in\mathbb{N}\backslash\{1\}\bigr). \tag{16}$$

Definition 1 Define the following weight coefficients:

$$\begin{aligned}\omega(\lambda_2,m) &:= \sum_{n=2}^{\infty} k_\lambda(\ln\alpha U_m,\ln\beta V_n)\frac{\upsilon_{n+1}\ln^{\lambda_1}\alpha U_m}{V_n\ln^{1-\lambda_2}\beta V_n}, \quad m\in\mathbb{N}\backslash\{1\},\\ \varpi(\lambda_1,n) &:= \sum_{m=2}^{\infty} k_\lambda(\ln\alpha U_m,\ln\beta V_n)\frac{\mu_{m+1}\ln^{\lambda_2}\beta V_n}{U_m\ln^{1-\lambda_1}\alpha U_m}, \quad n\in\mathbb{N}\backslash\{1\}.\end{aligned} \tag{17}$$

Lemma 2 *If $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$ are decreasing and $U_\infty = V_\infty = \infty$, then for $m,n\in\mathbb{N}\backslash\{1\}$, we have the following inequalities*:

$$\omega(\lambda_2,m) < K_\gamma(\lambda_1), \tag{18}$$

$$\varpi(\lambda_1,n) < K_\gamma(\lambda_1), \tag{19}$$

where $K_\gamma(\lambda_1)$ is determined by (12).

Proof For $y\in(n-\frac{1}{2},n+\frac{1}{2})\backslash\{n\}$, $\upsilon_{n+1}\le V'(y)$, by (16), we find

$$\begin{aligned}\omega(\lambda_2,m) &< \sum_{n=2}^{\infty}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} k_\lambda\bigl(\ln\alpha U_m,\ln\beta V(y)\bigr)\frac{\upsilon_{n+1}\ln^{\lambda_1}\alpha U_m}{V(y)\ln^{1-\lambda_2}\beta V(y)}\,dy\\ &\le \sum_{n=2}^{\infty}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} k_\lambda\bigl(\ln\alpha U_m,\ln\beta V(y)\bigr)\frac{V'(y)\ln^{\lambda_1}\alpha U_m}{V(y)\ln^{1-\lambda_2}\beta V(y)}\,dy\\ &= \int_{\frac{3}{2}}^{\infty} k_\lambda\bigl(\ln\alpha U_m,\ln\beta V(y)\bigr)\frac{V'(y)\ln^{\lambda_1}\alpha U_m}{V(y)\ln^{1-\lambda_2}\beta V(y)}\,dy.\end{aligned}$$

Setting $t=\frac{\ln\beta V(y)}{\ln\alpha U_m}$ in the above, since $\beta V(\frac{3}{2})=\beta(1+\frac{\upsilon_2}{2})\ge 1$ and $\frac{V'(y)}{V(y)}\,dy=(\ln\alpha U_m)\,dt$, we find

$$\omega(\lambda_2,m) < \int_0^{\infty} k_\lambda(1,t)t^{\lambda_2-1}\,dt = K_\gamma(\lambda_1).$$

Hence, we obtain (18). In the same way, we obtain (19). □

Note For example, $\mu_n=\upsilon_n=\frac{1}{n^\sigma}$ $(0\le\sigma\le 1)$ satisfies the conditions of Lemma 2.

Lemma 3 *With regard to the assumptions of Lemma* 2, (i) *for $m,n\in\mathbb{N}\backslash\{1\}$, we have*

$$K_\gamma(\lambda_1)\bigl(1-\theta(\lambda_2,m)\bigr) < \omega(\lambda_2,m), \tag{20}$$

$$K_\gamma(\lambda_1)\bigl(1-\vartheta(\lambda_1,n)\bigr) < \varpi(\lambda_1,n), \tag{21}$$

where

$$\begin{aligned}\theta(\lambda_2,m) &= \frac{k_\lambda(1,\frac{\ln\beta(1+\upsilon_2\theta(m))}{\ln\alpha U_m})}{\lambda_2 K_\gamma(\lambda_1)}\frac{\ln^{\lambda_2}\beta(1+\upsilon_2)}{\ln^{\lambda_2}\alpha U_m}\\ &= O\biggl(\frac{1}{\ln^{\lambda_2}\alpha U_m}\biggr)\in(0,1) \quad \biggl(\theta(m)\in\biggl(\frac{1-\beta}{\beta\upsilon_2},1\biggr)\biggr),\end{aligned} \tag{22}$$

$$\begin{aligned}\vartheta(\lambda_1, n) &= \frac{k_\lambda(\frac{\ln\alpha(1+\mu_2\vartheta(n))}{\ln\beta V_n})}{\lambda_1 K_\gamma(\lambda_1)} \frac{\ln^{\lambda_1}\alpha(1+\mu_2)}{\ln^{\lambda_1}\beta V_n} \\ &= O\left(\frac{1}{\ln^{\lambda_1}\beta V_n}\right) \in (0,1) \quad \left(\vartheta(n) \in \left(\frac{1-\alpha}{\alpha\mu_2}, 1\right)\right);\end{aligned} \tag{23}$$

(ii) *for any* $c > 0$, *we have*

$$\sum_{m=2}^{\infty} \frac{\mu_{m+1}}{U_m \ln^{1+c}\alpha U_m} = \frac{1}{c}\left[\frac{1}{\ln^c \alpha(1+\mu_2)} + cO(1)\right], \tag{24}$$

$$\sum_{n=2}^{\infty} \frac{\upsilon_{n+1}}{V_n \ln^{1+c}\beta V_n} = \frac{1}{c}\left[\frac{1}{\ln^c \beta(1+\upsilon_2)} + c\tilde{O}(1)\right]. \tag{25}$$

Proof In view of $\beta \le 1$ and $\beta \ge \frac{1}{1+\upsilon_2/2} > \frac{1}{1+\upsilon_2}$, it follows that $1 \le \frac{1-\beta}{\beta\upsilon_2} + 1 < 2$. Since, by Examples 1, $g(y)$ is strictly decreasing in $[n, n+1)$, then for $m \in \mathbb{N}\backslash\{1\}$, we find

$$\begin{aligned}\omega(\lambda_2, m) &> \sum_{n=2}^{\infty} \int_n^{n+1} k_\lambda(\ln\alpha U_m, \ln\beta V(y)) \frac{\upsilon_{n+1}\ln^{\lambda_1}\alpha U_m}{V(y)\ln^{1-\lambda_2}\beta V(y)}\,dy \\ &= \int_2^{\infty} k_\lambda(\ln\alpha U_m, \ln\beta V(y)) \frac{V'(y)\ln^{\lambda_1}\alpha U_m}{V(y)\ln^{1-\lambda_2}\beta V(y)}\,dy \\ &= \int_{\frac{1-\beta}{\beta\upsilon_2}+1}^{\infty} k_\lambda(\ln\alpha U_m, \ln\beta V(y)) \frac{V'(y)\ln^{\lambda_1}\alpha U_m}{V(y)\ln^{1-\lambda_2}\beta V(y)}\,dy \\ &\quad - \int_{\frac{1-\beta}{\beta\upsilon_2}+1}^{2} k_\lambda(\ln\alpha U_m, \ln\beta V(y)) \frac{V'(y)\ln^{\lambda_1}\alpha U_m}{V(y)\ln^{1-\lambda_2}\beta V(y)}\,dy.\end{aligned}$$

Setting $t = \frac{\ln\beta V(y)}{\ln\alpha U_m}$, we have $\ln\beta V(\frac{1-\beta}{\beta\upsilon_2} + 1) = \ln\beta(1 + \frac{1-\beta}{\beta\upsilon_2}\upsilon_2) = 0$ and

$$\begin{aligned}\omega(\lambda_2, m) &> \int_0^{\infty} k_\lambda(1,t)t^{\lambda_2-1}\,dt - \int_{\frac{1-\beta}{\beta\upsilon_2}+1}^{2} k_\lambda(\ln\alpha U_m, \ln\beta V(y)) \frac{V'(y)\ln^{\lambda_1}\alpha U_m}{V(y)\ln^{1-\lambda_2}\beta V(y)}\,dy \\ &= K_\gamma(\lambda_1)\bigl(1 - \theta(\lambda_2, m)\bigr),\end{aligned}$$

where

$$\theta(\lambda_2, m) := \frac{\ln^{\lambda_1}\alpha U_m}{K_\gamma(\lambda_1)} \int_{\frac{1-\beta}{\beta\upsilon_2}+1}^{2} k_\lambda(\ln\alpha U_m, \ln\beta V(y)) \frac{V'(y)}{V(y)\ln^{1-\lambda_2}\beta V(y)}\,dy \in (0,1).$$

In view of the integral mid-value theorem, for fixed $m \in \mathbb{N}\backslash\{1\}$, there exists $\theta(m) \in (\frac{1-\beta}{\beta\upsilon_2}, 1)$ such that

$$\begin{aligned}\theta(\lambda_2, m) &= \frac{\ln^{\lambda_1}\alpha U_m}{K_\gamma(\lambda_1)} k_\lambda\bigl(\ln\alpha U_m, \ln\beta V\bigl(1+\theta(m)\bigr)\bigr) \int_{\frac{1-\beta}{\beta\upsilon_2}+1}^{2} \frac{V'(y)}{V(y)\ln^{1-\lambda_2}\beta V(y)}\,dy \\ &= \frac{\ln^{\lambda_1}\alpha U_m}{\lambda_2 K_\gamma(\lambda_1)} k_\lambda\bigl(\ln\alpha U_m, \ln\beta V\bigl(1+\theta(m)\bigr)\bigr)\ln^{\lambda_2}\beta(1+\upsilon_2) \\ &= \frac{1}{\lambda_2 K_\gamma(\lambda_1)} k_\lambda\left(1, \frac{\ln\beta V(1+\theta(m))}{\ln\alpha U_m}\right) \frac{\ln^{\lambda_2}\beta(1+\upsilon_2)}{\ln^{\lambda_2}\alpha U_m}.\end{aligned}$$

Hence, we find

$$0 < \theta(\lambda_2, m) \leq \frac{1}{\lambda_2 K_\gamma(\lambda_1)} \frac{\ln^{\lambda_2} \beta(1 + \upsilon_2)}{(1 + \gamma) \ln^{\lambda_2} \alpha U_m},$$

namely, $\theta(\lambda_2, m) = O(\frac{1}{\ln^{\lambda_2} \alpha U_m})$. Then we obtain (20) and (22). In the same way, we obtain (21) and (23).

For $c > 0$, we find

$$\begin{aligned}
\sum_{m=2}^{\infty} \frac{\mu_{m+1}}{U_m \ln^{1+c} \alpha U_m} \leq \sum_{m=2}^{\infty} \frac{\mu_m}{U_m \ln^{1+c} \alpha U_m} &= \frac{\mu_2}{U_2 \ln^{1+c} \alpha U_2} + \sum_{m=3}^{\infty} \frac{\mu_m}{U_m \ln^{1+c} \alpha U_m} \\
&= \frac{\mu_2}{U_2 \ln^{1+c} \alpha U_2} + \sum_{m=3}^{\infty} \int_{m-1}^{m} \frac{U'(x)\,dx}{U_m \ln^{1+c} \alpha U_m} \\
&< \frac{\mu_2}{U_2 \ln^{1+c} \alpha U_2} + \sum_{m=3}^{\infty} \int_{m-1}^{m} \frac{U'(x)\,dx}{U(x) \ln^{1+c} \alpha U(x)} \\
&= \frac{\mu_2}{U_2 \ln^{1+c} \alpha U_2} + \int_{2}^{\infty} \frac{U'(x)\,dx}{U(x) \ln^{1+c} \alpha U(x)} \\
&= \frac{\mu_2}{U_2 \ln^{1+c} \alpha U_2} + \frac{1}{c \ln^{c} \alpha(1 + \mu_2)} \\
&= \frac{1}{c}\left[\frac{1}{\ln^{c} \alpha(1 + \mu_2)} + c \frac{\mu_2}{U_2 \ln^{1+c} \alpha U_2}\right], \\
\sum_{m=2}^{\infty} \frac{\mu_{m+1}}{U_m \ln^{1+c} \alpha U_m} = \sum_{m=2}^{\infty} \int_{m}^{m+1} \frac{U'(x)\,dx}{U_m \ln^{1+c} \alpha U_m} &> \sum_{m=2}^{\infty} \int_{m}^{m+1} \frac{U'(x)\,dx}{U(x) \ln^{1+c} \alpha U(x)} \\
&= \int_{2}^{\infty} \frac{U'(x)\,dx}{U(x) \ln^{1+c} \alpha U(x)} = \frac{1}{c \ln^{c} \alpha(1 + \mu_2)}.
\end{aligned}$$

Hence, we obtain (20). In the same way, we obtain (21). □

Lemma 4 *If* $-1 < \gamma \leq 0$, $0 < \lambda_1, \lambda_2 < 1$, $\lambda_1 + \lambda_2 \leq 1$, $K_\gamma(\lambda_1)$ *is determined by* (12), *then for* $0 < \delta < \min\{\lambda_1, \lambda_2\}$, *we have*

$$K_\gamma(\lambda_1 \pm \delta) = K_\gamma(\lambda_1) + o(1) \quad \bigl(\delta \to 0^+\bigr). \tag{26}$$

Proof We find, for $0 < \delta < \min\{\lambda_1, \lambda_2\}$,

$$\begin{aligned}
\bigl|K_\gamma(\lambda_1 + \delta) - K_\gamma(\lambda_1)\bigr| &\leq \int_0^\infty \frac{t^{\lambda_1 - 1}|t^\delta - 1|}{t^\lambda + 1 + \gamma|t^\lambda - 1|}\,dt \\
&= \int_0^1 \frac{t^{\lambda_1 - 1}(1 - t^\delta)}{1 + \gamma + (1 - \gamma)t^\lambda}\,dt + \int_1^\infty \frac{t^{\lambda_1 - 1}(t^\delta - 1)}{1 - \gamma + (1 + \gamma)t^\lambda}\,dt \\
&\leq \frac{1}{1 + \gamma}\left[\int_0^1 t^{\lambda_1 - 1}\bigl(1 - t^\delta\bigr)\,dt + \int_1^\infty \frac{t^{\lambda_1 - 1}(t^\delta - 1)}{t^\lambda}\,dt\right] \\
&= \frac{1}{1 + \gamma}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_1 + \delta} + \frac{1}{\lambda_2 - \delta} - \frac{1}{\lambda_2}\right) \to 0 \quad \bigl(\delta \to 0^+\bigr).
\end{aligned}$$

In the same way, we find

$$\begin{aligned}\left|K_\gamma(\lambda_1-\delta)-K_\gamma(\lambda_1)\right| &\le \frac{1}{1+\gamma}\left[\int_0^1 t^{\lambda_1-1}\left(t^{-\delta}-1\right)dt+\int_1^\infty \frac{t^{\lambda_1-1}(1-t^{-\delta})}{t^\lambda}\,dt\right]\\ &= \frac{1}{1+\gamma}\left(\frac{1}{\lambda_1-\delta}-\frac{1}{\lambda_1}+\frac{1}{\lambda_2}-\frac{1}{\lambda_2+\delta}\right)\to 0 \quad \left(\delta\to 0^+\right),\end{aligned}$$

and then we have (26). □

3 Main results

In the following, we also set

$$\begin{aligned}&\tilde{\Phi}_\lambda(m):=\omega(\lambda_2,m)\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1}(\ln\alpha U_m)^{p(1-\lambda_1)-1}\quad \left(m\in\mathbb{N}\backslash\{1\}\right),\\ &\tilde{\Psi}_\lambda(n):=\varpi(\lambda_1,n)\left(\frac{V_n}{\upsilon_{n+1}}\right)^{q-1}(\ln\beta V_n)^{q(1-\lambda_2)-1}\quad \left(n\in\mathbb{N}\backslash\{1\}\right).\end{aligned}\tag{27}$$

Theorem 1

(i) *For* $p>1$, *we have the following equivalent inequalities*:

$$I:=\sum_{n=2}^\infty\sum_{m=2}^\infty k_\lambda(\ln\alpha U_m,\ln\beta V_n)a_m b_n\le \|a\|_{p,\tilde{\Phi}_\lambda}\|b\|_{q,\tilde{\Psi}_\lambda},\tag{28}$$

$$J:=\left\{\sum_{n=2}^\infty \frac{\upsilon_{n+1}\ln^{p\lambda_2-1}\beta V_n}{(\varpi(\lambda_1,n))^{p-1}V_n}\left(\sum_{m=2}^\infty k_\lambda(\ln\alpha U_m,\ln\beta V_n)a_m\right)^p\right\}^{\frac{1}{p}}\le \|a\|_{p,\tilde{\Phi}_\lambda};\tag{29}$$

(ii) *for* $0<p<1$ (*or* $p<0$), *we have the equivalent reverse of* (28) *and* (29).

Proof (i) By Hölder's inequality with weight (cf. [20]) and (17), we have

$$\begin{aligned}&\left(\sum_{m=2}^\infty k_\lambda(\ln\alpha U_m,\ln\beta V_n)a_m\right)^p\\ &\quad=\left[\sum_{m=2}^\infty k_\lambda(\ln\alpha U_m,\ln\beta V_n)\left(\frac{U_m^{1/q}(\ln\alpha U_m)^{(1-\lambda_1)/q}\upsilon_{n+1}^{1/p}}{(\ln\beta V_n)^{(1-\lambda_2)/p}\mu_{m+1}^{1/q}}a_m\right)\right.\\ &\qquad\left.\times\left(\frac{(\ln\beta V_n)^{(1-\lambda_2)/p}\mu_{m+1}^{1/q}}{U_m^{1/q}(\ln\alpha U_m)^{(1-\lambda_1)/q}\upsilon_{n+1}^{1/p}}\right)\right]^p\\ &\quad\le\sum_{m=2}^\infty k_\lambda(\ln\alpha U_m,\ln\beta V_n)\frac{U_m^{p-1}(\ln\alpha U_m)^{(1-\lambda_1)p/q}\upsilon_{n+1}}{(\ln\beta V_n)^{1-\lambda_2}\mu_{m+1}^{p/q}}a_m^p\\ &\qquad\times\left[\sum_{m=2}^\infty k_\lambda(\ln\alpha U_m,\ln\beta V_n)\frac{(\ln\beta V_n)^{(1-\lambda_2)(q-1)}\mu_{m+1}}{U_m(\ln\alpha U_m)^{1-\lambda_1}\upsilon_{n+1}^{q-1}}\right]^{p-1}\\ &\quad=\frac{(\varpi(\lambda_1,n))^{p-1}V_n}{(\ln\beta V_n)^{p\lambda_2-1}\upsilon_{n+1}}\sum_{m=2}^\infty k_\lambda(\ln\alpha U_m,\ln\beta V_n)\frac{\upsilon_{n+1}U_m^{p-1}(\ln\alpha U_m)^{(1-\lambda_1)(p-1)}}{(\ln\beta V_n)^{1-\lambda_2}\mu_{m+1}^{p-1}}a_m^p.\end{aligned}\tag{30}$$

Then, by (16), we find

$$\begin{aligned} J &\le \left[\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} k_\lambda(\ln\alpha U_m,\ln\beta V_n)\frac{\upsilon_{n+1}U_m^{p-1}(\ln\alpha U_m)^{(1-\lambda_1)(p-1)}}{(\ln\beta V_n)^{1-\lambda_2}\mu_{m+1}^{p-1}}a_m^p\right]^{\frac{1}{p}} \\ &= \left[\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} k_\lambda(\ln\alpha U_m,\ln\beta V_n)\frac{\upsilon_{n+1}U_m^{p-1}(\ln\alpha U_m)^{(1-\lambda_1)(p-1)}}{(\ln\beta V_n)^{1-\lambda_2}\mu_{m+1}^{p-1}}a_m^p\right]^{\frac{1}{p}} \\ &= \left[\sum_{m=2}^{\infty}\omega(\lambda_2,m)\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1}(\ln\alpha U_m)^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}, \end{aligned} \tag{31}$$

and then (29) follows.

By Hölder's inequality (cf. [20]), we have

$$\begin{aligned} I &= \sum_{n=2}^{\infty}\left[\frac{(\ln\beta V_n)^{\lambda_2-\frac{1}{p}}\upsilon_{n+1}^{1/p}}{(\varpi(\lambda_1,n))^{1/q}V_n^{1/p}}\sum_{m=2}^{\infty}k_\lambda(\ln\alpha U_m,\ln\beta V_n)a_m\right]\left[\frac{(\varpi(\lambda_1,n))^{1/q}V_n^{1/p}}{(\ln\beta V_n)^{\lambda_2-\frac{1}{p}}\upsilon_{n+1}^{1/p}}b_n\right] \\ &\le J\|b\|_{q,\tilde{\Psi}_\lambda}. \end{aligned} \tag{32}$$

Then, by (29), we have (28).

On the other hand, assuming that (28) is valid, we set

$$b_n := \frac{\upsilon_{n+1}\ln^{p\lambda_2-1}\beta V_n}{(\varpi(\lambda_1,n))^{p-1}V_n}\left(\sum_{m=2}^{\infty}k_\lambda(\ln\alpha U_m,\ln\beta V_n)a_m\right)^{p-1},\quad n\in\mathbb{N}\backslash\{1\}. \tag{33}$$

Then we find $J^p = \|b\|_{q,\tilde{\Psi}_\lambda}^q$. If $J = 0$, then (29) is trivially valid; if $J = \infty$, then by (31), (29) takes the form of equality. Suppose that $0 < J < \infty$. By (28), it follows that

$$\|b\|_{q,\tilde{\Psi}_\lambda}^q = J^p = I \le \|a\|_{p,\tilde{\Phi}_\lambda}\|b\|_{q,\tilde{\Psi}_\lambda}, \tag{34}$$

$$\|b\|_{q,\tilde{\Psi}_\lambda}^{q-1} = J \le \|a\|_{p,\tilde{\Phi}_\lambda}, \tag{35}$$

and then (29) follows, which is equivalent to (28).

(ii) For $0 < p < 1$ (or $p < 0$), by the reverse Hölder's inequality with weight (cf. [20]) and (13), we obtain the reverse of (30) (or (30)), then we have the reverse of (31), and then the reverse of (29) follows. By Hölder's inequality (cf. [20]), we have the reverse of (32), and then by the reverse of (29), the reverse of (28) follows.

On the other hand, assuming that the reverse of (28) is valid, we set b_n as (33). Then we find $J^p = \|b\|_{q,\tilde{\Psi}_\lambda}^q$. If $J = \infty$, then the reverse of (29) is trivially valid; if $J = 0$, then by the reverse of (31), (29) takes the form of equality (= 0). Suppose that $0 < J < \infty$. By the reverse of (28), it follows that the reverses of (34) and (35) are valid, and then the reverse of (29) follows, which is equivalent to the reverse of (28). □

Theorem 2 *If $p > 1$, $\{\mu_m\}_{m=1}^\infty$ and $\{\upsilon_n\}_{n=1}^\infty$ are decreasing, $U_\infty = V_\infty = \infty$, $\|a\|_{p,\Phi_\lambda} \in \mathbb{R}_+$ and $\|b\|_{q,\Psi_\lambda}^q \in \mathbb{R}_+$, then we have the following equivalent inequalities*:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}k_\lambda(\ln\alpha U_m,\ln\beta V_n)a_m b_n < K_\gamma(\lambda_1)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{36}$$

$$J_1 := \left\{ \sum_{n=2}^{\infty} \frac{\upsilon_{n+1}}{V_n} \ln^{p\lambda_2 - 1} \beta V_n \left(\sum_{m=2}^{\infty} k_\lambda (\ln \alpha U_m, \ln \beta V_n) a_m \right)^p \right\}^{\frac{1}{p}} < K_\gamma(\lambda_1) \|a\|_{p,\Phi_\lambda}, \tag{37}$$

where the constant factor $K_\gamma(\lambda_1)$ *is the best possible.*

Proof Using (18) and (19) in (28) and (29), we obtain the equivalent inequalities (36) and (37).

For $\varepsilon \in (0, \min\{p\lambda_1, p(1-\lambda_2)\})$, we set $\tilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$ $(\in (0,1))$, $\tilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$ $(\in (0,1))$, and

$$\begin{aligned} \tilde{a}_m &:= \frac{\mu_{m+1}}{U_m} \ln^{\tilde{\lambda}_1 - 1} \alpha U_m = \frac{\mu_{m+1}}{U_m} \ln^{\lambda_1 - \frac{\varepsilon}{p} - 1} \alpha U_m, \\ \tilde{b}_n &:= \frac{\upsilon_{n+1}}{V_n} \ln^{\tilde{\lambda}_2 - \varepsilon - 1} \beta V_n = \frac{\upsilon_{n+1}}{V_n} \ln^{\lambda_2 - \frac{\varepsilon}{q} - 1} \beta V_n. \end{aligned} \tag{38}$$

Then, by (24), (25) and (21), we have

$$\begin{aligned} \|\tilde{a}\|_{p,\Phi_\lambda} \|\tilde{b}\|_{q,\Psi_\lambda} &= \left(\sum_{m=2}^{\infty} \frac{\mu_{m+1}}{U_m \ln^{1+\varepsilon} \alpha U_m} \right)^{\frac{1}{p}} \left(\sum_{n=2}^{\infty} \frac{\upsilon_{n+1}}{V_n \ln^{1+\varepsilon} \beta V_n} \right)^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} \left[\frac{1}{\ln^{\varepsilon} \alpha(1+\mu_2)} + \varepsilon O(1) \right]^{\frac{1}{p}} \left[\frac{1}{\ln^{\varepsilon} \beta(1+\upsilon_2)} + \varepsilon \tilde{O}(1) \right]^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} \tilde{I} &:= \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_\lambda (\ln \alpha U_m, \ln \beta V_n) \tilde{a}_m \tilde{b}_n \\ &= \sum_{n=2}^{\infty} \left[\sum_{m=2}^{\infty} k_\lambda (\ln \alpha U_m, \ln \beta V_n) \frac{\mu_{m+1} \ln^{\tilde{\lambda}_2} \beta V_n}{U_m \ln^{1-\tilde{\lambda}_1} \alpha U_m} \right] \frac{\upsilon_{n+1}}{V_n} \ln^{-\varepsilon - 1} \beta V_n \\ &= \sum_{n=2}^{\infty} \frac{\upsilon_{n+1} \varpi(\lambda_1, n)}{V_n \ln^{\varepsilon+1} \beta V_n} \geq K_\gamma(\tilde{\lambda}_1) \sum_{n=2}^{\infty} \left(1 - O\left(\frac{1}{\ln^{\tilde{\lambda}_1} \beta V_n} \right) \right) \frac{\upsilon_{n+1}}{V_n \ln^{\varepsilon+1} \beta V_n} \\ &= K_\gamma(\tilde{\lambda}_1) \left[\sum_{n=2}^{\infty} \frac{\upsilon_{n+1}}{V_n \ln^{\varepsilon+1} \beta V_n} - \sum_{n=2}^{\infty} O\left(\frac{1}{\ln^{\lambda_1 + \frac{\varepsilon}{q} + 1} \beta V_n} \right) \frac{\upsilon_{n+1}}{V_n} \right] \\ &= \frac{1}{\varepsilon} K_\gamma(\tilde{\lambda}_1) \left[\frac{1}{\ln^{\varepsilon} \beta(1+\upsilon_2)} + \varepsilon \big(\tilde{O}(1) - O(1) \big) \right]. \end{aligned}$$

If there exists a positive constant $K \leq K_\gamma(\lambda_1)$ such that (36) is valid when replacing $K_\gamma(\lambda_1)$ by K, then, in particular, we have $\varepsilon \tilde{I} < \varepsilon K \|\tilde{a}\|_{p,\Phi_\lambda} \|\tilde{b}\|_{q,\Psi_\lambda}$, namely,

$$\begin{aligned} & K_\gamma \left(\lambda_1 - \frac{\varepsilon}{p} \right) \left[\frac{1}{\ln^{\varepsilon} \beta(1+\upsilon_2)} + \varepsilon \big(\tilde{O}(1) - O(1) \big) \right] \\ & \quad < K \left[\frac{1}{\ln^{\varepsilon} \alpha(1+\mu_2)} + \varepsilon O(1) \right]^{\frac{1}{p}} \left[\frac{1}{\ln^{\varepsilon} \beta(1+\upsilon_2)} + \varepsilon \tilde{O}(1) \right]^{\frac{1}{q}}. \end{aligned}$$

In view of (26), it follows that $K_\gamma(\lambda_1) \leq K (\varepsilon \to 0^+)$. Hence, $K = K_\gamma(\lambda_1)$ is the best possible constant factor of (36).

Similarly to (32), we still can find the following inequality:

$$I \leq J_1 \|b\|_{q,\Psi_\lambda}. \tag{39}$$

Hence, we can prove that the constant factor $K_\gamma(\lambda_1)$ in (37) is the best possible. Otherwise, we would reach the contradiction by (39) that the constant factor in (36) is not the best possible. □

Remark 1 (i) For $\alpha = \beta = 1$ in (36) and (37), setting

$$\phi_\lambda(m) := \left(\frac{U_m}{\mu_{m+1}}\right)^{p-1}(\ln U_m)^{p(1-\lambda_1)-1},$$
$$\psi_\lambda(n) := \left(\frac{V_n}{\upsilon_{n+1}}\right)^{q-1}(\ln V_n)^{q(1-\lambda_2)-1} \quad \bigl(m, n \in \mathbb{N}\backslash\{1\}\bigr),$$

we have the following equivalent Mulholland-type inequalities:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln^\lambda U_m + \ln^\lambda V_n + \gamma|\ln^\lambda U_m - \ln^\lambda V_n|} < K_\gamma(\lambda_1)\|a\|_{p,\phi_\lambda}\|b\|_{q,\psi_\lambda}, \tag{40}$$

$$\left[\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}}{V_n}(\ln V_n)^{p\lambda_2-1}\left(\sum_{m=2}^{\infty}\frac{a_m}{\ln^\lambda U_m + \ln^\lambda V_n + \gamma|\ln^\lambda U_m - \ln^\lambda V_n|}\right)^p\right]^{\frac{1}{p}} < K_\gamma(\lambda_1)\|a\|_{p,\phi_\lambda}. \tag{41}$$

(40) is an extension of (7) and the following inequality (for $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, $\gamma = 0$):

$$\left[\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}}{V_n}(\ln V_n)^{p\lambda_2-1}\left(\sum_{m=2}^{\infty}\frac{a_m}{\ln U_m V_n}\right)^p\right]^{\frac{1}{p}} < \frac{\pi}{\sin(\frac{\pi}{p})}\left[\sum_{m=2}^{\infty}\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1}a_m^p\right]^{\frac{1}{p}}. \tag{42}$$

(ii) For $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$ in (36) and (37), we have the following equivalent inequalities:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln(\alpha\beta U_m V_n) + \gamma|\ln\frac{\alpha U_m}{\beta V_n}|} < K_{1,\gamma}\left(\frac{1}{q}\right)\left[\sum_{m=2}^{\infty}\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\left(\frac{V_n}{\upsilon_{n+1}}\right)^{q-1}b_n^q\right]^{\frac{1}{q}}, \tag{43}$$

$$\left\{\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}}{V_n}\left[\sum_{m=2}^{\infty}\frac{a_m}{\ln(\alpha\beta U_m V_n) + \gamma|\ln\frac{\alpha U_m}{\beta V_n}|}\right]^p\right\}^{\frac{1}{p}} < K_{1,\gamma}\left(\frac{1}{q}\right)\left[\sum_{m=2}^{\infty}\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1}a_m^p\right]^{\frac{1}{p}}, \tag{44}$$

where

$$K_{1,\gamma}\left(\frac{1}{q}\right) := \int_0^1 \frac{t^{\frac{-1}{p}-1} + t^{\frac{-1}{q}-1}}{1+\gamma+(1-\gamma)t}\,dt = \frac{1}{1+\gamma}\left[\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{1}{q}} + \left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{1}{p}}\right]\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{1}{1+\gamma}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1+\gamma}{1-\gamma}\right)^{k+1}\left(\frac{1}{k+\frac{1}{p}} + \frac{1}{k+\frac{1}{q}}\right). \tag{45}$$

(iii) For $\gamma = 0$, (43) reduces to the following more accurate Hardy-Mulholland-type inequality (7):

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln(\alpha\beta U_m V_n)} < \frac{\pi}{\sin(\frac{\pi}{p})}\left[\sum_{m=2}^{\infty}\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\left(\frac{V_n}{\upsilon_{n+1}}\right)^{q-1}b_n^q\right]^{\frac{1}{q}}. \tag{46}$$

In particular, for $\mu_i = \upsilon_j = 1$ ($i, j \in \mathbb{N}$), (46) reduces to the following more accurate Mulholland's inequality ($\frac{2}{3} \leq \alpha, \beta \leq 1$):

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln(\alpha\beta mn)} < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{m=2}^{\infty}m^{p-1}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}n^{q-1}b_n^q\right)^{\frac{1}{q}}. \tag{47}$$

For $p > 1$, $\Psi_\lambda^{1-p}(n) = \frac{\upsilon_{n+1}}{V_n}(\ln\beta V_n)^{p\lambda_2-1}$, we define the following normed spaces:

$$l_{p,\Phi_\lambda} := \left\{a = \{a_m\}_{m=2}^{\infty}; \|a\|_{p,\Phi_\lambda} < \infty\right\},$$
$$l_{q,\Psi_\lambda} := \left\{b = \{b_n\}_{n=2}^{\infty}; \|b\|_{q,\Psi_\lambda} < \infty\right\},$$
$$l_{p,\Psi_\lambda^{1-p}} := \left\{c = \{c_n\}_{n=2}^{\infty}; \|c\|_{p,\Psi_\lambda^{1-p}} < \infty\right\}.$$

Assuming that $a = \{a_m\}_{m=2}^{\infty} \in l_{p,\Phi_\lambda}$, setting

$$c = \{c_n\}_{n=2}^{\infty}, \quad c_n := \sum_{m=2}^{\infty}k_\lambda(\ln\alpha U_m, \ln\beta V_n)a_m, n \in \mathbb{N}\backslash\{1\},$$

we can rewrite (37) as follows:

$$\|c\|_{p,\Psi_\lambda^{1-p}} < K_\gamma(\lambda_1)\|a\|_{p,\Phi_\lambda} < \infty,$$

namely, $c \in l_{p,\Psi_\lambda^{1-p}}$.

Definition 2 Define a Hardy-Mulholland-type operator $T : l_{p,\Phi_\lambda} \to l_{p,\Psi_\lambda^{1-p}}$ as follows: For any $a = \{a_m\}_{m=2}^{\infty} \in l_{p,\Phi_\lambda}$, there exists a unique representation $Ta = c \in l_{p,\Psi_\lambda^{1-p}}$. Define the formal inner product of Ta and $b = \{b_n\}_{n=2}^{\infty} \in l_{q,\Psi_\lambda}$ as follows:

$$(Ta, b) := \sum_{n=2}^{\infty}\left(\sum_{m=2}^{\infty}k_\lambda(\ln\alpha U_m, \ln\beta V_n)a_m\right)b_n. \tag{48}$$

Then we can rewrite (36) and (37) as follows:

$$(Ta, b) < K_\gamma(\lambda_1)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{49}$$

$$\|Ta\|_{p,\Psi_\lambda^{1-p}} < K_\gamma(\lambda_1)\|a\|_{p,\Phi_\lambda}. \tag{50}$$

Define the norm of operator T as follows:

$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\Phi_\lambda}}\frac{\|Ta\|_{p,\Psi_\lambda^{1-p}}}{\|a\|_{p,\Phi_\lambda}}.$$

Then, by (50), we find $\|T\| \leq K_\gamma(\lambda_1)$. Since the constant factor in (50) is the best possible, we have

$$\|T\| = K_\gamma(\lambda_1) = \int_0^1 \frac{t^{\lambda_1-1} + t^{\lambda_2-1}}{1+\gamma+(1-\gamma)t^\lambda}\,dt. \tag{51}$$

4 Some reverses

In the following, we also set

$$\begin{aligned} &\tilde{\Omega}_\lambda(m) := \bigl(1-\theta(\lambda_2,m)\bigr)\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1}(\ln\alpha U_m)^{p(1-\lambda_1)-1}, \\ &\tilde{\Upsilon}_\lambda(n) := \bigl(1-\vartheta(\lambda_1,n)\bigr)\left(\frac{V_n}{\upsilon_{n+1}}\right)^{q-1}(\ln\beta V_n)^{q(1-\lambda_2)-1} \quad \bigl(m,n\in\mathbb{N}\backslash\{1\}\bigr). \end{aligned} \tag{52}$$

For $0<p<1$ or $p<0$, we still use the formal symbols $\|a\|_{p,\Phi_\lambda}$, $\|b\|_{q,\Psi_\lambda}$, $\|a\|_{p,\tilde{\Omega}_\lambda}$ and $\|b\|_{q,\tilde{\Upsilon}_\lambda}$ et al.

Theorem 3 *If $0<p<1$, $\{\mu_m\}_{m=1}^\infty$ and $\{\upsilon_n\}_{n=1}^\infty$ are decreasing, $U_\infty = V_\infty = \infty$, $\|a\|_{p,\Phi_\lambda} \in \mathbb{R}_+$ and $\|b\|_{q,\Psi_\lambda}^q \in \mathbb{R}_+$, then we have the following equivalent inequalities with the best possible constant factor $K_\gamma(\lambda_1)$:*

$$\sum_{n=2}^\infty\sum_{m=2}^\infty k_\lambda(\ln\alpha U_m, \ln\beta V_n)a_m b_n > K_\gamma(\lambda_1)\|a\|_{p,\tilde{\Omega}_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{53}$$

$$\left\{\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n}\ln^{p\lambda_2-1}\beta V_n\left(\sum_{m=2}^\infty k_\lambda(\ln\alpha U_m,\ln\beta V_n)a_m\right)^p\right\}^{\frac{1}{p}} > K_\gamma(\lambda_1)\|a\|_{p,\tilde{\Omega}_\lambda}. \tag{54}$$

Proof Using (20) and (19) in the reverses of (28) and (29), since

$$\begin{aligned} &\bigl(\omega(\lambda_2,m)\bigr)^{\frac{1}{p}} > \bigl(K_\gamma(\lambda_1)\bigr)^{\frac{1}{p}}\bigl(1-\theta(\lambda_2,m)\bigr)^{\frac{1}{p}} \quad (0<p<1), \\ &\bigl(\varpi(\lambda_1,n)\bigr)^{\frac{1}{q}} > \bigl(K_\gamma(\lambda_1)\bigr)^{\frac{1}{q}} \quad (q<0), \end{aligned}$$

and

$$\frac{1}{(K_\gamma(\lambda_1))^{p-1}} > \frac{1}{(\varpi(\lambda_1,n))^{p-1}} \quad (0<p<1),$$

we obtain equivalent inequalities (53) and (54).

For $\varepsilon\in(0,\min\{p\lambda_1, p(1-\lambda_2)\})$, we set $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{a}_m$ and $\tilde{b}_n$ as (38). Then, by (24), (25) and (19), we find

$$\begin{aligned} &\|\tilde{a}\|_{p,\tilde{\Omega}_\lambda}\|\tilde{b}\|_{q,\Psi_\lambda} \\ &\quad= \left(\sum_{m=2}^\infty \frac{(1-\theta(\lambda_2,m))\mu_{m+1}}{U_m\ln^{1+\varepsilon}\alpha U_m}\right)^{\frac{1}{p}}\left(\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n\ln^{1+\varepsilon}\beta V_n}\right)^{\frac{1}{q}} \\ &\quad= \left(\sum_{m=2}^\infty \frac{\mu_{m+1}}{U_m\ln^{1+\varepsilon}\alpha U_m} - \sum_{m=2}^\infty O\left(\frac{\mu_{m+1}}{U_m\ln^{1+(\lambda_2+\varepsilon)}\alpha U_m}\right)\right)^{\frac{1}{p}}\left(\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n\ln^{1+\varepsilon}\beta V_n}\right)^{\frac{1}{q}} \end{aligned}$$

$$= \frac{1}{\varepsilon}\left[\frac{1}{\ln^{\varepsilon}\alpha(1+\mu_2)} + \varepsilon(O(1) - O_1(1))\right]^{\frac{1}{p}}\left[\frac{1}{\ln^{\varepsilon}\beta(1+\upsilon_2)} + \varepsilon\tilde{O}(1)\right]^{\frac{1}{q}},$$

$$\tilde{I} := \sum_{n=2}^{\infty}\sum_{m=2}^{\infty} k_\lambda(\ln\alpha U_m, \ln\beta V_n)\tilde{a}_m\tilde{b}_n$$
$$= \sum_{n=2}^{\infty}\left[\sum_{m=2}^{\infty} k_\lambda(\ln\alpha U_m, \ln\beta V_n)\frac{\mu_{m+1}\ln^{\tilde{\lambda}_2}\beta V_n}{U_m\ln^{1-\tilde{\lambda}_1}\alpha U_m}\right]\frac{\upsilon_{n+1}}{V_n}\ln^{-\varepsilon-1}\beta V_n$$
$$= \sum_{n=2}^{\infty}\frac{\upsilon_{n+1}\varpi(\tilde{\lambda}_1, n)}{V_n\ln^{\varepsilon+1}\beta V_n} \le K_\gamma(\tilde{\lambda}_1)\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}}{V_n\ln^{\varepsilon+1}\beta V_n}$$
$$= \frac{1}{\varepsilon}K_\gamma(\tilde{\lambda}_1)\left[\frac{1}{\ln^{\varepsilon}\beta(1+\upsilon_2)} + \varepsilon\tilde{O}(1)\right].$$

If there exists a positive constant $K \ge K_\gamma(\lambda_1)$ such that (53) is valid when replacing $K_\gamma(\lambda_1)$ by K, then, in particular, we have $\varepsilon\tilde{I} > \varepsilon K\|\tilde{a}\|_{p,\Phi_\lambda}\|\tilde{b}\|_{q,\Psi_\lambda}$, namely,

$$K_\gamma\left(\lambda_1 - \frac{\varepsilon}{p}\right)\left[\frac{1}{\ln^{\varepsilon}\beta(1+\upsilon_2)} + \varepsilon\tilde{O}(1)\right]$$
$$> K\left[\frac{1}{\ln^{\varepsilon}\alpha(1+\mu_2)} + \varepsilon\big(O(1) - O_1(1)\big)\right]^{\frac{1}{p}}\left[\frac{1}{\ln^{\varepsilon}\beta(1+\upsilon_2)} + \varepsilon\tilde{O}(1)\right]^{\frac{1}{q}}.$$

It follows that $K_\gamma(\lambda_1) \ge K$ ($\varepsilon \to 0^+$). Hence, $K = K_\gamma(\lambda_1)$ is the best possible constant factor of (53).

The constant factor $K_\gamma(\lambda_1)$ in (54) is still the best possible. Otherwise, we would reach the contradiction by the reverse of (39) that the constant factor in (53) is not the best possible. □

Remark 2 For $\alpha = \beta = 1$, set

$$\tilde{\theta}(\lambda_2, m) = \frac{k_\lambda(1, \frac{\ln(1+\upsilon_2\theta(m))}{\ln U_m})}{\lambda_2 K_\gamma(\lambda_1)}\frac{\ln^{\lambda_2}(1+\upsilon_2)}{ln^{\lambda_2}U_m} = O\left(\frac{1}{ln^{\lambda_2}U_m}\right) \in (0,1) \quad \big(\theta(m) \in (0,1)\big),$$
$$\tilde{\phi}_\lambda(m) := \big(1 - \tilde{\theta}(\lambda_2, m)\big)\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1}(\ln U_m)^{p(1-\lambda_1)-1}.$$

It is evident that (53) and (54) are extensions of the following equivalent inequalities:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} k_\lambda(\ln U_m, \ln V_n)a_m b_n > K_\gamma(\lambda_1)\|a\|_{p,\tilde{\phi}_\lambda}\|b\|_{q,\psi_\lambda}, \tag{55}$$

$$\left\{\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}}{V_n}\ln^{p\lambda_2-1}V_n\left(\sum_{m=2}^{\infty} k_\lambda(\ln U_m, \ln V_n)a_m\right)^p\right\}^{\frac{1}{p}} > K_\gamma(\lambda_1)\|a\|_{p,\tilde{\phi}_\lambda}, \tag{56}$$

where the constant factor $K_\gamma(\lambda_1)$ is the best possible.

Theorem 4 *If $p < 0$, $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$ are decreasing, $U_\infty = V_\infty = \infty$, $\|a\|_{p,\Phi_\lambda} \in \mathbb{R}_+$ and $\|b\|_{q,\Psi_\lambda}^q \in \mathbb{R}_+$, then we have the following equivalent inequalities with the best possible* *constant factor* $K_\gamma(\lambda_1)$:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} k_\lambda(\ln\alpha U_m, \ln\beta V_n)a_m b_n > K_\gamma(\lambda_1)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\tilde{\Upsilon}_\lambda}, \tag{57}$$

$$J_2 := \left\{\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}\ln^{p\lambda_2-1}\beta V_n}{(1-\vartheta(\lambda_1,n))^{p-1}V_n}\left(\sum_{m=2}^{\infty}k_\lambda(\ln\alpha U_m, \ln\beta V_n)a_m\right)^p\right\}^{\frac{1}{p}} > K_\gamma(\lambda_1)\|a\|_{p,\Phi_\lambda}. \tag{58}$$

Proof Using (18) and (21) in the reverses of (28) and (29), since

$$\begin{aligned}
&\big(\omega(\lambda_2,m)\big)^{\frac{1}{p}} > \big(K_\gamma(\lambda_1)\big)^{\frac{1}{p}} \quad (p<0),\\
&\big(\varpi(\lambda_1,n)\big)^{\frac{1}{q}} > \big(K_\gamma(\lambda_1)\big)^{\frac{1}{q}}\big(1-\vartheta(\lambda_1,n)\big)^{\frac{1}{q}} \quad (0<q<1),
\end{aligned}$$

and

$$\left[\frac{1}{(K_\gamma(\lambda_1))^{p-1}(1-\vartheta(\lambda_1,n))^{p-1}}\right]^{\frac{1}{p}} > \left[\frac{1}{(\varpi(\lambda_1,n))^{p-1}}\right]^{\frac{1}{p}} \quad (p<0),$$

we obtain equivalent inequalities (57) and (58).

For $\varepsilon \in (0,\min\{q\lambda_2, q(1-\lambda_1)\})$, we set $\tilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}$ $(\in(0,1))$, $\tilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}$ $(\in(0,1))$, and

$$\begin{aligned}
\tilde{a}_m &:= \frac{\mu_{m+1}}{U_m}\ln^{\tilde{\lambda}_1-\varepsilon-1}\alpha U_m = \frac{\mu_{m+1}}{U_m}\ln^{\lambda_1-\frac{\varepsilon}{p}-1}\alpha U_m,\\
\tilde{b}_n &:= \frac{\upsilon_{n+1}}{V_n}\ln^{\tilde{\lambda}_2-\varepsilon-1}\beta V_n = \frac{\upsilon_{n+1}}{V_n}\ln^{\lambda_2-\frac{\varepsilon}{q}-1}\beta V_n.
\end{aligned}$$

Then, by (24), (25) and (18), we have

$$\begin{aligned}
&\|\tilde{a}\|_{p,\Phi_\lambda}\|\tilde{b}\|_{q,\tilde{\Upsilon}_\lambda}\\
&\quad= \left(\sum_{m=2}^{\infty}\frac{\mu_{m+1}}{U_m\ln^{1+\varepsilon}\alpha U_m}\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\frac{(1-\vartheta(\lambda_1,n))\upsilon_{n+1}}{V_n\ln^{1+\varepsilon}\beta V_n}\right)^{\frac{1}{q}}\\
&\quad= \left(\sum_{m=2}^{\infty}\frac{\mu_{m+1}}{U_m\ln^{1+\varepsilon}\alpha U_m}\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}}{V_n\ln^{1+\varepsilon}\beta V_n} - \sum_{n=2}^{\infty}O\left(\frac{\upsilon_{n+1}}{V_n\ln^{1+\varepsilon}\beta V_n}\right)\right)^{\frac{1}{q}}\\
&\quad= \frac{1}{\varepsilon}\left[\frac{1}{\ln^{\varepsilon}\alpha(1+\mu_2)} + \varepsilon O(1)\right]^{\frac{1}{p}}\left[\frac{1}{\ln^{\varepsilon}\beta(1+\upsilon_2)} + \varepsilon\big(\tilde{O}(1) - O_1(1)\big)\right]^{\frac{1}{q}},\\
&\tilde{I} = \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}k_\lambda(\ln\alpha U_m,\ln\beta V_n)\tilde{a}_m\tilde{b}_n\\
&\quad= \sum_{m=2}^{\infty}\left[\sum_{n=2}^{\infty}k_\lambda(\ln\alpha U_m,\ln\beta V_n)\frac{\upsilon_{n+1}\ln^{\tilde{\lambda}_1}\alpha U_m}{V_n\ln^{1-\tilde{\lambda}_2}\beta V_n}\right]\frac{\mu_{m+1}}{U_m\ln^{1+\varepsilon}\alpha U_m}\\
&\quad= \sum_{m=2}^{\infty}\frac{\mu_{m+1}\omega(\tilde{\lambda}_2,m)}{U_m\ln^{\varepsilon+1}\alpha U_m} \le K_\gamma(\tilde{\lambda}_1)\sum_{m=2}^{\infty}\frac{\mu_{m+1}}{U_m\ln^{\varepsilon+1}\alpha U_m}\\
&\quad= \frac{1}{\varepsilon}K_\gamma(\tilde{\lambda}_1)\left[\frac{1}{\ln^{\varepsilon}\beta(1+\upsilon_2)} + \varepsilon O(1)\right].
\end{aligned}$$

If there exists a positive constant $K \geq K_\gamma(\lambda_1)$ such that (57) is valid when replacing $K_\gamma(\lambda_1)$ by K, then, in particular, we have $\varepsilon\tilde{I} > \varepsilon K\|\tilde{a}\|_{p,\Phi_\lambda}\|\tilde{b}\|_{q,\tilde{\Upsilon}_\lambda}$, namely,

$$\begin{aligned} & K_\gamma\left(\lambda_1+\frac{\varepsilon}{q}\right)\left[\frac{1}{\ln^\varepsilon \alpha(1+\mu_2)}+\varepsilon O(1)\right] \\ & \quad > K\left[\frac{1}{\ln^\varepsilon \alpha(1+\mu_2)}+\varepsilon O(1)\right]^{\frac{1}{p}}\left[\frac{1}{\ln^\varepsilon \beta(1+\upsilon_2)}+\varepsilon\bigl(\tilde{O}(1)-O_1(1)\bigr)\right]^{\frac{1}{q}}. \end{aligned}$$

It follows that $K_\gamma(\lambda_1) \geq K$ $(\varepsilon \to 0^+)$. Hence, $K = K_\gamma(\lambda_1)$ is the best possible constant factor of (57).

Similarly to the reverse of (32), we still can find that

$$I \geq J_2\|b\|_{q,\tilde{\Upsilon}_\lambda}. \tag{59}$$

Hence, the constant factor $K_\gamma(\lambda_1)$ in (58) is still the best possible. Otherwise, we would reach the contradiction by (59) that the constant factor in (57) is not the best possible. □

Remark 3 For $\alpha = \beta = 1$, set

$$\begin{aligned} & \tilde{\vartheta}(\lambda_1,n) = \frac{k_\lambda(\frac{\ln(1+\mu_2\vartheta(n))}{\ln U_m},1)}{\lambda_1 K_\gamma(\lambda_1)}\frac{\ln^{\lambda_1}(1+\mu_2)}{ln^{\lambda_1}\beta V_n} = O\left(\frac{1}{ln^{\lambda_2}U_m}\right) \in (0,1) \quad \bigl(\vartheta(n) \in (0,1)\bigr), \\ & \tilde{\psi}_\lambda(n) := \bigl(1-\tilde{\vartheta}(\lambda_1,n)\bigr)\left(\frac{V_n}{\upsilon_{n+1}}\right)^{q-1}(\ln V_n)^{q(1-\lambda_2)-1}. \end{aligned}$$

It is evident that (57) and (58) are extensions of the following equivalent inequalities:

$$\sum_{n=2}^\infty\sum_{m=2}^\infty k_\lambda(\ln U_m,\ln V_n)a_m b_n > K_\gamma(\lambda_1)\|a\|_{p,\phi_\lambda}\|b\|_{q,\tilde{\psi}_\lambda}, \tag{60}$$

$$\left\{\sum_{n=2}^\infty \frac{\upsilon_{n+1}\ln^{p\lambda_2-1}V_n}{V_n(1-\tilde{\vartheta}(\lambda_1,n))}\left(\sum_{m=2}^\infty k_\lambda(\ln U_m,\ln V_n)a_m\right)^p\right\}^{\frac{1}{p}} > K_\gamma(\lambda_1)\|a\|_{p,\phi_\lambda}, \tag{61}$$

where the constant factor $K_\gamma(\lambda_1)$ is the best possible.

5 Conclusions

In this paper, by using the way of weight coefficients, the technique of real analysis, and Hermite-Hadamard's inequality, a more accurate Hardy-Mulholland-type inequality with multi-parameters and a best possible constant factor is given by Theorems 1, 2, and the equivalent forms are considered. The equivalent reverses with the best possible constant factor are obtained by Theorems 3, 4. Moreover, the operator expressions and some particular cases are considered. The method of weight coefficients is very important, which helps us to prove the main inequalities with the best possible constant factor. The lemmas and theorems provide an extensive account of this type of inequalities.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. QC participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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